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Autor:	Morier-Genoud, Sophie / Ovsienko, Valentin / Tabachnikow, Serge
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# $SL_2(\mathbb{Z})$ -tilings of the torus, Coxeter–Conway friezes and Farey triangulations

Sophie Morier-Genoud, Valentin Ovsienko and Serge TABACHNIKOV

Abstract. The notion of SL<sub>2</sub>-tiling is a generalization of that of classical Coxeter–Conway frieze pattern. We classify doubly antiperiodic SL<sub>2</sub>-tilings that contain a rectangular domain of positive integers. Every such SL<sub>2</sub>-tiling corresponds to a pair of frieze patterns and a unimodular  $2 \times 2$ -matrix with positive integer coefficients. We relate this notion to triangulated *n*-gons in the Farey graph.

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Keywords. Frieze pattern, SL<sub>2</sub>-tiling, Farey graph, Modular group.

## 1. Introduction

Frieze patterns were introduced and studied by Coxeter and Conway, [Co, CC], in the 70's. A frieze pattern is an infinite array of numbers, bounded by two diagonals of 1's, such that every four adjacent numbers a, b, c, d forming a "small" square satisfy the relation ad - bc = 1 called *the unimodular rule*; for an example see Figure 1. The *width* of the frieze is the number of diagonals between the bounding diagonals of 1's.

The fundamental Conway–Coxeter theorem [CC] offers the following classification: frieze patterns with positive integer entries of width n-3, are in one-to-one correspondence with triangulations of a convex n-gon; for a simple proof see [Hen]. More precisely, given a triangulated n-gon in the oriented plane, one constructs a frieze of width n-3 as follows. The diagonal next to the diagonal of 1's is formed by the numbers of triangles incident at each vertex (taken cyclically). This, in particular, implies that every diagonal in a frieze of width n-3 is n-periodic. Throughout this paper, we will be considering frieze patterns with positive integer entries.

The following terminology is due to Conway and Coxeter [CC]. A sequence of *n* positive integers  $q = (q_0, \ldots, q_{n-1})$  is called a *quiddity* of order *n*, if there



FIGURE 1 A 7-periodic frieze pattern and the corresponding triangulated heptagon

exists a triangulated *n*-gon such that every  $q_i$  is equal to the number of incident triangles at *i*-th vertex. For instance, the example in Figure 1 corresponds to the following quiddities of order 7: (1,3,2,2,1,4,2), (3,2,2,1,4,2,1),... (cyclic permutation).

Every quiddity of order n determines a unique positive integer frieze pattern. Two quiddities correspond to the same positive integer frieze pattern if and only if they differ by a cyclic permutation. According to the Conway–Coxeter theorem, positive integer frieze patterns can be enumerated by the Catalan numbers.

**Example 1.0.1.** For each case n = 3, 4 and 5, there is a unique (up to cyclic permutation) quiddity: (1, 1, 1), (1, 2, 1, 2) and (1, 3, 1, 2, 2), respectively.

For n = 6, there are four different quiddities:

$$(1, 3, 1, 3, 1, 3), (1, 4, 1, 2, 2, 2), (1, 2, 3, 1, 2, 3), (1, 3, 2, 1, 3, 2)$$

and their cyclic permutations.

We can also consider the "degenerate" case n = 2, where the corresponding "degenerate" quiddity is (0,0).

Examples of frieze patterns can be constructed using the computer program [Scha].

Among many beautiful properties of Coxeter–Conway friezes, the property of periodicity and so-called Laurent phenomenon are particularly important. They relate frieze patterns to the theory of cluster algebras developed by Fomin and Zelevinsky, [FZ1, FZ2].

Various generalizations of Coxeter–Conway friezes have recently been introduced and studied, see [CaCh, Pro, BM, ARS, MOT]. One of the generalizations, called  $SL_2$ -*tiling*, was first considered by Assem, Reutenauer and Smith [ARS], and further developed by Bergeron and Reutenauer [BR]. An  $SL_2$ -tiling is an infinite array of numbers satisfying the above unimodular rule, without the condition of bounding diagonals of 1's. Unlike the frieze patterns,  $SL_2$ -tilings are not necessarily periodic. Nevertheless, correspondences between  $SL_2$ -tilings and triangulations can be established, [HJ, BHJ].

The case of (n, m)-antiperiodic, or "toric" SL<sub>2</sub>-tilings was suggested in [BR]. In this paper, we study such tilings.

The main results of the paper are the following.

We classify doubly antiperiodic  $SL_2$ -tilings that contain a rectangular fundamental domain of positive integers. We show that every such  $SL_2$ -tiling is generated by a pair of quiddities and a unimodular  $2 \times 2$ -matrix with positive

	÷	÷	÷	÷	÷	÷	÷	÷	÷	÷	
• • •	2	5	8	11	3	-2	-5	-8	-11	-3	•••
•••	7	18	29	40	11	-7	-18	-29	-40	-11	•••
•••	5	13	21	29	8	-5	-13	-21	-29	-8	•••
•••	3	8	13	18	5	-3	-8	-13	-18	-5	•••
•••	-2	-5	-8	-11	-3	2	5	8	11	3	•••
•••	-7	-18	-29	-40	-11	7	18	29	40	11	•••
•••	-5	-13	-21	-29	-8	5	13	21	29	8	•••
•••	-3	-8	-13	-18	-5	3	8	13	18	5	•••
	÷	÷	÷	÷	÷	:	÷	÷	÷	÷	



integer coefficients. Although there are infinitely many such  $SL_2$ -tilings, their description is very explicit.

