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Mathematical Structure of the Non-Symmetric Field Theory

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In the gravitational theory the field variables (g_{ik} and Γ^s_{ik}) are taken to be symmetric in their subscripts. This symmetry property is natural if we think of g_{ik} as a metric tensor, and consider it to be the primitive concept in the theory.

However, it is known that one can approach the theory from a different point-of-view, in which the 'displacement field' Γ^s_{ik} is the primary concept. One sees then that the RIEMANN and RICCI tensors can be constructed without making use of a metric tensor, and that at no point in this procedure is symmetry in the indices required. In this sense, the gravitational theory is a specialization of a more general theory – that of the non-symmetric field.

I would like to give an account of the logical steps through which one goes when trying to set up this generalization. The present account will be based on recent work[1]) in which I participated with Prof. EINSTEIN, and in which the theory of the non-symmetric field is presented in a new form.

A. The Formalism of the Theory

1. The primary concept is the *parallel displacement of a (contravariant) vector*: When a vector $\vec{A}$ is displaced parallel to itself by an infinitesimal distance dx^i the change in its components is to be given by

$$\delta A^s = -\Gamma^s_{ik}\, dx^i\, A^k$$

We see, from the way the coefficients Γ enter here, that it would be an unwarranted specialization to take Γ^s_{ik} as symmetric in its lower indices. *Γ^s_{ik} will, then, be considered as a non-symmetrical quantity.*

[1]) Annals of Mathematics, *62* (1955), 128.

When we displace the vector $\vec{A}$ parallel to itself around a closed infinitesimally small cycle, we find the total displacement

$$\Delta A^s = R^s{}_{kmn} A^k f^{mn}$$

(where f^{mn} is the infinitesimal surface element). The coefficient $R^s{}_{kmn}$ is the Riemann curvature tensor, and it has formally the same appearance as in the symmetric field theory, except that it is now constructed from non-symmetric Γ^s_{ik}.

The curvature tensor can be contracted in two different ways. One of these contractions gives a tensor analogous to the Ricci tensor of gravitational theory:

$$R^s{}_{kms} \equiv R_{km} = \Gamma^s_{ks,m} - \Gamma^s_{km,s} - \Gamma^s_{kt}\,\Gamma^t_{sm} + \Gamma^s_{km}\,\Gamma^t_{st}\,.$$

The other contraction (which vanishes identically in the gravitational theory) is:

$$R^s{}_{smn} \equiv V_{mn} = \Gamma^s_{sm,n} - \Gamma^s_{sn,m}\,.$$

From its definition, it is clear that V_{mn} is an antisymmetrical tensor.

2. In the definition of parallel displacement a certain duality enters. One can displace vectors according to the definition given above; but, with the same coefficients Γ^s_{ik}, one can also define a displacement 'dual' to the previous one:

$$\delta A^s = -\Gamma^s_{ik}\,dx^k A^i\,.$$

We can say that we have here two displacement fields: Γ^s_{ik} and $\widetilde{\Gamma}^s_{ik} \equiv \Gamma^s_{ki}$. The second displacement field is obtained from the first by the operation of '*transposition*'[1].

A 'dual' curvature tensor can be constructed from the 'dual' displacement field; this dual tensor and its contractions differ from the corresponding tensors in the original displacement field:

$$R^s{}_{kmn}(\widetilde{\Gamma}) \neq R^s{}_{kmn}(\Gamma)\,, \quad R_{km}(\widetilde{\Gamma}) \neq R_{km}(\Gamma)\,.$$

A duality is thus introduced into the mathematical apparatus, and with it an arbitrariness in the whole scheme. We avoid this arbitrariness by postulating that *all equations of the theory shall be invariant under the operation of transposition*. In other words, one would get to the same field-equations whether one starts with the displacement-field Γ or its transpose $\widetilde{\Gamma}$.

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1) Sometimes referred to as 'Hermitian conjugation'.

It is natural to define for tensors and other field quantities the property of 'transposition symmetry'. $M^{\cdot\cdot\cdot}_{\cdot\cdot i\cdot\cdot k\cdot\cdot}(\Gamma)$ will be called transposition-symmetric in the indices i, k if

$$M^{\cdot\cdot\cdot}_{\cdot\cdot i\cdot\cdot k\cdot\cdot}(\tilde{\Gamma}) = M^{\cdot\cdot\cdot}_{\cdot\cdot k\cdot\cdot i\cdot\cdot}(\Gamma)\,.$$

If the tensor $Q_{ik}(\tilde{\Gamma})$ is transposition-*symmetric*, then the system of equations: $Q_{ik}(\Gamma) = 0$ entails the system: $Q_{ik}(\tilde{\Gamma}) = 0$, i.e., this system of equations is transposition-*invariant*. Conversely, we will expect the left-hand sides of our field-equations to be transposition-symmetric tensors. The property of transposition-invariance is thus seen to be in some sense a weaker form of the property of symmetry.

