The Ideal Standard Clocks in the General Theory of Relativity

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According to basic assumptions in the general theory of relativity the rate of an ideal standard clock moving with the velocity v through a gravitational field with the scalar potential χ is given by the formula

$$d\tau = dt \sqrt{1 + \frac{2\,\chi}{c^2} - \frac{v^2}{c^2}}\,, \tag{1}$$

where τ is the proper time of the standard clock and t is the coordinate time in a *time-orthogonal* system of space-time coordinates. It is of some didactical interest to investigate the conditions which a real clock must satisfy in order to be an ideal clock in the sense of the formula (1). This question is also of a more practical interest in view of the fact that the construction of accurate time measuring instruments in recent years has made such progress that a direct verification of (1) with $v = 0$ by terrestrial experiments is in sight. The 'atomic clocks' constructed so far, in which atomic systems like ammonia molecules act as the balance of the clock, have already an accuracy of the order 3×10^{-10}, while the relative difference in rate, according to (1), of two clocks placed at rest at suitable places on the earth may be of the order 10^{-12}.

Since the oscillating system which contributes the essential part of the clock approximately may be treated as a harmonic oscillator, we shall first investigate the motion of a particle of proper mass $\overset{\circ}{m}_0$ which is elastically bound to a fixed point O in the gravitational field. The non-gravitationed force F is then of the form

$$F = -k\,s\,, \tag{2}$$

where s is the distance of the particle from O and k is the elastic constant. The equations of motion of the particle may be written in the form of three-dimensional vector equations

$$\frac{d_c\,\boldsymbol{p}}{dt} = -\,m\,\mathrm{grad}\,\chi + \boldsymbol{F}, \tag{3}$$

where

$$m = \frac{\overset{\circ}{m}_0}{\sqrt{1 + \dfrac{2\,\chi}{c^2} - \dfrac{u^2}{c^2}}} \tag{4}$$

is the mass of the particle and

$$\boldsymbol{p} = m\,\boldsymbol{u} \tag{5}$$

is the momentum vector. $d_c\,\boldsymbol{p}/dt$ is the three-dimensionally covariant time derivative of the momentum. We have now to establish the conditions under which the time shown by this oscillator-clock is given by (1) with $v = 0$, i.e. by

$$d\tau = dt\,\sqrt{1 + \frac{2\,\chi}{c^2}}\,. \tag{6}$$

For a sufficiently small amplitude and for sufficiently small velocities in the oscillation, the mass in (3) may be treated as a constant

$$m_0 = \frac{\overset{\circ}{m}_0}{\sqrt{1 + \dfrac{2\,\chi}{c^2}}}, \tag{7}$$

where χ here is the value of the potential at the point O. Further in the immediate surroundings of O the spatial geometry may be treated as Euclidean; therefore, if we also can neglect the gravitational force as compared with F, the equations (3) take the form of the usual oscillator equation

$$m_0\,\frac{d^2x}{dt^2} = -\,k\,x\,. \tag{8}$$

In this approximation the frequency of the oscillator is $\omega = \sqrt{k/m_0}$. From (7) and the transformation formula for the elastic constant [1]

$$k = \overset{\circ}{k}\,\sqrt{1 + \frac{2\,\chi}{c^2}} \tag{9}$$

we therefore get

$$\omega = \overset{\circ}{\omega}\,\sqrt{1 + \frac{2\,\chi}{c^2}}, \tag{10}$$

where $\overset{\circ}{\omega} = \sqrt{\overset{\circ}{k}/\overset{\circ}{m}_0}$ is the frequency of the same oscillator when it is placed at rest in a system of inertia. Now ω is a measure of the rate of the oscillator-clock and therefore (6) is a consequence of (10).

