# SL2(Z)-tillings of the torus, Coxeter-Conway friezes and Farey triangulations 

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# $\mathrm{SL}_{2}(\mathbb{Z})$-tilings of the torus, Coxeter-Conway friezes and Farey triangulations 

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#### Abstract

The notion of $\mathrm{SL}_{2}$-tiling is a generalization of that of classical Coxeter-Conway frieze pattern. We classify doubly antiperiodic $\mathrm{SL}_{2}$-tilings that contain a rectangular domain of positive integers. Every such $\mathrm{SL}_{2}$-tiling corresponds to a pair of frieze patterns and a unimodular $2 \times 2$-matrix with positive integer coefficients. We relate this notion to triangulated $n$-gons in the Farey graph.


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## 1. Introduction

Frieze patterns were introduced and studied by Coxeter and Conway, [Co, CC], in the 70 's. A frieze pattern is an infinite array of numbers, bounded by two diagonals of 1's, such that every four adjacent numbers $a, b, c, d$ forming a "small" square satisfy the relation $a d-b c=1$ called the unimodular rule; for an example see Figure 1. The width of the frieze is the number of diagonals between the bounding diagonals of 1 's.

The fundamental Conway-Coxeter theorem [CC] offers the following classification: frieze patterns with positive integer entries of width $n-3$, are in one-to-one correspondence with triangulations of a convex $n$-gon; for a simple proof see [Hen]. More precisely, given a triangulated $n$-gon in the oriented plane, one constructs a frieze of width $n-3$ as follows. The diagonal next to the diagonal of 1 's is formed by the numbers of triangles incident at each vertex (taken cyclically). This, in particular, implies that every diagonal in a frieze of width $n-3$ is $n$-periodic. Throughout this paper, we will be considering frieze patterns with positive integer entries.

The following terminology is due to Conway and Coxeter [CC]. A sequence of $n$ positive integers $q=\left(q_{0}, \ldots, q_{n-1}\right)$ is called a quiddity of order $n$, if there


Figure 1
A 7-periodic frieze pattern and the corresponding triangulated heptagon
exists a triangulated $n$-gon such that every $q_{i}$ is equal to the number of incident triangles at $i$-th vertex. For instance, the example in Figure 1 corresponds to the following quiddities of order $7:(1,3,2,2,1,4,2),(3,2,2,1,4,2,1), \ldots$ (cyclic permutation).

Every quiddity of order $n$ determines a unique positive integer frieze pattern. Two quiddities correspond to the same positive integer frieze pattern if and only if they differ by a cyclic permutation. According to the Conway-Coxeter theorem, positive integer frieze patterns can be enumerated by the Catalan numbers.

Example 1.0.1. For each case $n=3,4$ and 5, there is a unique (up to cyclic permutation) quiddity: $(1,1,1),(1,2,1,2)$ and $(1,3,1,2,2)$, respectively.

For $n=6$, there are four different quiddities:

$$
\begin{equation*}
(1,3,1,3,1,3), \quad(1,4,1,2,2,2), \quad(1,2,3,1,2,3) \tag{1,3,2,1,3,2}
\end{equation*}
$$

and their cyclic permutations.

We can also consider the "degenerate" case $n=2$, where the corresponding "degenerate" quiddity is $(0,0)$.

Examples of frieze patterns can be constructed using the computer program [Scha]

Among many beautiful properties of Coxeter-Conway friezes, the property of periodicity and so-called Laurent phenomenon are particularly important. They relate frieze patterns to the theory of cluster algebras developed by Fomin and Zelevinsky, [FZ1, FZ2].

Various generalizations of Coxeter-Conway friezes have recently been introduced and studied, see [CaCh, Pro, BM, ARS, MOT]. One of the generalizations, called $\mathrm{SL}_{2}$-tiling, was first considered by Assem, Reutenauer and Smith [ARS], and further developed by Bergeron and Reutenauer [BR]. An $\mathrm{SL}_{2}$-tiling is an infinite array of numbers satisfying the above unimodular rule, without the condition of bounding diagonals of 1's. Unlike the frieze patterns, $\mathrm{SL}_{2}$-tilings are not necessarily periodic. Nevertheless, correspondences between $\mathrm{SL}_{2}$-tilings and triangulations can be established, [HJ, BHJ].

The case of $(n, m)$-antiperiodic, or "toric" $\mathrm{SL}_{2}$-tilings was suggested in [BR]. In this paper, we study such tilings.

The main results of the paper are the following.
We classify doubly antiperiodic $\mathrm{SL}_{2}$-tilings that contain a rectangular fundamental domain of positive integers. We show that every such $\mathrm{SL}_{2}$-tiling is generated by a pair of quiddities and a unimodular $2 \times 2$-matrix with positive

|  | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |  |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | :--- |
| $\cdots$ | 2 | 5 | 8 | 11 | 3 | -2 | -5 | -8 | -11 | -3 | $\cdots$ |
| $\cdots$ | 7 | 18 | 29 | 40 | 11 | -7 | -18 | -29 | -40 | -11 | $\cdots$ |
| $\cdots$ | 5 | 13 | 21 | 29 | 8 | -5 | -13 | -21 | -29 | -8 | $\cdots$ |
| $\cdots$ | 3 | 8 | 13 | 18 | 5 | -3 | -8 | -13 | -18 | -5 | $\cdots$ |
| $\cdots$ | -2 | -5 | -8 | -11 | -3 | 2 | 5 | 8 | 11 | 3 | $\cdots$ |
| $\cdots$ | -7 | -18 | -29 | -40 | -11 | 7 | 18 | 29 | 40 | 11 | $\cdots$ |
| $\cdots$ | -5 | -13 | -21 | -29 | -8 | 5 | 13 | 21 | 29 | 8 | $\cdots$ |
| $\cdots$ | -3 | -8 | -13 | -18 | -5 | 3 | 8 | 13 | 18 | 5 | $\cdots$ |
|  | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |  |

Figure 2
A $(4,5)$-antiperiodic $\mathrm{SL}_{2}$-tiling with positive rectangular domain
integer coefficients. Although there are infinitely many such $\mathrm{SL}_{2}$-tilings, their description is very explicit.