Following the original idea of Coxeter [Co], we also interpret the entries of a doubly periodic  $SL_2$ -tiling that contain a rectangular fundamental domain of positive integers in terms of the Farey graph of rational numbers. Every such  $SL_2$ -tiling corresponds to a triple: an *n*-gon, an *m*-gon in the Farey graph, and a totally positive matrix from  $SL_2(\mathbb{Z})$  relating them. We also obtain an explicit formula for the entries of the tiling.

# 2. Farey graph and the Conway-Coxeter theorem

In this section, we give an explanation of the relation between the Coxeter frieze patterns and triangulated n-gons.

It was already noticed by Coxeter [Co] that a Farey series (of arbitrary order N) defines a frieze pattern. Moreover, every frieze pattern corresponds to an n-gon (i.e., an n-cycle) in the Farey graph. A Farey n-gon always carries a triangulation; we will prove that this triangulation is precisely that of Conway–Coxeter theorem. This statement seems to be new and to extend the observation illustrated in [Scha].

**2.1. Farey graph, Farey series and Farey** *n*-gons. For two rational numbers,  $v_1, v_2 \in \mathbb{Q}$ , written as irreducible fractions  $v_1 = \frac{a_1}{b_1}$  and  $v_2 = \frac{a_2}{b_2}$ , the *Farey* "*distance*" is defined by

$$d(v_1, v_2) := |a_1b_2 - a_2b_1|.$$

Note that the above "distance" does not satisfy the triangle inequality. Recall the definition of the *Farey graph*.

- (1) The set of vertices of the Farey graph is  $\mathbb{Q} \cup \{\infty\}$ , with  $\infty$  represented by  $\frac{1}{0}$ .
- (2) Two vertices,  $v_1, v_2$  are joined by a (non-oriented) edge  $(v_1, v_2)$  whenever  $d(v_1, v_2) = 1$ .

The Farey graph is often embedded into the hyperbolic half-plane, the edges being realized as geodesics joining rational points on the ideal boundary.

The following classical properties of the Farey graph can be found in [HW] (the proof is elementary).

**Proposition 2.1.1.** (i) Every 3-cycle of the Farey graph is of the form

(2.1) 
$$\left\{\frac{a_1}{b_1}, \frac{a_1+a_2}{b_1+b_2}, \frac{a_2}{b_2}\right\}.$$

- (ii) Every edge of the Farey graph belongs to a 3-cycle.
- (iii) Edges in the Farey graph do not cross, i.e., for a quadruple  $v_1 > v_2 > v_3 > v_4$ it is not possible to have edges  $(v_1, v_3)$  and  $(v_2, v_4)$ .

**Definition 2.1.2.** The *Farey series* (also called *Farey sequence*) of order N is the sequence of irreducible fractions in [0, 1] whose denominators do not exceed N.

We will write the sequences in decreasing order; see Figure 3.



FIGURE 3 The Farey series of order 5 embedded in the Farey graph

The following fundamental property of Farey series is also proved in [HW]. It shows that every Farey series is a cycle in the Farey graph.

**Proposition 2.1.3.** Every two consecutive numbers in a Farey series are joined by an edge in the Farey graph.

This is less elementary than Proposition 2.1.1, so we propose here a short proof. Our proof is different from the well-known one, it is based on the classical Pick formula.

*Proof.* Consider two consecutive numbers  $\frac{a}{b} > \frac{c}{d}$ , in a Farey series of some order N. Suppose that  $ad - bc \ge 2$ . The quantity  $A = \frac{1}{2}(ad - bc)$  is the area of the Euclidean triangle spanned by the vertices (0,0), (a,b), (c,d). Pick's formula states:

$$A = I + \frac{B}{2} - 1,$$



The case of interior point

where *I* is the number of integer points in the interior of the triangle, and *B* the number of integer points on the border. By assumption,  $A \ge 1$ , and therefore  $I + \frac{B}{2} \ge 2$ . It follows that there exists a point (x, y), which is either inside the triangle, or on the segment between (a, b) and (c, d) (since the fractions  $\frac{a}{b}$  and  $\frac{c}{d}$  are irreducible). One then has:

$$y \le \max(b, d) \le N$$
 and  $\frac{a}{b} > \frac{x}{y} > \frac{c}{d}$ 

This contradicts the assumption that  $\frac{a}{b}$  and  $\frac{c}{d}$  are consecutive numbers in the Farey series.

Proposition 2.1.3 is used three times to prove the following.

**Corollary 2.1.4.** Every Farey series forms a triangulated polygon in the Farey graph.

*Proof.* We prove this statement by induction on N (the order of Farey series). Assume that the series of order N - 1 is triangulated. The series of order N is obtained from that of order N - 1 by adding points of the form  $\frac{k}{N}$ .