To give a physical interpretation of the duality which arises in the non-symmetric field, we can say that it corresponds to the double sign of the electric charge: $+$ or $-$. The postulate of transposition-invariance would then be interpreted to mean: all equations of the theory shall be invariant under change of the sign of the electric charge.

3. In order to determine the behavior of the field-variables, we postulate, as usual, that *the equations of the theory shall be derived from a variational principle*. In other words, we construct from our field-variables a 'variational function' $\mathfrak{H}$; a variation on the field-variables induces a variation in $\mathfrak{H}$, and we demand that

$$\delta \int \mathfrak{H}\, d\tau = 0\,,$$

when the (independent) variations of the field-variables vanish on the boundaries of integration. This demand will have an invariant meaning if $\mathfrak{H}$ transform like a scalar density under coordinate-transformations. Now, a scalar density can be constructed from the contracted RIEMANN tensor if we multiply it by a contravariant tensor density (of rank 2). In this way we are led to the introduction of new field variables $\mathfrak{g}^{ik}$, by the side of the Γ^s_{ik}; we then have the scalar densities $\mathfrak{g}^{ik} R_{ik}$, $\mathfrak{g}^{ik} V_{ik}$, and others from which to form $\mathfrak{H}$.

All this is entirely analogous to the procedure used in the gravitational theory, except that in that theory R_{ik} is the only available 2-index covariant tensor formed from the Γ^s_{ik}. In the present theory $R_{ik}(\Gamma)$ is a nonsymmetric tensor, and it would be an unjustified specialization to take $\mathfrak{g}^{ik}$ as symmetric, since in that case the antisymmetric part of R_{ik} would drop out of $\mathfrak{g}^{ik} R_{ik}$ (and $\mathfrak{g}^{ik} V_{ik}$ would vanish altogether). Therefore, $\mathfrak{g}^{ik}$ *is taken to be a non-symmetric tensor-density*. $\mathfrak{g}^{ik}$ and Γ^s_{ik} are $16 + 64$ field variables which are to be determined by the differential equations derived from the variational principle.

4. The last remaining question is that of the particular choice of the function $\mathfrak{H}$. Here the postulate of transposition-invariance plays a decisive rôle. In order to obtain transposition-invariant equations from the variational principle, we choose $\mathfrak{H}$ so that it itself is invariant under transposition. It is in this step that the new formulation of the theory appears. In previous versions of the theory, the final field-equations were brought into a transposition-invariant form; however, the variational function from which these equations were derived was not itself invariant under transposition, and this necessitated various artifices in the procedure of the derivation. These artifices are now avoided by the introduction of more natural field-variables U^s_{ik}, instead of the Γ^s_{ik}; and we understand the transposition of indices to refer to the U's rather than to the Γ's.

We can define the U's from the Γ's so:

$$U^s_{ik} \equiv \Gamma^s_{ik} - \delta^s_k \, \Gamma^t_{it} \, .$$

And now we can replace the Γ by the U in the RICCI tensor, and we find:

$$R_{ik} = U^s_{ik,s} - U^s_{it} \, U^t_{sk} + \frac{1}{3} U^s_{is} \, U^t_{tk} \, .$$

When the U^s_{ik} in this expression are replaced by their transposes, we find:

$$R_{ik}\left(\widetilde{U}\right) = R_{ki}\,(U)$$

That is to say: $R_{ik}\,(U)$ *is symmetric with respect to the transposition of the variables* U.

What we have done here is to change our understanding of the duality discussed in § 2. From now on we will understand a 'dual quantity' to mean: a quantity obtained by the transposition of the variables U (and *not* Γ). Similarly, in the postulate of transposition-invariance, we will understand that the U^s_{ik} are the variables which are being transposed. (The $\mathfrak{g}^{ik}$ are assumed to be transposition-symmetric: $\tilde{\mathfrak{g}}^{ik} = \mathfrak{g}^{ki}$). With this in mind we can rephrase (most of) the preceding discussion, so that it refers to the variables $(\mathfrak{g}, U)$ rather than to the variables $(\mathfrak{g}, \Gamma)$.