Following the original idea of Coxeter [Co], we also interpret the entries of a doubly periodic $\mathrm{SL}_{2}$-tiling that contain a rectangular fundamental domain of positive integers in terms of the Farey graph of rational numbers. Every such $\mathrm{SL}_{2}$-tiling corresponds to a triple: an $n$-gon, an $m$-gon in the Farey graph, and a totally positive matrix from $\mathrm{SL}_{2}(\mathbb{Z})$ relating them. We also obtain an explicit formula for the entries of the tiling.

## 2. Farey graph and the Conway-Coxeter theorem

In this section, we give an explanation of the relation between the Coxeter frieze patterns and triangulated $n$-gons.

It was already noticed by Coxeter [Co] that a Farey series (of arbitrary order $N$ ) defines a frieze pattern. Moreover, every frieze pattern corresponds to an $n$-gon (i.e., an $n$-cycle) in the Farey graph. A Farey $n$-gon always carries a triangulation; we will prove that this triangulation is precisely that of ConwayCoxeter theorem. This statement seems to be new and to extend the observation illustrated in [Scha].
2.1. Farey graph, Farey series and Farey $\boldsymbol{n}$-gons. For two rational numbers, $v_{1}, v_{2} \in \mathbb{Q}$, written as irreducible fractions $v_{1}=\frac{a_{1}}{b_{1}}$ and $v_{2}=\frac{a_{2}}{b_{2}}$, the Farey "distance" is defined by

$$
d\left(v_{1}, v_{2}\right):=\left|a_{1} b_{2}-a_{2} b_{1}\right| .
$$

Note that the above "distance" does not satisfy the triangle inequality. Recall the definition of the Farey graph.
(1) The set of vertices of the Farey graph is $\mathbb{Q} \cup\{\infty\}$, with $\infty$ represented by $\frac{1}{0}$.
(2) Two vertices, $v_{1}, v_{2}$ are joined by a (non-oriented) edge $\left(v_{1}, v_{2}\right)$ whenever $d\left(v_{1}, v_{2}\right)=1$.

The Farey graph is often embedded into the hyperbolic half-plane, the edges being realized as geodesics joining rational points on the ideal boundary.

The following classical properties of the Farey graph can be found in [HW] (the proof is elementary).

Proposition 2.1.1. (i) Every 3-cycle of the Farey graph is of the form

$$
\begin{equation*}
\left\{\frac{a_{1}}{b_{1}}, \frac{a_{1}+a_{2}}{b_{1}+b_{2}}, \frac{a_{2}}{b_{2}}\right\} . \tag{2.1}
\end{equation*}
$$

(ii) Every edge of the Farey graph belongs to a 3-cycle.
(iii) Edges in the Farey graph do not cross, i.e., for a quadruple $v_{1}>v_{2}>v_{3}>v_{4}$ it is not possible to have edges $\left(v_{1}, v_{3}\right)$ and $\left(v_{2}, v_{4}\right)$.

Definition 2.1.2. The Farey series (also called Farey sequence) of order $N$ is the sequence of irreducible fractions in $[0,1]$ whose denominators do not exceed $N$.

We will write the sequences in decreasing order; see Figure 3.


Figure 3
The Farey series of order 5 embedded in the Farey graph

The following fundamental property of Farey series is also proved in [HW]. It shows that every Farey series is a cycle in the Farey graph.

Proposition 2.1.3. Every two consecutive numbers in a Farey series are joined by an edge in the Farey graph.

This is less elementary than Proposition 2.1.1, so we propose here a short proof. Our proof is different from the well-known one, it is based on the classical Pick formula.

Proof. Consider two consecutive numbers $\frac{a}{b}>\frac{c}{d}$, in a Farey series of some order $N$. Suppose that $a d-b c \geq 2$. The quantity $A=\frac{1}{2}(a d-b c)$ is the area of the Euclidean triangle spanned by the vertices $(0,0),(a, b),(c, d)$. Pick's formula states:

$$
A=I+\frac{B}{2}-1,
$$



Figure 4
The case of interior point
where $I$ is the number of integer points in the interior of the triangle, and $B$ the number of integer points on the border. By assumption, $A \geq 1$, and therefore $I+\frac{B}{2} \geq 2$. It follows that there exists a point $(x, y)$, which is either inside the triangle, or on the segment between $(a, b)$ and $(c, d)$ (since the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are irreducible). One then has:

$$
y \leq \max (b, d) \leq N \quad \text { and } \quad \frac{a}{b}>\frac{x}{y}>\frac{c}{d} .
$$

This contradicts the assumption that $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive numbers in the Farey series.

Proposition 2.1.3 is used three times to prove the following.
Corollary 2.1.4. Every Farey series forms a triangulated polygon in the Farey graph.

Proof. We prove this statement by induction on $N$ (the order of Farey series). Assume that the series of order $N-1$ is triangulated. The series of order $N$ is obtained from that of order $N-1$ by adding points of the form $\frac{k}{N}$.

First, we observe that two points, $\frac{k_{1}}{N}$ and $\frac{k_{2}}{N}$ cannot be consecutive. Indeed, $d\left(\frac{k_{1}}{N}, \frac{k_{2}}{N}\right) \neq 1$ : that would contradict Proposition 2.1.3; therefore, every new point $\frac{k}{N}$ appears between two "old" points:

$$
\begin{equation*}
\frac{p_{1}}{q_{1}}>\frac{k}{N}>\frac{p_{2}}{q_{2}} \tag{2.2}
\end{equation*}
$$

Second, by Proposition 2.1.3, $\frac{k}{N}$ is joined by edges with $\frac{p_{1}}{q_{1}}$ and $\frac{p_{2}}{q_{2}}$. Third, $\frac{p_{1}}{q_{1}}$ and $\frac{p_{2}}{q_{2}}$ are joined by an edge, according to Proposition 2.1.3 applied to the series of order $N-1$. We conclude that (2.2) is a triangle.