First, we observe that two points,  $\frac{k_1}{N}$  and  $\frac{k_2}{N}$  cannot be consecutive. Indeed,  $d(\frac{k_1}{N}, \frac{k_2}{N}) \neq 1$ : that would contradict Proposition 2.1.3; therefore, every new point  $\frac{k}{N}$  appears between two "old" points:

(2.2) 
$$\frac{p_1}{q_1} > \frac{k}{N} > \frac{p_2}{q_2}$$

Second, by Proposition 2.1.3,  $\frac{k}{N}$  is joined by edges with  $\frac{p_1}{q_1}$  and  $\frac{p_2}{q_2}$ . Third,  $\frac{p_1}{q_1}$  and  $\frac{p_2}{q_2}$  are joined by an edge, according to Proposition 2.1.3 applied to the series of order N - 1. We conclude that (2.2) is a triangle.

We will be interested in n-cycles (or "n-gons") in the Farey graph that are more general than Farey series.

**Definition 2.1.5.** (1) An *n*-gon in the Farey graph, or a *Farey n*-gon is a decreasing sequence of rationals  $(v_0, \ldots, v_{n-1})$ :

$$\infty \geq v_0 > v_1 > \ldots > v_{n-1} \geq 0,$$

such that every pair of consecutive numbers  $v_i, v_{i+1}$ , as well as  $v_{n-1}, v_0$ , are joined by an edge.

(2) The *n*-gon is called *normalized* if  $v_0 = \infty$  and  $v_{n-1} = 0$ .

Since every n-gon can be embedded in a Farey series, Corollary 2.1.4 implies the following.

Corollary 2.1.6. Every Farey n-gon is triangulated.

We thus can speak of the quiddity of a Farey n-gon.

*Proof.* A Farey *n*-gon is obtained from a Farey series which is a triangulated polygon, by cutting along diagonals of the triangulation.  $\Box$ 

We define the notion of *cyclic equivalence* of Farey *n*-gons. Given an *n*-gon  $(v_0, \ldots, v_{n-1})$ , consider the *n*-cycle  $(v_1, \ldots, v_{n-1}, v_0)$ , and renormalize it using the SL<sub>2</sub>( $\mathbb{Z}$ )-action so that  $v_1 = \infty$  and  $v_0 = 0$ . The obtained *n*-gon is called cyclically equivalent to the given one. For an example, see Figure 5.



FIGURE 5 Two cyclically equivalent normalized heptagons in the Farey graph corresponding to the frieze of Figure 1

**2.2. Farey** *n***-gons and Coxeter–Conway friezes.** Proposition 2.1.3 leads to the following observation due to Coxeter [Co]: *every Farey series gives rise to a Coxeter–Conway frieze pattern of positive integers.* Along the same lines, we have the following strengthened statement.

**Proposition 2.2.1.** The Coxeter–Conway frieze patterns of positive integers of width n-3 are in one-to-one correspondence with the normalized Farey n-gons, up to cyclic equivalence.

*Proof.* The correspondence is given by considering the ratios of two consecutive rows of the frieze patterns. The sequence

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{a_1}{1}, \quad \dots, \quad v_i = \frac{a_i}{b_i}, \quad \dots, \quad v_{n-2} = \frac{1}{b_{n-2}}, \quad v_{n-1} = \frac{0}{1}$$

corresponds to the frieze determined by the rows

 $\square$ 

and vice versa.

The Conway–Coxeter theorem mentioned in the introduction provides a relation between frieze patterns and triangulations. The following result somewhat "demystifies" this relation and provides an alternative proof of the Conway–Coxeter theorem.

**Theorem 1.** The quiddity of a Farey n-gon coincides with the quiddity of the corresponding Coxeter–Conway frieze pattern.

*Proof.* Consider a frieze pattern, and denote by  $c_{i,j}$  its entries:

0	1	$c_{1,1}$	$c_{1,2}$	• • •	$c_{1,n-3}$	1	0
	0	1	<i>c</i> <sub>2,2</sub>		•••	$c_{2,n-2}$	1
		۰.	۰.				۰.

where

$$\begin{cases} c_{i,j} = 1, & i - j = 1 \text{ or } 3 - n, \\ c_{i,j} = 0, & i - j = 2 \text{ or } 2 - n. \end{cases}$$

The quiddity of the frieze pattern reads in the *n*-periodic line  $(c_{i,i})$ .

Clearly, two consecutive rows determine the rest of the frieze; the following formula was proved in [Co], formula (5.6):

 $SL_2(\mathbb{Z})$ -tilings of the torus

$$c_{i,j} = c_{1,i-2}c_{2,j} - c_{1,j}c_{2,i-2}.$$

In particular, we have:

(2.3) 
$$c_{i,i} = c_{1,i-2}c_{2,i} - c_{1,i}c_{2,i-2}.$$

The corresponding Farey n-gon has the following vertices

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{c_{1,1}}{1}, \quad \dots \quad v_i = \frac{c_{1,i}}{c_{2,i}}, \quad \dots \quad v_{n-2} = \frac{1}{c_{2,n-2}}, \quad v_{n-1} = \frac{0}{1}.$$

Therefore, the expression (2.3) reads:  $c_{i,i} = d(v_{i-2}, v_i)$ . It remains to calculate the Farey distance between pairs of vertices  $v_{i-2}$  and  $v_i$  in a Farey *n*-gon.

**Lemma 2.2.2.** Given a (triangulated) Farey n-gon

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{a_1}{1}, \quad \dots, \quad v_i = \frac{a_i}{b_i}, \quad \dots, \quad v_{n-2} = \frac{1}{b_{n-2}}, \quad v_{n-1} = \frac{0}{1},$$

the Farey distance  $d(v_{i-1}, v_{i+1})$  coincides with the number of triangles incident at  $v_i$ .