One might then ask: why not introduce the variables U right from the start, rather than define them through the Γ? Indeed, this is what we will proceed to do, – and it makes the procedure more transparent and natural. The one advantage which the variables Γ have over the U's is a more direct geometrical meaning, in terms of the parallel displacement of vectors[1]).

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[1]) V. BARGMANN has pointed out that the variables U^s_{ik} are related to the displacement-field of a vector-*density*.

We can, however, define the U's formally through their transformations law under coordinate-transformations[1]):

$$U^{l^*}_{i^* k^*} = \frac{\partial x^{l^*}}{\partial x^l} \frac{\partial x^i}{\partial x^{i^*}} \frac{\partial x^k}{\partial x^{k^*}} U^{l}_{ik} + \frac{\partial x^{l^*}}{\partial x^s} \frac{\partial^2 x^s}{\partial x^{i^*} \partial x^{k^*}} - \frac{1}{2} \delta^{l^*}_{i^*} \frac{\partial x^{t^*}}{\partial x^s} \frac{\partial^2 x^s}{\partial x^{t^*} \partial x^{k^*}}$$

$$- \frac{1}{2} \delta^{l^*}_{k^*} \frac{\partial x^{t^*}}{\partial x^s} \frac{\partial^2 x^s}{\partial x^{i^*} \partial x^{t^*}} . \tag{1}$$

We then show, in a straightforward manner, that the quantity

$$R_{ik} \equiv U^{s}_{ik,s} - U^{t}_{is} U^{s}_{tk} + \frac{1}{3} U^{s}_{is} U^{t}_{tk} \tag{2}$$

transforms like a tensor. Furthermore, $R_{ik}(U)$ is transposition-symmetric, and therefore

$$\mathfrak{H} \equiv \mathfrak{g}^{ik} R_{ik} \tag{3}$$

is a transposition-invariant scalar density, and can be used as a variational function.

5. These few steps sketched above contain the complete formalism of the theory. The rest (field-equations, conservation-laws and identities) follows from the variational principle by straightforward, classical, methods.

First one has for the variation of $\mathfrak{H}$

$$\delta \mathfrak{H} = (\mathfrak{g}^{ik} \delta U^{s}_{ik})_{,s} + \mathfrak{N}^{ik}_{s} \delta U^{s}_{ik} + R_{ik} \delta \mathfrak{g}_{ik} \tag{4}$$

where

$$\mathfrak{N}^{ik}_{s} \equiv \mathfrak{g}^{ik}{}_{,s} + \mathfrak{g}^{it} \left(U^{k}_{st} - \frac{1}{3} \delta^{k}_{s} U^{m}_{mt}\right) + \mathfrak{g}^{tk} \left(U^{i}_{ts} - \frac{1}{3} \delta^{i}_{s} U^{m}_{tm}\right) . \tag{5}$$

Next, one requires $\delta \int \mathfrak{H}\, d\tau = 0$, under the condition that the independent variations δU^{s}_{ik} and $\delta \mathfrak{g}^{ik}$ vanish on the boundary. This gives the 'Field Equations':

$$\left.\begin{aligned} R_{ik} &= 0 \\ \mathfrak{N}^{ik}_{s} &= 0 \end{aligned}\right\} \tag{6}$$

Assuming that the field-equations are satisfied, we find from (4) that

$$(\mathfrak{g}^{ik} \delta U^{s}_{ik})_{,s} = 0 . \tag{7}$$

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[1]) From the transformation-law one might think that we are here introducing a different connection between U and Γ than the one defined above. However, these relations are seen to be identical, when one takes the λ-transformation into account. See below; cf. also the paper cited above.

Different specializations of δU^{s}_{ik} (no longer required to vanish on the boundary!) give us the 'Conservation Law':

$$\mathfrak{T}^{\,s}_{a,s} \equiv (\mathfrak{g}^{ik}\, U^{s}_{ik,a})_{,s} = 0 \tag{8}$$

as well as

$$\mathfrak{T}^{\,s}_{a} = (\mathfrak{g}^{ik}\, U^{s}_{ik}\, \delta^{t}_{a} - \mathfrak{g}^{sk}\, U^{t}_{ak} - \mathfrak{g}^{ks}\, U^{t}_{ka})_{,t} \tag{9}$$

and the 'Divergence Equation'[1]:

$$\mathfrak{g}^{\underset{\smile}{ik}}{}_{,k} = 0\,. \tag{10}$$

((8) and (9) are due to infinitesimal coordinate transformations; (10) – to an infinitesimal λ-transformation).