We will be interested in $n$-cycles (or " $n$-gons") in the Farey graph that are more general than Farey series.

Definition 2.1.5. (1) An $n$-gon in the Farey graph, or a Farey $n$-gon is a decreasing sequence of rationals $\left(v_{0}, \ldots, v_{n-1}\right)$ :

$$
\infty \geq v_{0}>v_{1}>\ldots>v_{n-1} \geq 0
$$

such that every pair of consecutive numbers $v_{i}, v_{i+1}$, as well as $v_{n-1}, v_{0}$, are joined by an edge.
(2) The $n$-gon is called normalized if $v_{0}=\infty$ and $v_{n-1}=0$.

Since every $n$-gon can be embedded in a Farey series, Corollary 2.1.4 implies the following.

Corollary 2.1.6. Every Farey $n$-gon is triangulated.

We thus can speak of the quiddity of a Farey $n$-gon.

Proof. A Farey $n$-gon is obtained from a Farey series which is a triangulated polygon, by cutting along diagonals of the triangulation.

We define the notion of cyclic equivalence of Farey $n$-gons. Given an $n$-gon $\left(v_{0}, \ldots, v_{n-1}\right)$, consider the $n$-cycle $\left(v_{1}, \ldots, v_{n-1}, v_{0}\right)$, and renormalize it using the $\mathrm{SL}_{2}(\mathbb{Z})$-action so that $v_{1}=\infty$ and $v_{0}=0$. The obtained $n$-gon is called cyclically equivalent to the given one. For an example, see Figure 5.


Figure 5
Two cyclically equivalent normalized heptagons in the Farey graph corresponding to the frieze of Figure 1
2.2. Farey $\boldsymbol{n}$-gons and Coxeter-Conway friezes. Proposition 2.1.3 leads to the following observation due to Coxeter [Co]: every Farey series gives rise to a Coxeter-Conway frieze pattern of positive integers. Along the same lines, we have the following strengthened statement.

Proposition 2.2.1. The Coxeter-Conway frieze patterns of positive integers of width $n-3$ are in one-to-one correspondence with the normalized Farey $n$-gons, up to cyclic equivalence.

Proof. The correspondence is given by considering the ratios of two consecutive rows of the frieze patterns. The sequence

$$
v_{0}=\frac{1}{0}, \quad v_{1}=\frac{a_{1}}{1}, \quad \ldots, \quad v_{i}=\frac{a_{i}}{b_{i}}, \quad \ldots, \quad v_{n-2}=\frac{1}{b_{n-2}}, \quad v_{n-1}=\frac{0}{1}
$$

corresponds to the frieze determined by the rows

$$
\begin{array}{ccccccc}
1 & a_{1} & a_{2} & \cdots & a_{n-3} & 1 & 0 \\
0 & 1 & b_{2} & & \cdots & b_{n-2} & 1
\end{array}
$$

and vice versa.
The Conway-Coxeter theorem mentioned in the introduction provides a relation between frieze patterns and triangulations. The following result somewhat "demystifies" this relation and provides an alternative proof of the Conway-Coxeter theorem.

Theorem 1. The quiddity of a Farey $n$-gon coincides with the quiddity of the corresponding Coxeter-Conway frieze pattern.

Proof. Consider a frieze pattern, and denote by $c_{i, j}$ its entries:

$$
\begin{array}{llllllll}
0 & 1 & c_{1,1} & c_{1,2} & \cdots & c_{1, n-3} & 1 & 0 \\
& 0 & 1 & c_{2,2} & & \cdots & c_{2, n-2} & 1 \\
& & \ddots & \ddots & & & & \ddots
\end{array}
$$

where

$$
\begin{cases}c_{i, j}=1, & i-j=1 \text { or } 3-n \\ c_{i, j}=0, & i-j=2 \text { or } 2-n\end{cases}
$$

The quiddity of the frieze pattern reads in the $n$-periodic line $\left(c_{i, i}\right)$.
Clearly, two consecutive rows determine the rest of the frieze; the following formula was proved in [Co], formula (5.6):

$$
c_{i, j}=c_{1, i-2} c_{2, j}-c_{1, j} c_{2, i-2} .
$$

In particular, we have:

$$
\begin{equation*}
c_{i, i}=c_{1, i-2} c_{2, i}-c_{1, i} c_{2, i-2} . \tag{2.3}
\end{equation*}
$$

The corresponding Farey $n$-gon has the following vertices

$$
v_{0}=\frac{1}{0}, \quad v_{1}=\frac{c_{1,1}}{1}, \quad \ldots \quad v_{i}=\frac{c_{1, i}}{c_{2, i}}, \quad \ldots \quad v_{n-2}=\frac{1}{c_{2, n-2}}, \quad v_{n-1}=\frac{0}{1} .
$$

Therefore, the expression (2.3) reads: $c_{i, i}=d\left(v_{i-2}, v_{i}\right)$. It remains to calculate the Farey distance between pairs of vertices $v_{i-2}$ and $v_{i}$ in a Farey $n$-gon.

Lemma 2.2.2. Given a (triangulated) Farey $n$-gon

$$
v_{0}=\frac{1}{0}, \quad v_{1}=\frac{a_{1}}{1}, \quad \ldots, \quad v_{i}=\frac{a_{i}}{b_{i}}, \quad \ldots, \quad v_{n-2}=\frac{1}{b_{n-2}}, \quad v_{n-1}=\frac{0}{1}
$$

the Farey distance $d\left(v_{i-1}, v_{i+1}\right)$ coincides with the number of triangles incident at $v_{i}$.

Proof. Among all the vertices of the $n$-gon $\left(v_{i}\right)$, let us select those connected to $v_{i}$ by edges of the Farey graph. Denote by $\left\{v_{i_{1}}, \ldots, v_{i_{k}}\right\}$, resp. $\left\{v_{i_{k+1}}, \ldots, v_{i_{k+\ell}}\right\}$ the vertices at the left, resp. right, of $v_{i}$, so that

$$
v_{i_{1}}>\ldots>v_{i_{k}}>v_{i}>v_{i_{k+1}}>\ldots>v_{i_{k+\ell}}
$$

(note that $v_{i_{k}}=v_{i-1}$ and $v_{i_{k+1}}=v_{i+1}$ ). The number of triangles incident at $v_{i}$ is then equal to $k+\ell-1$.

