Gravitational waves

Gravitational Waves

by N. ROSEN (Haifa)

Let us consider first the case of a weak gravitational field, so that by a suitable choice of coordinates ($x^1, x^2, x^3, x^4 = x, y, z, t$), one can write for the metric tensor

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}$$

where $\gamma_{\mu\nu}$ is the metric tensor of the special relativity theory, and the $h_{\mu\nu}$ are small quantities. If their squares and products can be neglected, one obtains the linear approximation of the field equations. It is well known [1] that, to describe gravitational waves travelling in the positive x-direction, one can take all the components $h_{\mu\nu}$ to vanish, except possibly

$$h_{22} = -h_{33} \neq 0 \,,$$

$$h_{23} \neq 0$$

where the non-vanishing components are arbitrary functions of the argument $x - t$. One finds then that in the linear approximation there exists the possibility of plane transverse gravitational waves propagating with the speed of light. Since these involve arbitrary functions, one can have plane monochromatic, or sinusoidal, waves as a special case. These waves are analogous to the electromagnetic waves in the MAXWELL theory.

The situation is quite different when one makes use of the exact gravitational field equations. It was shown some time ago [2] that the exact equations exclude the possibility of plane periodic waves, since one of the variables describing the field is not oscillatory, but rather has a monotonic behavior. If one attempts to set up a solution describing a plane wave train of infinite extent one finds that somewhere in space-time a singularity will be present. Thus it follows that there are no solutions of the exact equations corresponding to the monochromatic plane wave solutions of the linear equations.

It was suggested by H. P. Robertson [2] that the field equations set up for plane waves be reinterpreted in a cylindrical polar coordinate system (x^1, x^2, x^3, $x^4 = \varrho$, z, φ, t). In this case the singularity can be located on the polar axis, where it is unobjectionable (since it can be regarded as describing a material source), and the solutions obtained can be considered to describe cylindrical, rather than plane, waves.

By a suitable choice of coordinates one can write

$$ds^2 = -e^{2\gamma-2\psi}\, d\varrho^2 - e^{2\psi}\, dz^2 - e^{-2\psi}\, \varrho^2\, d\varphi^2 + e^{2\gamma-2\psi}\, dt^2.$$

The field equation can then be put into the form

$$\psi_{\varrho\varrho} + \frac{1}{\varrho}\,\psi_\varrho - \psi_{tt} = 0$$

$$\gamma_\varrho = \varrho\,(\psi_\varrho^2 + \psi_t^2), \quad \gamma_t = 2\,\varrho\,\psi_\varrho\psi_t,$$

where a subscript now denotes partial differentiation. In this case oscillatory wave solutions exist. In order that solutions sinusoidal in the time should exist, it is obviously necessary that

$$\int_0^\tau \psi_\varrho\psi_t\, dt = 0,$$

where τ is the period. One readily verifies that this condition is fulfilled by a standing wave, but not by a progressive wave. This is sometimes explained by saying that a progressive wave carries energy, which produces a secular change in the metric.

However, it is interesting to look into the question of the energy in some detail. For this purpose we make use of the pseudo-tensor t_μ^ν which occurs in the conservation relation

$$[\sqrt{-g}\,(T_\mu^\nu + t_\mu^\nu)]_{,\nu} = 0,$$

where T_μ^ν is the energy-stress tensor of the matter or other non-gravitational fields. This can be written [3]

$$16\,\pi\sqrt{-g}\; t_\mu^\nu = \left\{ {\nu \atop \alpha\beta} \right\} (\sqrt{-g}\; g^{\alpha\beta})_{,\mu} - (\ln\sqrt{-g})_{,\alpha}(\sqrt{-g}\; g^{\nu\alpha})_{,\mu}$$
$$+\,\delta_\mu^\nu \left[\left\{ {\alpha \atop \lambda\beta} \right\} \left\{ {\beta \atop \sigma\alpha} \right\} g^{\lambda\sigma}\sqrt{-g} - g^{\lambda\sigma}\left\{ {\alpha \atop \lambda\sigma} \right\} (\sqrt{-g})_{,\alpha} \right].$$

In the case of the weak field in the form of a transverse plane wave travelling in the positive x-direction, one finds for the non-vanishing components

$$-t_1^1 = -t_1^4 = t_4^1 = t_4^4 = \frac{1}{16\pi}(\dot{h}_{22}^2 + \dot{h}_{23}^2),$$

where a dot denotes differentiation with respect to the argument. On the other hand, an exact calculation in the case of the cylindrical waves gives

$$t_1^1 = t_1^4 = t_4^1 = t_4^4 = 0$$

$$t_2^2 = t_3^3 = -\frac{\varrho\,\psi_t^2}{4\pi\sqrt{-g}}.$$

Thus we find that the cylindrical waves carry no energy or momentum.