One sees that the conservation law (8) has a particularly simple form in the new variables.

Finally we get the 'Bianchi Identities' (which are given here modulo the field-equations $\mathfrak{N}^{ik}_{s} = 0$)

$$\mathfrak{g}^{ik}\, R_{ik,t} - (\mathfrak{g}^{is}\, R_{it} + \mathfrak{g}^{si}\, R_{ti})_{,s} \equiv 0$$

and another differential identity

$$(\mathfrak{N}^{\underset{\smile}{sk}}{}_{s})_{,k} \equiv 0\,.$$

(This identity is a trivial consequence of (5), since $\mathfrak{N}^{\underset{\smile}{sk}}{}_{s} = \mathfrak{g}^{\underset{\smile}{sk}}{}_{,s}$ as one can readily verify).

The existence of identities is due to the invariance properties of $\mathfrak{H}$. On the one hand $\int \mathfrak{H}\, d\tau$ is invariant under coordinate transformations, so that its variation vanishes *identically* under these transformations, and gives rise to four identities among the field equations. As a result, four of the field variables remain undetermined, so that four arbitrary coordinate choices can be made; fields which differ from one another only by a coordinate-transformation are thus essentially the same.

On the other hand, it is easy to verify that $\mathfrak{H}$ remains invariant under the so-called 'λ-transformations' defined by

$$\begin{aligned} \lambda :\; & U^{s}_{ik} \to U^{s}_{ik} + (\delta^{s}_{i}\, \lambda_{,k} - \delta^{s}_{k}\, \lambda_{,i}) \\ & \mathfrak{g}^{ik} \to \mathfrak{g}^{ik}. \end{aligned} \tag{11}$$

This invariance leads to one more identity among the field equations. Similar to the case of the 4 Bianchi identities, it suggests that U-fields

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[1]) This equation is also a consequence of the system $\mathfrak{N}^{ik}_{s} = 0$.

which differ from one another only by a λ-transformation are to be considered as the same field.

What we have done is to extend the group of transformations of general relativity. Under the extended group, the variables U^s_{ik} no longer decompose into symmetric and anti-symmetric parts which transform separately.

In concluding this summary of the formalism of the theory, it is important to remark that the system of field-equations (6) is entirely equivalent to the system

$$g^{ik}{}_{,s} + g^{il}\,\Gamma^k_{st} + g^{tk}\,\Gamma^i_{ts} = 0 \tag{11a}$$

$$\Gamma_i = \frac{1}{2}\,(\Gamma^s_{is} - \Gamma^s_{si}) = 0 \tag{11b}$$

$$R_{\underline{ik}} = 0 \qquad R_{\underset{\vee}{ik},l} + R_{\underset{\vee}{kl},i} + R_{\underset{\vee}{li},k} = 0 \tag{11c}$$

in EINSTEIN's previous version of the theory; and one can pass from (6) to (11) by a suitable substitution of variables.

B. A Few Remarks concerning Physical Interpretation

6. From the relation (9) which gives the components of the density $\mathfrak{T}^s_a$ one can calculate $\int \mathfrak{T}^4_4\, d\tau$, and we find (assuming that the field behaves as a SCHWARZSCHILD solution for large distances) that

$$\int \mathfrak{T}^4_4\, d\tau \sim m\,.$$

It is then natural to look upon $\mathfrak{T}^s_a$ as an 'energy-momentum tensor-density' (really a pseudo-tensor).

The Divergence Equation (10) corresponds to the vanishing of the magnetic current-density in MAXWELL's theory – provided one identifies $\mathfrak{g}^{i4}$ ($i = 1, 2, 3$) with the components of the magnetic field.

To satisfy the continuity equation for electric charge, one identifies the electric current-density with the vector density[1])

$$\mathfrak{I}^s = \frac{1}{6}\,\eta^{ikls}\,(g_{\underset{\vee}{ik},l} + g_{\underset{\vee}{kl},i} + g_{\underset{\vee}{li},k})\,.$$

We have then identically:

$$\mathfrak{I}^s{}_{,s} = 0\,.$$

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[1]) η^{ikls} is the LEVI-CIVITA tensor density, antisymmetric in all indices.