Two consecutive selected vertices, $v_{i_{j}}$ and $v_{i_{j+1}}$ are connected by an edge. Indeed, this follows from the fact that every Farey polygon is triangulated. Therefore, the vertices $\left(v_{i_{j}}, v_{i_{j+1}}, v_{i}\right)$ form a triangle (a 3-cycle) in the Farey graph. Using Eq. (2.1), we obtain by induction:

$$
v_{i-1}\left(=v_{i_{k}}\right)=\frac{a_{i_{1}}+(k-1) a_{i}}{b_{i_{1}}+(k-1) b_{i}}, \quad v_{i+1}\left(=v_{i_{k+1}}\right)=\frac{a_{i_{k+\ell}}+(\ell-1) a_{i}}{b_{i_{k+\ell}}+(\ell-1) b_{i}}
$$

We have:
$d\left(v_{i-1}, v_{i+1}\right)=a_{i_{1}} b_{i_{k+\ell}}-b_{i_{1}} a_{i_{k+\ell}}+(k-1)\left(a_{i} b_{i_{k+\ell}}-b_{i} a_{i_{k+\ell}}\right)+(\ell-1)\left(a_{i_{1}} b_{i}-b_{i_{1}} a_{i}\right)$.
By assumption, $v_{i}$ is joined by edges with $v_{i_{1}}$ and $v_{i_{k+\ell}}$, hence $a_{i} b_{i_{k+\ell}}-$ $b_{i} a_{i_{k+\ell}}=1$, and $a_{i_{1}} b_{i}-b_{i_{1}} a_{i}=1$. Furthermore, $\left(v_{i_{1}}, v_{i}, v_{i_{k+\ell}}\right)$ is also a triangle, therefore $a_{i_{1}} b_{i_{k+\ell}}-b_{i_{1}} a_{i_{k+\ell}}=1$. We have finally:

$$
\begin{equation*}
d\left(v_{i-1}, v_{i+1}\right)=k+\ell-1 . \tag{2.4}
\end{equation*}
$$

Hence the lemma.
Theorem 1 is proved.
2.3. Entries of the frieze pattern. Coxeter's formula (5.6) in [Co] for the entries of the frieze pattern translates into our language as the following general expression:

$$
\begin{equation*}
c_{i, j}=d\left(v_{i-2}, v_{j}\right) \tag{2.5}
\end{equation*}
$$

where, as above, $\left(v_{i}\right)$ is the Farey $n$-gon corresponding to the frieze pattern.

## 3. $\mathrm{SL}_{2}$-tilings

In this section, we introduce the main notions studied in this paper.
3.1. Tame $\mathbf{S L}_{\mathbf{2}}$-tilings. Let us first recall the notion of $\mathrm{SL}_{2}$-tiling introduced in [BR].
(1) An $\mathrm{SL}_{2}$-tiling, is an infinite matrix $\mathcal{A}=\left(a_{i, j}\right)_{(i, j) \in \mathbb{Z} \times \mathbb{Z}}$, such that every adjacent $2 \times 2$-minor equals 1 :

$$
\left|\begin{array}{cc}
a_{i, j} & a_{i, j+1} \\
a_{i+1, j} & a_{i+1, j+1}
\end{array}\right|=1,
$$

for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.
(2) The tiling is called tame if every adjacent $3 \times 3$-minor equals 0 :

$$
\left|\begin{array}{ccc}
a_{i, j} & a_{i, j+1} & a_{i, j+2} \\
a_{i+1, j} & a_{i+1, j+1} & a_{i+1, j+2} \\
a_{i+2, j} & a_{i+2, j+1} & a_{i+2, j+2}
\end{array}\right|=0
$$

for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.
Let us stress the fact that a generic $\mathrm{SL}_{2}$-tiling is tame.
3.2. Antiperiodicity. The following condition was also suggested in [BR].

An $\mathrm{SL}_{2}$-tiling is called $(n, m)$-antiperiodic if every row is $n$-antiperiodic, and every column is $m$-antiperiodic:

$$
\begin{aligned}
a_{i, j+n} & =-a_{i, j}, \\
a_{i+m, j} & =-a_{i, j},
\end{aligned}
$$

for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.
The following relation between $(n, m)$-antiperiodic $\mathrm{SL}_{2}$-tilings and the classical Coxeter-Conway frieze patterns shows that the antiperiodicity condition for the $\mathrm{SL}_{2}$-tilings is natural and interesting.
3.3. Frieze patterns and $(\boldsymbol{n}, \boldsymbol{n})$-antiperiodic $\mathbf{S L}_{\mathbf{2}}$-tilings. As explained in $[\mathrm{BR}]$, every Coxeter-Conway frieze pattern of width $n-3$ can be extended to a tame ( $n, n$ )-antiperiodic $\mathrm{SL}_{2}$-tiling, in a unique way.

The construction is as follows. One adds two diagonals of 0 's next to the diagonals of 1 's, and then continues by antiperiodicity.