*Proof.* Among all the vertices of the *n*-gon  $(v_i)$ , let us select those connected to  $v_i$  by edges of the Farey graph. Denote by  $\{v_{i_1}, \ldots, v_{i_k}\}$ , resp.  $\{v_{i_{k+1}}, \ldots, v_{i_{k+\ell}}\}$  the vertices at the left, resp. right, of  $v_i$ , so that

$$v_{i_1} > \ldots > v_{i_k} > v_i > v_{i_{k+1}} > \ldots > v_{i_{k+\ell}}$$

(note that  $v_{i_k} = v_{i-1}$  and  $v_{i_{k+1}} = v_{i+1}$ ). The number of triangles incident at  $v_i$  is then equal to  $k + \ell - 1$ .

Two consecutive selected vertices,  $v_{ij}$  and  $v_{ij+1}$  are connected by an edge. Indeed, this follows from the fact that every Farey polygon is triangulated. Therefore, the vertices  $(v_{ij}, v_{ij+1}, v_i)$  form a triangle (a 3-cycle) in the Farey graph. Using Eq. (2.1), we obtain by induction:

$$v_{i-1}(=v_{i_k}) = \frac{a_{i_1} + (k-1)a_i}{b_{i_1} + (k-1)b_i}, \qquad v_{i+1}(=v_{i_{k+1}}) = \frac{a_{i_{k+\ell}} + (\ell-1)a_i}{b_{i_{k+\ell}} + (\ell-1)b_i}.$$

We have:

$$d(v_{i-1}, v_{i+1}) = a_{i_1}b_{i_{k+\ell}} - b_{i_1}a_{i_{k+\ell}} + (k-1)(a_ib_{i_{k+\ell}} - b_ia_{i_{k+\ell}}) + (\ell-1)(a_{i_1}b_i - b_{i_1}a_i).$$

By assumption,  $v_i$  is joined by edges with  $v_{i_1}$  and  $v_{i_{k+\ell}}$ , hence  $a_i b_{i_{k+\ell}} - b_i a_{i_{k+\ell}} = 1$ , and  $a_{i_1} b_i - b_{i_1} a_i = 1$ . Furthermore,  $(v_{i_1}, v_i, v_{i_{k+\ell}})$  is also a triangle, therefore  $a_{i_1} b_{i_{k+\ell}} - b_{i_1} a_{i_{k+\ell}} = 1$ . We have finally:

(2.4) 
$$d(v_{i-1}, v_{i+1}) = k + \ell - 1.$$

Hence the lemma.

Theorem 1 is proved.

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**2.3. Entries of the frieze pattern.** Coxeter's formula (5.6) in [Co] for the entries of the frieze pattern translates into our language as the following general expression:

(2.5) 
$$c_{i,j} = d(v_{i-2}, v_j),$$

where, as above,  $(v_i)$  is the Farey *n*-gon corresponding to the frieze pattern.

# 3. SL<sub>2</sub>-tilings

In this section, we introduce the main notions studied in this paper.

**3.1. Tame SL<sub>2</sub>-tilings.** Let us first recall the notion of SL<sub>2</sub>-tiling introduced in [BR].

(1) An SL<sub>2</sub>-tiling, is an infinite matrix  $\mathcal{A} = (a_{i,j})_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}$ , such that every adjacent  $2 \times 2$ -minor equals 1:

$$\begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} = 1,$$

for all  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ .

(2) The tiling is called *tame* if every adjacent  $3 \times 3$ -minor equals 0:

$$\begin{vmatrix} a_{i,j} & a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j} & a_{i+1,j+1} & a_{i+1,j+2} \\ a_{i+2,j} & a_{i+2,j+1} & a_{i+2,j+2} \end{vmatrix} = 0,$$

for all  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ .

Let us stress the fact that a generic SL<sub>2</sub>-tiling is tame.

**3.2.** Antiperiodicity. The following condition was also suggested in [BR].

An SL<sub>2</sub>-tiling is called (n,m)-antiperiodic if every row is *n*-antiperiodic, and every column is *m*-antiperiodic:

$$a_{i,j+n} = -a_{i,j} ,$$
  
$$a_{i+m,j} = -a_{i,j} ,$$

for all  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ .

The following relation between (n,m)-antiperiodic SL<sub>2</sub>-tilings and the classical Coxeter–Conway frieze patterns shows that the antiperiodicity condition for the SL<sub>2</sub>-tilings is natural and interesting.

**3.3. Frieze patterns and** (n, n)-antiperiodic SL<sub>2</sub>-tilings. As explained in [BR], every Coxeter–Conway frieze pattern of width n - 3 can be extended to a tame (n, n)-antiperiodic SL<sub>2</sub>-tiling, in a unique way.

The construction is as follows. One adds two diagonals of 0's next to the diagonals of 1's, and then continues by antiperiodicity.

**Example 3.3.1.** The frieze pattern in Figure 1 corresponds to the following (7, 7)-antiperiodic tame SL<sub>2</sub>-tiling.