Since F is of the order kA where A is the amplitude in the oscillation, the conditions for the exact equations (3) to reduce to the simple equation (8) are obviously

$$\left.\begin{array}{ll} \dfrac{u^2}{c^2} \ll 1 & \text{a} \\[2ex] \dfrac{dm_0}{dt}\, u \ll kA & \text{b} \\[2ex] \dfrac{m_0 G}{kA} \ll 1 & \text{c} \\[2ex] KA^2 \ll 1 & \text{d} \end{array}\right\} \quad (11)$$

where $G = |-\operatorname{grad}\chi\,|$ is the gravitational acceleration and K is the Riemann curvature constant for any 'plane' surface of three-dimensional geodesics through O. However, a closer consideration shows [1] that the first of these conditions, i.e. $u^2/c^2 \ll 1$ is not necessary for the validity of the formula (6) for the rate of the oscillator-clock.

Finally, if the centre O of the clock is moving with the velocity v and accelerated with the acceleration $\boldsymbol{a}$, a similar consideration shows that the formula (1) follows from the equations of motion (3) if the further condition

$$m_0\, \frac{a}{kA} \ll 1 \tag{12}$$

is satisfied. When the conditions (11c) and (12) are not well satisfied, an extra force $m_0\,(\boldsymbol{G}-\boldsymbol{a})$ will appear on the right hand side of the equation of motion (8) causing in general a change in the frequency, i.e. in the rate which is not contained in the expression (1). Only if $\boldsymbol{a} = \boldsymbol{G}$, as is the case for a freely falling clock, the two effects dealt with in (11c) and (12) will practically cancel.

For given G, K and a, it is always possible to choose the parameters of the clock, i.e. $\mathring{k}$, $\mathring{m}_0$, and A such that the conditions (11) and (12) are satisfied to any degree of accuracy. Thus it is always possible to construct clocks which are 'ideal' under given circumstances. On the other hand, the degree of accuracy to which a given clock (given $\mathring{k}$, $\mathring{m}_0$, A) may be regarded as ideal depends of course on the use we want to make of it (i. e. on G, K and a). It seems that the 'atomic clocks' constructed so far in this respect may be regarded as ideal to a very high degree of accuracy [1].

Diskussion – Discussion

V. Fock: Since the formula $d\tau = \sqrt{1 + [2\chi - v^2]/c^2}\, dt$ is approximate it is perhaps useless to write the square root. (The gravitation potential contains 10 components and to express it by a single function is possible only in an approximate treatment or in some particular cases).

C. Møller: The formula is exact for arbitrarily large χ in any time-orthogonal system of space-time coördinates, where $g_{\iota 4} = \iota$, $\iota = 1, 2, 3$. Besides the scalar potential χ defined by $g_{44} = -(1 + 2\chi/c^2)$ the spatial metric tensor components $g_{\iota\varkappa}$ enter the formula through $v^2 = g_{\iota K}\, v^\iota\, v^K$. (Greek indices like ι and $\varkappa$ are running from 1 to 3.) In the most general system of coördinates, the exact formula [2] for $d\tau$ is.

$$d\tau = dt \sqrt{\left(\sqrt{1 + \frac{2\chi}{c^2}} - \gamma_\iota v^\iota\right)^2 - \frac{v^2}{c^2}}$$

with $\gamma_\iota = g_{\iota 4}/\sqrt{-g_{44}}$ and $v^2 = \gamma_{\iota K}\, v^\iota\, v^K$, $\gamma_{\iota K} = g_{\iota K} + \gamma_\iota \gamma_K$.

W. H. McCrea: In the standard derivation of the gravitational redshift the radiation-source is implicitly assumed to be falling freely in the field; in this case the relativistic result must be exact. Can Prof. Møller's result be interpreted as showing that the result continues to hold to a good approximation if the source is *supported* in the gravitational field?

C. Møller: Yes, if the conditions (11) are statisfied.

References

[1] See, f. inst., C. Møller, *Old Problems in the General Theory of Relativity viewed from a New Angle.* Dan. Mat. Fys. Medd. vol. *30*, no. 10 *(1955)*.

[2] See, f. inst., C. Møller, *The Theory of Relativity*, Oxford *1952* (pp. 247 and 238).