Example 3.3.1. The frieze pattern in Figure 1 corresponds to the following (7,7)antiperiodic tame $\mathrm{SL}_{2}$-tiling.

$$
\begin{array}{lrrrrrrrrrrrrr} 
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 2 & 3 & 1 & 1 & 1 & 0 & -1 & -2 & -3 & -1 & -1 & \cdots \\
\cdots & 0 & 1 & 2 & 1 & 2 & 3 & 1 & 0 & -1 & -2 & -1 & -2 & \cdots \\
\cdots & -1 & 0 & 1 & 1 & 3 & 5 & 2 & 1 & 0 & -1 & -1 & -3 & \cdots \\
\cdots & -2 & -1 & 0 & 1 & 4 & 7 & 3 & 2 & 1 & 0 & -1 & -4 & \cdots \\
\cdots & -1 & -1 & -1 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & -1 & \cdots \\
\cdots & -2 & -3 & -4 & -1 & 0 & 1 & 1 & 2 & 3 & 4 & 1 & 0 & \cdots \\
\cdots & -3 & -5 & -7 & -2 & -1 & 0 & 1 & 3 & 5 & 7 & 2 & 1 & \cdots \\
\cdots & -1 & -2 & -3 & -1 & -1 & -1 & 0 & 1 & 2 & 3 & 1 & 1 & \cdots \\
\cdots & 0 & -1 & -2 & -1 & -2 & -1 & -1 & 0 & 1 & 2 & 1 & 2 & \cdots
\end{array}
$$

For the details of the above construction and the "antiperiodic nature" of Conway-Coxeter's friezes; see [BR, MOST].
3.4. Positive rectangular domain. In this paper, we are considering $(n, m)$ antiperiodic $\mathrm{SL}_{2}$-tilings that contain an $m \times n$-rectangular domain of positive integers.

More precisely, we are interested in $\mathrm{SL}_{2}$-tilings of the following form:

|  | $\vdots$ | $\vdots$ | $\vdots$ |  |
| :---: | :---: | :---: | :---: | :---: |
| $\ldots$ | $P$ | $-P$ | $P$ | $\ldots$ |
| $\ldots$ | $-P$ | $P$ | $-P$ | $\ldots$ |
|  | $\vdots$ | $\vdots$ | $\vdots$ |  |

where $P$ is an $m \times n$-matrix with entries in $\mathbb{Z}_{>0}$. An example of such an $\mathrm{SL}_{2}$-tilling is presented in Figure 2.

The following property is important for us.

Proposition 3.4.1. An ( $n, m$ )-antiperiodic $\mathrm{SL}_{2}$-tiling that contains a positive $m \times n$-rectangular domain is tame.

Proof. This is a consequence of the Jacobi identity or Dodgson formula for determinants:

$$
\left|\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array}\right|\left|\begin{array}{ccc}
\circ & \circ & 0 \\
\circ & \bullet & 0 \\
\circ & \circ & 0
\end{array}\right|=\left|\begin{array}{lll}
\bullet & \bullet & 0 \\
\bullet & \bullet & 0 \\
\circ & \circ & 0
\end{array}\right|\left|\begin{array}{lll}
\circ & \circ & 0 \\
\circ & \bullet & \bullet \\
\circ & \bullet & \bullet
\end{array}\right|-\left|\begin{array}{lll}
\circ & \circ & 0 \\
\bullet & \bullet & 0 \\
\bullet & \bullet & \circ
\end{array}\right|\left|\begin{array}{lll}
\circ & \bullet & \bullet \\
0 & \bullet & \bullet \\
\circ & \circ & 0
\end{array}\right|
$$

where the white dots represent deleted entries, and the black dots initial entries.
Since the values are non zero and the $2 \times 2$-minors all equal to 1 , the above identity implies that all the $3 \times 3$-minors vanish.

## 4. The main theorem

In this section, we formulate our main result. The proof will be given in Section 6.
4.1. Classification. It turns out that every $\mathrm{SL}_{2}$-tiling corresponds to a pair of frieze patterns and a positive integer $2 \times 2$-matrix $M$ satisfying some conditions.

Theorem 2. The set of ( $n, m$ )-antiperiodic $\mathrm{SL}_{2}$-tilings containing a fundamental rectangular domain of positive integers is in a one-to-one correspondence with the set of triples $\left(q, q^{\prime}, M\right)$, where

$$
q=\left(q_{0}, \ldots, q_{n-1}\right), \quad q^{\prime}=\left(q_{0}^{\prime}, \ldots, q_{m-1}^{\prime}\right)
$$

are quiddities of order $n$ and $m$, respectively, and where $M=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ is a unimodular $2 \times 2$-matrix with positive integer coefficients, such that the inequalities

$$
\begin{equation*}
q_{0}<\frac{b}{a}, \quad q_{0}^{\prime}<\frac{c}{a} \tag{4.1}
\end{equation*}
$$

are satisfied.
Remark 4.1.1. It is important to notice that inequalities (4.1) also imply

$$
\begin{equation*}
q_{0}<\frac{d}{c}, \quad q_{0}^{\prime}<\frac{d}{b} \tag{4.2}
\end{equation*}
$$

Indeed, the unimodular condition $a d-b c=1$ and the assumption that $a, b, c, d$ are positive integers imply that $\frac{b}{a}<\frac{d}{c}$ and $\frac{c}{a}<\frac{d}{b}$.

Corollary 4.1.2. For every pair of quiddities $q, q^{\prime}$, there exist infinitely many ( $n, m$ )-antiperiodic $\mathrm{SL}_{2}$-tilings containing a fundamental rectangular domain of positive integers.

Proof. Given arbitrary pair of quiddities $q$ and $q^{\prime}$, the matrices:

$$
\left(\begin{array}{cc}
1 & b \\
c & b c+1
\end{array}\right)
$$

satisfy (4.1) for sufficiently large $b, c$.
4.2. The semigroup $\mathcal{S}$. Consider the set of $2 \times 2$-matrices with positive integral entries satisfying the following conditions of positivity:

$$
\mathcal{S}=\left\{\left.\left(\begin{array}{ll}
a & b  \tag{4.3}\\
c & d
\end{array}\right) \in \operatorname{SL}_{2}(\mathbb{Z}) \right\rvert\, \begin{array}{l}
0<a<b<d \\
0<a<c<d
\end{array}\right\}
$$

Note that the inequalities $b<d$ and $c<d$ are included for the sake of completeness. These inequalities actually follow from $a<b, a<c$ together with $a d-b c=1$ and the assumption that $a, b, c, d$ are positive.

We have the following property.

Proposition 4.2.1. The set $\mathcal{S} \subset \mathrm{SL}_{2}(\mathbb{Z})$ is a semigroup, i.e., it is stable by multiplication.

Proof. Straightforward.