	•	÷	:		:	:	:	:			÷		
	1	2	3	1	1	1	0	-1	-2	-3	_1	-1	
•••	0	1	2	1	2	3	1	0	-1	-2	-1	-2	•••
•••	-1	0	1	1	3	5	2	1	0	-1	-1	-3	•••
•••	-2	-1	0	1	4	7	3	2	1	0	-1	-4	•••
•••	-1	-1	-1	0	1	2	1	1	1	1	0	-1	•••
•••	-2	-3	-4	-1	0	1	1	2	3	4	1	0	•••
•••	-3	-5	-7	-2	-1	0	1	3	5	7	2	1	•••
•••	-1	-2	-3	-1	-1	-1	0	1	2	3	1	1	•••
•••	0	-1	-2	-1	-2	-1	-1	0	1	2	1	2	•••
	:	:	:	:	:	:	:	:	:	:	:	:	
	•	•	•	•	•	•	•	•	•	•	•	•	

For the details of the above construction and the "antiperiodic nature" of Conway–Coxeter's friezes; see [BR, MOST].

**3.4.** Positive rectangular domain. In this paper, we are considering (n,m)-antiperiodic SL<sub>2</sub>-tilings that contain an  $m \times n$ -rectangular domain of positive integers.

More precisely, we are interested in SL<sub>2</sub>-tilings of the following form:

		•	:	•	
(3.1)		Р	- <i>P</i>	Р	•••
(3.1)	•••	- <i>P</i>	Р	- <i>P</i>	•••
		:	•		

where *P* is an  $m \times n$ -matrix with entries in  $\mathbb{Z}_{>0}$ . An example of such an SL<sub>2</sub>-tilling is presented in Figure 2.

The following property is important for us.

**Proposition 3.4.1.** An (n,m)-antiperiodic SL<sub>2</sub>-tiling that contains a positive  $m \times n$ -rectangular domain is tame.

*Proof.* This is a consequence of the Jacobi identity or Dodgson formula for determinants:

 $\begin{vmatrix} \bullet & \bullet & \bullet & 0 \\ \hline \end{array} = \begin{vmatrix} \bullet & \bullet & \circ & 0 \\ \bullet & \bullet & \circ & 0 \\ \circ & \bullet & \circ & 0 \\ \hline \end{array} = \begin{vmatrix} \bullet & \bullet & \circ & 0 \\ \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & 0 \\ \hline \end{array} = \begin{vmatrix} \circ & \circ & \circ & 0 \\ \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & 0 \\ \hline \end{array} = \begin{vmatrix} \circ & \circ & \circ & 0 \\ \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & 0 \\ \hline \end{array} = \begin{vmatrix} \circ & \circ & \circ & 0 \\ \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & 0 \\ \hline \end{array} = \begin{vmatrix} \circ & \circ & \circ & 0 \\ \circ & \bullet & \bullet \\ \bullet & \bullet & \circ & 0 \\ \hline \end{array} = \begin{vmatrix} \circ & \circ & \circ & 0 \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & 0 \\ \hline \end{array} = \begin{vmatrix} \circ & \circ & \circ & 0 \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & 0 \\ \hline$ 

where the white dots represent deleted entries, and the black dots initial entries.

Since the values are non zero and the  $2 \times 2$ -minors all equal to 1, the above identity implies that all the  $3 \times 3$ -minors vanish.

#### 4. The main theorem

In this section, we formulate our main result. The proof will be given in Section 6.

**4.1. Classification.** It turns out that every  $SL_2$ -tiling corresponds to a pair of frieze patterns and a positive integer  $2 \times 2$ -matrix M satisfying some conditions.

**Theorem 2.** The set of (n,m)-antiperiodic  $SL_2$ -tilings containing a fundamental rectangular domain of positive integers is in a one-to-one correspondence with the set of triples (q,q',M), where

$$q = (q_0, \dots, q_{n-1}), \qquad q' = (q'_0, \dots, q'_{m-1})$$

are quiddities of order n and m, respectively, and where  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is a unimodular 2×2-matrix with positive integer coefficients, such that the inequalities

$$(4.1) q_0 < \frac{b}{a}, q'_0 < \frac{c}{a}$$

are satisfied.

**Remark 4.1.1.** It is important to notice that inequalities (4.1) also imply

$$(4.2) q_0 < \frac{d}{c}, q'_0 < \frac{d}{b}.$$

Indeed, the unimodular condition ad - bc = 1 and the assumption that a, b, c, d are positive integers imply that  $\frac{b}{a} < \frac{d}{c}$  and  $\frac{c}{a} < \frac{d}{b}$ .

**Corollary 4.1.2.** For every pair of quiddities q,q', there exist infinitely many (n,m)-antiperiodic SL<sub>2</sub>-tilings containing a fundamental rectangular domain of positive integers.

*Proof.* Given arbitrary pair of quiddities q and q', the matrices:

$$\begin{pmatrix} 1 & b \\ c & bc+1 \end{pmatrix}$$

satisfy (4.1) for sufficiently large b, c.

**4.2. The semigroup** S. Consider the set of  $2 \times 2$ -matrices with positive integral entries satisfying the following conditions of positivity:

(4.3) 
$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z}) \middle| \begin{array}{c} 0 < a < b < d, \\ 0 < a < c < d. \end{array} \right\}$$

Note that the inequalities b < d and c < d are included for the sake of completeness. These inequalities actually follow from a < b, a < c together with ad - bc = 1 and the assumption that a, b, c, d are positive.