As a check one can use, in place of the unsymmetrical t_μ^ν, the symmetrical pseudo-tensor $t^{\mu\nu}$ proposed by LANDAU and LIFSHITZ [4], satisfying the conservation relation in the form

$$[(-g)(T^{\mu\nu} + t^{\mu\nu})]_{,\nu} = 0,$$

and given by the expression

$$\begin{aligned}
16\pi t^{\mu\nu} = {} & \\
& (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma}) \left[2 \left\{ {\tau \atop \lambda\sigma} \right\} \left\{ {\varrho \atop \tau\varrho} \right\} - \left\{ {\tau \atop \lambda\varrho} \right\} \left\{ {\varrho \atop \sigma\tau} \right\} - \left\{ {\tau \atop \lambda\tau} \right\} \left\{ {\varrho \atop \sigma\varrho} \right\} \right] \\
& + g^{\mu\nu} g^{\sigma\tau} \left[\left\{ {\nu \atop \lambda\varrho} \right\} \left\{ {\varrho \atop \sigma\tau} \right\} + \left\{ {\nu \atop \sigma\tau} \right\} \left\{ {\varrho \atop \lambda\varrho} \right\} - \left\{ {\nu \atop \tau\varrho} \right\} \left\{ {\varrho \atop \lambda\sigma} \right\} - \left\{ {\nu \atop \lambda\sigma} \right\} \left\{ {\varrho \atop \tau\varrho} \right\} \right] \\
& + g^{\nu\lambda} g^{\sigma\tau} \left[\left\{ {\mu \atop \lambda\varrho} \right\} \left\{ {\varrho \atop \sigma\tau} \right\} + \left\{ {\mu \atop \sigma\tau} \right\} \left\{ {\varrho \atop \lambda\varrho} \right\} - \left\{ {\mu \atop \tau\varrho} \right\} \left\{ {\varrho \atop \lambda\sigma} \right\} - \left\{ {\mu \atop \lambda\sigma} \right\} \left\{ {\varrho \atop \tau\varrho} \right\} \right] \\
& + g^{\lambda\sigma} g^{\tau\varrho} \left[\left\{ {\mu \atop \lambda\tau} \right\} \left\{ {\nu \atop \sigma\varrho} \right\} - \left\{ {\mu \atop \lambda\sigma} \right\} \left\{ {\nu \atop \tau\varrho} \right\} \right].
\end{aligned}$$

One finds that for the cylindrical waves

$$t^{11} = t^{14} = 0,$$

$$(-g)\,t^{44} = -\frac{1}{8\pi}.$$

Since the right-hand members do not depend on the nature of the solution, one can say that no energy or momentum is carried by the waves.

The results obtained for the cylindrical waves fit in with the conjecture [5] that a physical system cannot radiate gravitational energy, but, of course, they do not represent a proof. The fact that the exact equations do not admit solutions corresponding to plane monochromatic waves appears to raise some doubt as to the physical significance of gravitons arising from the quantization of the linear equations.

Diskussion – Discussion

M. Fierz: 1. Physikalisch ist es wohl richtiger aperiodische Lösungen der Gleichung

$$\psi_{\varrho\varrho} + \frac{1}{\varrho}\psi_{\varrho} - \psi_{tt} = 0$$

zu betrachten, die einem einfallenden Wellenpaket entsprechen, das im Koordinatenursprung reflektiert wird. Die Metrik ist dann überall regulär. Freilich ist im Unendlichen ($\varrho \to \infty$) die Geometrie nicht die euklidische, sondern diejenige auf einer Kegeloberfläche. In diesem Sinne ist sie dort singulär.

2. Die linearisierte Theorie ist hochgradig irreführend und lehrt gar nichts über den Charakter der strengen Lösungen. Insbesondere haben die, durch Quantisierung der linearisierten Theorie abgeleiteten „Gravitonen" keinen Sinn.

3. Da Zylinderwellen unphysikalisch sind, sollte man Lösungen vom Typus „Kugelwellen" finden. Diese Aufgabe übersteigt aber meine Kunst.

V. Fock: 1. The solution that corresponds to cylindrical waves has a singularity on the axis. Is such a singularity admissible?

2. There exist spherical waves emitted by a system of bodies of total mass M. If $\alpha = \gamma\, M/c^2$ is the gravitational radius then the phase of the waves is

$$t - \frac{1}{c}\,(r - 2\,\alpha \lg r)\,.$$

The amplitude contains terms of the order $1/r$ and $(\lg r)/r$. The non linear terms are also taken into account. Possibly the spherical waves have more physical significance than cylindrical ones.

N. Rosen: 1. A singularity on the axis is admissible since it can be considered as describing a physical system emitting or absorbing the wave.

2. Physically, spherical waves are much more important than cylindrical waves. However, it is necessary to obtain exact solutions of the field equations.

P. G. BERGMANN: Should the linearized version of cylindrical waves not yield correct results near infinity, especially with regard to energy-momentum?

N. ROSEN: The energy-momentum pseudo-tensor in cylindrical coördinates contains terms linear in the variables and these cancel the quadratic terms.

References

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[5] SCHEIDEGGER, A. E., Rev. Mod. Phys. *25*, 451 (1953).