The semigroup $\mathcal{S}$ naturally appears in our context. Indeed, if $n, m \geq 3$, then the inequalities (4.1) imply $M \in \mathcal{S}$. Moreover every quiddity $q$ contains a unit entry, so that after a cyclic permutation of any quiddity one can obtain $q_{0}=1$. The inequalities (4.1) then coincide with the conditions (4.3).

### 4.3. Examples. Let us give two simple examples of $\mathrm{SL}_{2}$-tilings.

Example 4.3.1. There is a one-to-one correspondence between ( 3,3 )-antiperiodic $\mathrm{SL}_{2}$-tilings containing a fundamental domain of positive integers and elements of the semigroup $\mathcal{S}$. Indeed, the only quiddity of order 3 is $q=(1,1,1)$. To
every matrix (4.3) there corresponds the following $\mathrm{SL}_{2}$-tiling:

$$
\begin{array}{ccccc} 
& \vdots & \vdots & \vdots & \\
\cdots & a & b & b-a & \cdots \\
\cdots & c & d & d-c & \cdots \\
\cdots & c-a & d-b & d-b-c+a & \cdots
\end{array}
$$

It is a good exercise to check that the positivity condition $d-b-c+a>0$ follows from (4.3) together with $a d-b c=1$.

Example 4.3.2. In the case $n=2$ or $m=2$, the conditions (4.1) become trivial.
Consider also the simplest (degenerate) case of (2,2)-antiperiodic $\mathrm{SL}_{2}$-tilings.
A $(2,2)$-antiperiodic $\mathrm{SL}_{2}$-tiling containing a fundamental domain of positive integers is of the form:

$$
\begin{array}{cccccc} 
& \vdots & \vdots & \vdots & \vdots & \\
\cdots & a & b & -a & -b & \cdots \\
\cdots & c & d & -c & -d & \cdots \\
& \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

where $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ is an arbitrary unimodular matrix with positive integer coefficients. Note that this case corresponds to the "degenerate quiddity" of order 2, namely $q=(0,0)$.

## 5. Frieze patterns and linear recurrence equations

We will recall here a remarkable and well-known property of Coxeter-Conway frieze patterns. It concerns a relation of frieze patterns and linear recurrence equations. The statement presented in this subsection was implicitly obtained in [CC]; for details see [MOST]. We recall this statement without proof.

### 5.1. Discrete non-oscillating Hill equations.

Definition 5.1.1. Let $\left(c_{i}\right)_{i \in \mathbb{Z}}$ be an arbitrary $n$-periodic sequence of numbers.
(a) A linear difference equation

$$
\begin{equation*}
V_{i+1}=c_{i} V_{i}-V_{i-1}, \tag{5.1}
\end{equation*}
$$

where the sequence $\left(c_{i}\right)$ is given (the coefficients) and where $\left(V_{i}\right)$ is unknown (the solution), is called a discrete Hill, or Sturm-Liouville, or one-dimensional Schrödinger equation.
(b) The equation (5.1) is called non-oscillating if every solution $\left(V_{i}\right)$ is antiperiodic:

$$
V_{i+n}=-V_{i},
$$

for all $i$, and has exactly one sign change in any sequence $\left(V_{i}, V_{i+1}, \ldots V_{i+n}\right)$.
In other words, every solution of a non-oscillating equation must have nonnegative intervals of length $n$, that is, $n$ consecutive non-negative values: $\left(V_{k}, \ldots, V_{k+n-1}\right)$.

Moreover, for a generic solution of (5.1), all the elements $V_{j}$ of a non-negative interval are strictly positive. Zero values can only occur at the endpoints: $V_{k}=0$, or $V_{k+n-1}=0$.

Note also that the coefficients in a non-oscillating equation are necessarily positive.
5.2. Frieze patterns and difference equations. The relation between the equations (5.1) and Coxeter-Conway frieze patterns is as follows.

Proposition 5.2.1. Given an equation (5.1) with integer coefficients, it is a nonoscillating equation if and only if the coefficients $\left(c_{0}, c_{1}, \ldots, c_{n-1}\right)$ form a quiddity.

Proof. This is an immediate consequence of properties established by Coxeter and Conway. Indeed, it was proved in [Co] (see also [CC] property (17)) that the entries in any row of the pattern (extended by antiperiodicity) form a solution of an equation (5.1), where the coefficients $c_{i}$ are given by the sequence on the first non-trivial diagonal. Thus, from an non-oscillating equation one can write down a frieze, and vice versa.

$$
\begin{array}{cccccccccc}
\ddots & \ddots & & \ddots & \ddots & \ddots & & & & \\
& 1 & c_{0} & \cdots & 1 & 0 & -1 & \cdots & & \\
& 1 & c_{1} & \cdots & 1 & 0 & -1 & \cdots & \\
& & 1 & c_{2} & \cdots & 1 & 0 & -1 & \cdots \\
& & & & \ddots & \ddots & & \ddots & \ddots & \ddots
\end{array}
$$

Finally, the integer condition establishes the correspondence with quiddities.

Of course, for an arbitrary non-oscillating equation (5.1), the corresponding frieze pattern does not necessarily have integer entries. In [MOST], the space of frieze patterns and the space of non-oscillating equation (5.1) are identified in a more general setting.

Example 5.2.2. (a) The simplest quiddity $q=(1,1,1)$ corresponds to the nonoscillating equation with all $c_{i}=1$. Every solution of this equation is 3antiperiodic and can be obtained as a linear combination of the following two solutions:

$$
\left(V_{i}^{(1)}\right)=(\ldots, 0,1,1,0,-1,-1, \ldots), \quad\left(V_{i}^{(2)}\right)=(\ldots, 1,1,0,-1,-1,0 \ldots)
$$

This corresponds to a degenerate frieze of Coxeter-Conway of width 0.
(b) The frieze from Figure 1 corresponds to the non-oscillating equation with 7 -antiperiodic solutions that are linear combinations of the following two:

$$
\left(V_{i}^{(1)}\right)=(\ldots, 1,2,3,1,1,1,0, \ldots), \quad\left(V_{i}^{(2)}\right)=(\ldots, 0,1,2,1,2,3,1, \ldots)
$$

The above two solutions are exactly the first two rows of the frieze in Figure 1. One can of course choose different rows for a basis.