We have the following property.

**Proposition 4.2.1.** The set  $S \subset SL_2(\mathbb{Z})$  is a semigroup, i.e., it is stable by *multiplication*.

Proof. Straightforward.

The semigroup S naturally appears in our context. Indeed, if  $n, m \ge 3$ , then the inequalities (4.1) imply  $M \in S$ . Moreover every quiddity q contains a unit entry, so that after a cyclic permutation of any quiddity one can obtain  $q_0 = 1$ . The inequalities (4.1) then coincide with the conditions (4.3).

**4.3. Examples.** Let us give two simple examples of  $SL_2$ -tilings.

**Example 4.3.1.** There is a one-to-one correspondence between (3, 3)-antiperiodic SL<sub>2</sub>-tilings containing a fundamental domain of positive integers and elements of the semigroup S. Indeed, the only quiddity of order 3 is q = (1, 1, 1). To

 $\square$ 

every matrix (4.3) there corresponds the following SL<sub>2</sub>-tiling:

It is a good exercise to check that the positivity condition d - b - c + a > 0 follows from (4.3) together with ad - bc = 1.

**Example 4.3.2.** In the case n = 2 or m = 2, the conditions (4.1) become trivial. Consider also the simplest (degenerate) case of (2, 2)-antiperiodic SL<sub>2</sub>-tilings. A (2, 2)-antiperiodic SL<sub>2</sub>-tiling containing a fundamental domain of positive integers is of the form:

where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is an arbitrary unimodular matrix with positive integer coefficients. Note that this case corresponds to the "degenerate quiddity" of order 2, namely q = (0, 0).

# 5. Frieze patterns and linear recurrence equations

We will recall here a remarkable and well-known property of Coxeter–Conway frieze patterns. It concerns a relation of frieze patterns and linear recurrence equations. The statement presented in this subsection was implicitly obtained in [CC]; for details see [MOST]. We recall this statement without proof.

### 5.1. Discrete non-oscillating Hill equations.

**Definition 5.1.1.** Let  $(c_i)_{i \in \mathbb{Z}}$  be an arbitrary *n*-periodic sequence of numbers.

(a) A linear difference equation

(5.1) 
$$V_{i+1} = c_i V_i - V_{i-1},$$

where the sequence  $(c_i)$  is given (the coefficients) and where  $(V_i)$  is unknown (the solution), is called a discrete Hill, or Sturm-Liouville, or one-dimensional Schrödinger equation.

(b) The equation (5.1) is called *non-oscillating* if every solution  $(V_i)$  is antiperiodic:

$$V_{i+n} = -V_i,$$

for all *i*, and has exactly one sign change in any sequence  $(V_i, V_{i+1}, \dots, V_{i+n})$ .

In other words, every solution of a non-oscillating equation must have non-negative intervals of length n, that is, n consecutive non-negative values:  $(V_k, \ldots, V_{k+n-1})$ .

Moreover, for a *generic* solution of (5.1), all the elements  $V_j$  of a non-negative interval are *strictly positive*. Zero values can only occur at the endpoints:  $V_k = 0$ , or  $V_{k+n-1} = 0$ .

Note also that the coefficients in a non-oscillating equation are necessarily positive.

**5.2. Frieze patterns and difference equations.** The relation between the equations (5.1) and Coxeter–Conway frieze patterns is as follows.

**Proposition 5.2.1.** Given an equation (5.1) with integer coefficients, it is a nonoscillating equation if and only if the coefficients  $(c_0, c_1, \ldots, c_{n-1})$  form a quiddity.

*Proof.* This is an immediate consequence of properties established by Coxeter and Conway. Indeed, it was proved in [Co] (see also [CC] property (17)) that the entries in any row of the pattern (extended by antiperiodicity) form a solution of an equation (5.1), where the coefficients  $c_i$  are given by the sequence on the first non-trivial diagonal. Thus, from an non-oscillating equation one can write down a frieze, and vice versa.

Finally, the integer condition establishes the correspondence with quiddities.  $\Box$ 

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Of course, for an arbitrary non-oscillating equation (5.1), the corresponding frieze pattern does not necessarily have integer entries. In [MOST], the space of frieze patterns and the space of non-oscillating equation (5.1) are identified in a more general setting.

**Example 5.2.2.** (a) The simplest quiddity q = (1, 1, 1) corresponds to the nonoscillating equation with all  $c_i = 1$ . Every solution of this equation is 3antiperiodic and can be obtained as a linear combination of the following two solutions:

$$(V_i^{(1)}) = (\dots, 0, 1, 1, 0, -1, -1, \dots), \qquad (V_i^{(2)}) = (\dots, 1, 1, 0, -1, -1, 0 \dots).$$

This corresponds to a degenerate frieze of Coxeter–Conway of width 0.(b) The frieze from Figure 1 corresponds to the non-oscillating equation with 7-antiperiodic solutions that are linear combinations of the following two:

$$(V_i^{(1)}) = (\dots, 1, 2, 3, 1, 1, 1, 0, \dots), \qquad (V_i^{(2)}) = (\dots, 0, 1, 2, 1, 2, 3, 1, \dots).$$

The above two solutions are exactly the first two rows of the frieze in Figure 1. One can of course choose different rows for a basis.