Note that, in the both cases, the basis solutions $\left(V_{i}^{(1)}\right),\left(V_{i}^{(2)}\right)$ are not generic since they contain zeros.

## 6. Proof of Theorem 2

6.1. The construction. Given a triple $\left(q, q^{\prime}, M\right)$ as in Theorem 2 , we will construct an $\mathrm{SL}_{2}$-tiling satisfying the above conditions. Define $T=\left(a_{i, j}\right)$ using the following recurrence relations:

$$
\begin{align*}
a_{i, j+1} & :=q_{j} a_{i, j}-a_{i, j-1},  \tag{6.1}\\
a_{i+1, j} & :=q_{i}^{\prime} a_{i, j}-a_{i-1, j},
\end{align*}
$$

for all $i, j \in \mathbb{Z}$, where the quiddities are periodically extended, i.e $q_{i}=q_{i+n}, q_{i}^{\prime}=$ $q_{i+m}^{\prime}$, and taking the initial conditions

$$
\left(\begin{array}{ll}
a_{0,0} & a_{0,1}  \tag{6.2}\\
a_{1,0} & a_{1,1}
\end{array}\right):=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) .
$$

It is very easy to check that the tiling $T$ is well-defined, i.e., the two recurrences commute and the calculations along the rows and columns give the same result. We show that the defined tiling $T$ contains a fundamental rectangular domain of positive integers.

By Proposition 5.2.1, the defined tiling $T$ is $(n, m)$-antiperiodic. Consider the following $m \times n$-subarray of $T$

$$
P=\left(\begin{array}{cccc}
a_{0,0} & a_{0,1} & \cdots & a_{0, n-1}  \tag{6.3}\\
a_{1,0} & a_{1,1} & \cdots & a_{1, n-1} \\
\cdots & & & \\
a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1, n-1}
\end{array}\right) .
$$

The main step of the proof of Theorem 2 is the following lemma.

Lemma 6.1.1. The entries of $P$ are positive integers.

Proof. It turns out that thanks to Proposition 5.2.1 we will only need to perform "local" calculation of the elements neighboring to the initial ones:

| $a_{-1,-1}$ | $a_{-1,0}$ | $a_{-1,1}$ |
| :---: | :---: | :---: |
| $a_{0,-1}$ | $a$ | $b$ |
| $a_{1,-1}$ | $c$ | $d$ |

The conditions (4.1) imply: $a_{0,-1}<0$ and $a_{-1,0}<0$. Indeed, from (6.1) and (6.2), one has

$$
a_{0,-1}=q_{0} a-b, \quad a_{-1,0}=q_{0}^{\prime} a-c .
$$

Since the rows and the columns of $P$ are solutions of non-oscillating equations, and $a$ is positive, this implies that all the values of the first row and the first column of $P$ are positive.

Furthermore, again from the recurrence (6.1), one has

$$
a_{-1,-1}=q_{0} q_{0}^{\prime} a-q_{0} c-q_{0}^{\prime} b+d .
$$

The condition (4.1) then implies $a_{-1,-1}>0$. Indeed, one establishes

$$
\begin{aligned}
0<q_{0} & =a q_{0}\left(d-q_{0}^{\prime} b\right)-b q_{0}\left(c-q_{0}^{\prime} a\right)<b\left(d-q_{0}^{\prime} b\right)-b q_{0}\left(c-q_{0}^{\prime} a\right) \\
& =b\left(q_{0} q_{0}^{\prime} a-q_{0} c-q_{0}^{\prime} b+d\right) .
\end{aligned}
$$

Proposition 5.2.1 then guarantees that

$$
\begin{array}{lll}
a_{0,-1}<0, & \ldots, & a_{m-1,-1}<0, \\
a_{-1,0}<0, & \ldots, & a_{-1, n-1}<0,
\end{array}
$$

and applying again Proposition 5.2.1, we deduce that all the entries in $P$ are positive.
6.2. From tilings to triples. Conversely, consider an ( $n, m$ ) -periodic $\mathrm{SL}_{2}$-tiling $T=\left(a_{i, j}\right)_{(i, j) \in \mathbb{Z} \times \mathbb{Z}}$ such that the $m \times n$-subarray $P$ given by (6.3) consists of positive integers. We claim that $T$ can be obtained by the above construction.

Lemma 6.2.1. The ratios of the first two rows of $P$ form a decreasing sequence:

$$
\frac{a_{0,0}}{a_{1,0}}>\frac{a_{0,1}}{a_{1,1}}>\ldots>\frac{a_{0, n-1}}{a_{1, n-1}}
$$

and similarly for the ratios of the first two columns of $P$ :

$$
\frac{a_{0,1}}{a_{0,0}}>\frac{a_{1,1}}{a_{1,0}}>\ldots>\frac{a_{m-1,1}}{a_{m-1,0}}
$$

Proof. This follows from the unimodular conditions $a_{0, j} a_{1, j+1}-a_{0, j+1} a_{1, j}=1$ and the assumption that all the entries of $P$ are positive.

Lemma 6.2.2. The entries of $T$ satisfy the recurrence relations (6.1) where $q=\left(q_{j}\right)$ and $q^{\prime}=\left(q_{i}^{\prime}\right)$ are $n$-periodic and $m$-periodic sequences of positive integers, respectively.

Proof. Given $(i, j)$, there is a linear relation

$$
\binom{a_{i, j+1}}{a_{i+1, j+1}}=\lambda_{i, j}\binom{a_{i, j}}{a_{i+1, j}}+\mu_{i, j}\binom{a_{i, j-1}}{a_{i+1, j-1}} .
$$

Using the $\mathrm{SL}_{2}$ conditions one immediately obtains the values

$$
\lambda_{i, j}=a_{i, j-1} a_{i+1, j+1}-a_{i, j+1} a_{i+1, j-1}, \quad \mu_{i, j}=-1 .
$$

From Lemma 6.2.1, one has $\lambda_{i, j}>0$. Furthermore, it readily follows from the tameness property (see Proposition 3.4.1) that $\lambda_{i, j}$ actually does not depend on $i$, so we use the notation $q_{j}:=\lambda_{i, j}$.

The arguments for the rows are similar.
Lemma 6.2.3. The above sequences $\left(q_{0}, \ldots, q_{m-1}\right)$ and $\left(q_{0}^{\prime}, \ldots, q_{n-1}\right)$ are quiddities.

Proof. The rows, resp. columns, of $T$ are antiperiodic solutions of an equation (5.1) with $c_{i}=c_{i+n}=q_{i}$, resp. $c_{i}=c_{i+m}=q_{i}^{\prime}$. It follows from Proposition 5.2.1 that the coefficients are quiddities.

Lemma 6.2.4. The $2 \times 2$ left upper block of $P$, satisfies

$$
\begin{aligned}
q_{0} a_{0,0} & <a_{0,1} \\
q_{0}^{\prime} a_{0,0} & <a_{1,0} .
\end{aligned}
$$

Proof. By antiperiodicity, $a_{0,-1}<0$. One has from (6.1): $a_{0,1}=q_{0} a_{0,0}-a_{0,-1}$, and similarly for $q_{0}^{\prime}$. Hence the result.

In other words, the elements of the matrix

$$
\left(\begin{array}{ll}
a_{0,0} & a_{0,1} \\
a_{1,0} & a_{1,1}
\end{array}\right)=:\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)
$$

satisfy (4.1).
Theorem 2 is proved.

## 7. $\mathrm{SL}_{2}$-tilings and the Farey graph

In this section, we give an interpretation of the entries $a_{i, j}$ of a doubly periodic $\mathrm{SL}_{2}$-tiling. We follow the idea of Coxeter [Co] and consider $n$-gons in the classical Farey graph.
7.1. The distance between two $\boldsymbol{n}$-gons. Consider a doubly periodic $\mathrm{SL}_{2}$-tiling $T=\left(a_{i, j}\right)$ and the corresponding triple $\left(q, q^{\prime}, M\right)$ (see Theorem 2). Our next goal is to give an explicit expression for the numbers $a_{i, j}$ similar to (2.5).

From the triple $\left(q, q^{\prime}, M\right)$ we construct the unique $n$-gon $\left(v_{0}, v_{1}, \ldots, v_{n-1}\right)$ and the unique $m$-gon $\left(v_{0}^{\prime}, v_{1}^{\prime}, \ldots, v_{m-1}^{\prime}\right)$ with the "initial" conditions:

$$
\left(v_{0}, v_{1}\right):=\left(\frac{a}{c}, \frac{b}{d}\right), \quad\left(v_{0}^{\prime}, v_{m-1}^{\prime}\right):=\left(\frac{1}{0}, \frac{0}{1}\right)
$$

and with the quiddities $\left(q_{0}, \ldots, q_{n-1}\right)$ and $\left(q_{1}^{\prime}, \ldots, q_{m}^{\prime}\right)$, respectively. Notice that the quiddity $q^{\prime}$ is shifted cyclically.

Theorem 3. The entries of the $\mathrm{SL}_{2}$-tiling $T=\left(a_{i, j}\right)$ are given by

$$
a_{i, j}=d\left(v_{i-1}^{\prime}, v_{j}\right)
$$

for all $0 \leq i \leq m-1,0 \leq j \leq n-1$.
Proof. The main idea of the proof is to include the $n$-gon $v$ and the $m$-gon $v^{\prime}$ into a bigger $N$-gon in a Farey graph, and then apply Eq. (2.5). In other words, we will include the fundamental domain $P$ into a (bigger) frieze pattern.

First, let us show that

$$
v_{m-2}^{\prime}>v_{0}>v_{1}>\ldots>v_{n-1}>v_{m-1}^{\prime}
$$

Indeed, the vertices $v_{m-2}^{\prime}, v_{m-1}^{\prime}, v_{0}^{\prime}$ are consecutive vertices of the $m$-gon $v^{\prime}$. By assumption, $v_{m-1}^{\prime}=\frac{0}{1}$, so that the condition

$$
d\left(v_{m-2}^{\prime}, v_{m-1}^{\prime}\right)=1
$$

implies $v_{m-2}^{\prime}=\frac{1}{\ell}$ for some $\ell$. By Lemma 2.2.2, the distance $d\left(v_{0}^{\prime}, v_{m-2}^{\prime}\right)$ coincides with the number of triangles at the vertex $v_{m-1}^{\prime}$ which is, by construction, equal to $q_{0}^{\prime}$. We finally have:

$$
d\left(v_{0}^{\prime}, v_{m-2}^{\prime}\right)=\ell=q_{0}^{\prime},
$$

so that $v_{m-2}^{\prime}=\frac{1}{q_{0}^{\prime}}$. The inequality $v_{m-1}^{\prime}>v_{0}$ then follows from the second inequality (4.1).

It is well-known that the Farey graph is connected; see [HW]. Therefore, two disjoint polygons, $v$ and $v^{\prime}$, belong to some $N$-gon that contains the $n$-gon $v$ and the $m$-gon $v^{\prime}$.

Theorem 3 then follows from formula (2.5).
Example 7.1.1. Consider the tiling given in Figure 2. It corresponds to the following data:

$$
q=(1,2,2,1,3), \quad q^{\prime}=(2,1,2,1), \quad M=\left(\begin{array}{cc}
2 & 5 \\
7 & 18
\end{array}\right) .
$$

The associated 5 -gon and 4 -gon in the Farey graph are as follows:

$$
v=\left(\frac{2}{7}, \frac{5}{18}, \frac{8}{29}, \frac{11}{40}, \frac{3}{11}\right), \quad \text { and } \quad v^{\prime}=\left(\frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{0}{1}\right),
$$

respectively. They can be included in an 11-gon; see Figure 6.


Figure 6
The subgraph associated with the tiling in Figure 2

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