Note that, in the both cases, the basis solutions  $(V_i^{(1)}), (V_i^{(2)})$  are not generic since they contain zeros.

#### 6. Proof of Theorem 2

**6.1. The construction.** Given a triple (q, q', M) as in Theorem 2, we will construct an SL<sub>2</sub>-tiling satisfying the above conditions. Define  $T = (a_{i,j})$  using the following recurrence relations:

(6.1) 
$$\begin{aligned} a_{i,j+1} &:= q_j a_{i,j} - a_{i,j-1}, \\ a_{i+1,j} &:= q'_i a_{i,j} - a_{i-1,j}, \end{aligned}$$

for all  $i, j \in \mathbb{Z}$ , where the quiddities are periodically extended, i.e  $q_i = q_{i+n}, q'_i = q'_{i+m}$ , and taking the initial conditions

(6.2) 
$$\begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is very easy to check that the tiling T is well-defined, i.e., the two recurrences commute and the calculations along the rows and columns give the same result. We show that the defined tiling T contains a fundamental rectangular domain of positive integers.

By Proposition 5.2.1, the defined tiling T is (n,m)-antiperiodic. Consider the following  $m \times n$ -subarray of T

(6.3) 
$$P = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ \cdots & & & & \\ a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1,n-1} \end{pmatrix}.$$

The main step of the proof of Theorem 2 is the following lemma.

Lemma 6.1.1. The entries of P are positive integers.

*Proof.* It turns out that thanks to Proposition 5.2.1 we will only need to perform "local" calculation of the elements neighboring to the initial ones:

$$\begin{array}{c|c|c} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ \hline a_{0,-1} & a & b \\ a_{1,-1} & c & d \\ \end{array}$$

The conditions (4.1) imply:  $a_{0,-1} < 0$  and  $a_{-1,0} < 0$ . Indeed, from (6.1) and (6.2), one has

$$a_{0,-1} = q_0 a - b, \qquad a_{-1,0} = q'_0 a - c.$$

Since the rows and the columns of P are solutions of non-oscillating equations, and a is positive, this implies that all the values of the first row and the first column of P are positive.

Furthermore, again from the recurrence (6.1), one has

$$a_{-1,-1} = q_0 q'_0 a - q_0 c - q'_0 b + d.$$

The condition (4.1) then implies  $a_{-1,-1} > 0$ . Indeed, one establishes

$$0 < q_0 = aq_0(d - q'_0b) - bq_0(c - q'_0a) < b(d - q'_0b) - bq_0(c - q'_0a)$$
  
=  $b(q_0q'_0a - q_0c - q'_0b + d).$ 

Proposition 5.2.1 then guarantees that

$$a_{0,-1} < 0, \dots, a_{m-1,-1} < 0,$$
  
 $a_{-1,0} < 0, \dots, a_{-1,n-1} < 0,$ 

and applying again Proposition 5.2.1, we deduce that all the entries in P are positive.

**6.2. From tilings to triples.** Conversely, consider an (n,m)-periodic SL<sub>2</sub>-tiling  $T = (a_{i,j})_{(i,j)\in\mathbb{Z}\times\mathbb{Z}}$  such that the  $m \times n$ -subarray P given by (6.3) consists of positive integers. We claim that T can be obtained by the above construction.

Lemma 6.2.1. The ratios of the first two rows of P form a decreasing sequence:

$$\frac{a_{0,0}}{a_{1,0}} > \frac{a_{0,1}}{a_{1,1}} > \ldots > \frac{a_{0,n-1}}{a_{1,n-1}},$$

and similarly for the ratios of the first two columns of P:

$$\frac{a_{0,1}}{a_{0,0}} > \frac{a_{1,1}}{a_{1,0}} > \ldots > \frac{a_{m-1,1}}{a_{m-1,0}}$$

*Proof.* This follows from the unimodular conditions  $a_{0,j}a_{1,j+1} - a_{0,j+1}a_{1,j} = 1$  and the assumption that all the entries of *P* are positive.

**Lemma 6.2.2.** The entries of T satisfy the recurrence relations (6.1) where  $q = (q_j)$  and  $q' = (q'_i)$  are n-periodic and m-periodic sequences of positive integers, respectively.

*Proof.* Given (i, j), there is a linear relation

$$\begin{pmatrix} a_{i,j+1} \\ a_{i+1,j+1} \end{pmatrix} = \lambda_{i,j} \begin{pmatrix} a_{i,j} \\ a_{i+1,j} \end{pmatrix} + \mu_{i,j} \begin{pmatrix} a_{i,j-1} \\ a_{i+1,j-1} \end{pmatrix}.$$

Using the  $SL_2$  conditions one immediately obtains the values

$$\lambda_{i,j} = a_{i,j-1}a_{i+1,j+1} - a_{i,j+1}a_{i+1,j-1}, \qquad \mu_{i,j} = -1.$$

From Lemma 6.2.1, one has  $\lambda_{i,j} > 0$ . Furthermore, it readily follows from the tameness property (see Proposition 3.4.1) that  $\lambda_{i,j}$  actually does not depend on i, so we use the notation  $q_j := \lambda_{i,j}$ .

 $\square$ 

The arguments for the rows are similar.

**Lemma 6.2.3.** The above sequences  $(q_0, \ldots, q_{m-1})$  and  $(q'_0, \ldots, q_{n-1})$  are quiddities.

*Proof.* The rows, resp. columns, of *T* are antiperiodic solutions of an equation (5.1) with  $c_i = c_{i+n} = q_i$ , resp.  $c_i = c_{i+m} = q'_i$ . It follows from Proposition 5.2.1 that the coefficients are quiddities.

**Lemma 6.2.4.** The  $2 \times 2$  left upper block of P, satisfies

$$\begin{array}{rcl} q_0 \, a_{0,0} & < & a_{0,1}, \\ q_0' \, a_{0,0} & < & a_{1,0}. \end{array}$$

*Proof.* By antiperiodicity,  $a_{0,-1} < 0$ . One has from (6.1):  $a_{0,1} = q_0 a_{0,0} - a_{0,-1}$ , and similarly for  $q'_0$ . Hence the result.

In other words, the elements of the matrix

$$\begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfy (4.1).

Theorem 2 is proved.

#### 7. SL<sub>2</sub>-tilings and the Farey graph

In this section, we give an interpretation of the entries  $a_{i,j}$  of a doubly periodic SL<sub>2</sub>-tiling. We follow the idea of Coxeter [Co] and consider *n*-gons in the classical Farey graph.

**7.1. The distance between two** *n***-gons.** Consider a doubly periodic SL<sub>2</sub>-tiling  $T = (a_{i,j})$  and the corresponding triple (q, q', M) (see Theorem 2). Our next goal is to give an explicit expression for the numbers  $a_{i,j}$  similar to (2.5).

From the triple (q, q', M) we construct the unique *n*-gon  $(v_0, v_1, \ldots, v_{n-1})$ and the unique *m*-gon  $(v'_0, v'_1, \ldots, v'_{m-1})$  with the "initial" conditions:

$$(v_0, v_1) := \left(\frac{a}{c}, \frac{b}{d}\right), \qquad \left(v'_0, v'_{m-1}\right) := \left(\frac{1}{0}, \frac{0}{1}\right)$$

and with the quiddities  $(q_0, \ldots, q_{n-1})$  and  $(q'_1, \ldots, q'_m)$ , respectively. Notice that the quiddity q' is shifted cyclically.

**Theorem 3.** The entries of the  $SL_2$ -tiling  $T = (a_{i,j})$  are given by

$$a_{i,j} = d(v_{i-1}', v_j),$$

for all  $0 \le i \le m - 1$ ,  $0 \le j \le n - 1$ .

*Proof.* The main idea of the proof is to include the *n*-gon v and the *m*-gon v' into a bigger *N*-gon in a Farey graph, and then apply Eq. (2.5). In other words, we will include the fundamental domain *P* into a (bigger) frieze pattern.

First, let us show that

$$v'_{m-2} > v_0 > v_1 > \ldots > v_{n-1} > v'_{m-1}$$

Indeed, the vertices  $v'_{m-2}, v'_{m-1}, v'_0$  are consecutive vertices of the *m*-gon v'. By assumption,  $v'_{m-1} = \frac{0}{1}$ , so that the condition

$$d(v'_{m-2}, v'_{m-1}) = 1$$

implies  $v'_{m-2} = \frac{1}{\ell}$  for some  $\ell$ . By Lemma 2.2.2, the distance  $d(v'_0, v'_{m-2})$  coincides with the number of triangles at the vertex  $v'_{m-1}$  which is, by construction, equal to  $q'_0$ . We finally have:

$$d(v'_0, v'_{m-2}) = \ell = q'_0,$$

so that  $v'_{m-2} = \frac{1}{q'_0}$ . The inequality  $v'_{m-1} > v_0$  then follows from the second inequality (4.1).

It is well-known that the Farey graph is connected; see [HW]. Therefore, two disjoint polygons, v and v', belong to some N-gon that contains the n-gon v and the m-gon v'.

Theorem 3 then follows from formula (2.5).

**Example 7.1.1.** Consider the tiling given in Figure 2. It corresponds to the following data:

$$q = (1, 2, 2, 1, 3),$$
  $q' = (2, 1, 2, 1),$   $M = \begin{pmatrix} 2 & 5 \\ 7 & 18 \end{pmatrix}.$ 

The associated 5-gon and 4-gon in the Farey graph are as follows:

$$v = \left(\frac{2}{7}, \frac{5}{18}, \frac{8}{29}, \frac{11}{40}, \frac{3}{11}\right), \text{ and } v' = \left(\frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{0}{1}\right),$$

respectively. They can be included in an 11-gon; see Figure 6.



FIGURE 6 The subgraph associated with the tiling in Figure 2

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Sophie MORIER-GENOUD, Sorbonne Universités, UPMC Univ Paris 06, UMR 7586, Institut de Mathématiques de Jussieu- Paris Rive Gauche, Case 247, 4 place Jussieu, 75005, Paris, France

e-mail: sophie.morier-genoud@imj-prg.fr

Valentin OVSIENKO, CNRS, Laboratoire de Mathématiques U.F.R. Sciences Exactes et Naturelles Moulin de la Housse – BP 1039 51687 Reims cedex 2, France

e-mail: valentin.ovsienko@univ-reims.fr

Serge TABACHNIKOV, Pennsylvania State University, Department of Mathematics, University Park, PA 16802, USA

e-mail: tabachni@math.psu.edu