In order to make further connections with electromagnetic theory, one has to use approximation methods. We assume that $g_{\underset{\vee}{ik}}$ is a weak field of first order, and that $g_{\underline{ik}}$ differs from the MINKOWSKI values by quantities of the first order. When the field equations are written out to first order, we find that they decompose into two sets: (1) 'gravitational equations' which are identical with the symmetric-field equations (to that order of approximation); (2) 'MAXWELL equations'[1])

$$\eta_s \, g_{\underset{\vee}{is},s} = 0 \quad \text{and} \quad \eta^s \, (g_{\underset{\vee}{ik},l} + g_{\underset{\vee}{kl},i} + g_{\underset{\vee}{li},k})_{,ss} = 0 \, .$$

The second set of these equations is weaker than the corresponding one in MAXWELL's theory. Of course, this first approximation, in a non-linear theory, tells us nothing about the interaction of the symmetric and anti-symmetric fields. For that one has to make complicated calculations to higher orders of approximation.

C. Results in the Theory

7. When we attempt to solve the equations in this theory, we are faced with difficulties which are even greater than those of the gravitational theory. The usual approach is to treat the system of equations as consisting of two parts. The first part ($\mathfrak{N}_s^{ik} = 0$ in our presentation) is quite simple in principle. It is a system of linear, non-homogeneous algebraic equations for the U^s_{ik} (or correspondingly the Γ^s_{ik}) as unknown variables, to be solved in terms of $\mathfrak{g}^{ik}$ and $\mathfrak{g}^{ik}{}_{,s}$. In principle, one has only to invert the matrix of coefficients of the unknown U's (or Γ's), and to state the exceptional cases when this inversion cannot be carried out (due to the vanishing of the determinant of the coefficients). In practice, however, the inversion is quite a laborious task. Several papers have appeared [1], expressing the inverted matrix in different forms.

The complexity of the expressions for U in terms of $\mathfrak{g}^{ik}$, $\mathfrak{g}^{ik}{}_{,s}$ makes it impossible in general to substitute for U in the other part of the system of equations ($R_{ik} = 0$). Such substitutions have only been carried out in very specialized cases.

Nevertheless, some general information about the system $R_{ik} = 0$ can be obtained by analyzing the way in which the derivatives $g_{ik,4}$ and $g_{ik,44}$ enter into the equations. One can then treat the Cauchy problem relative to this system, and it has been shown [2] that, just as in the gravitational field theory, so also in the non-symmetrical theory, the Cauchy problem (the question of 'relativitic determinism') has a unique solution.

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[1]) $\eta_1 = \eta_2 = \eta_3 = -1 = -\eta_4$.

A considerable amount of work has been done on special solutions in the theory. Rigorous solutions in various forms have been given for the static, spherically-symmetric case [3]. All of these solutions show singularities. Similarly, for the axially-symmetric static case we have shown that the assumption of regularity at the origin is incompatible with the field-equations. For time dependent fields, a rigorous special solution is known, which is everywhere regular (the 'plane electromagnetic wave') [4]. This solution however is not Euclidean at infinity.

Singular solutions are inadmissible in a complete field theory which does not make an artificial separation between matter and the field produced by it. Acceptable solutions, according to this viewpoint, must be everywhere regular[1]). In addition, the solutions are assumed to be asymptotically Euclidean, in a suitable coordinate-system.

8. The requirement that the field variables shall be everywhere regular has several important consequences, both locally and globally.

a) The space-time signature. In the gravitational theory one requires that

$$det\,(g_{i\,k}) \neq 0 \tag{12}$$

everywhere, so that the contravariant quantities g^{ik} are nowhere singular. Taking into account the boundary conditions, which require the field to be imbedded in a Euclidean space, we see that this determinant is everywhere negative. The matrix g_{ik} can be transformed locally into a diagonal form with the signature $(-, -, -, +)$, and this gives us the basis for distinguishing time-like and space-like directions at each point of the continuum.

In the non-symmetric theory, the g_{ik} matrix cannot be transformed into a diagonal form by any *real* coordinate transformation. The simplest form to which g_{ik} can be transformed locally is

$$g_{ik} \sim \begin{pmatrix} -1 & \underset{\vee}{g_{12}} & & \\ -\underset{\vee}{g_{12}} & -1 & & \\ & & -1 & \underset{\vee}{g_{34}} \\ & & -\underset{\vee}{g_{34}} & +1 \end{pmatrix} \tag{13}$$

where $\underset{\vee}{g_{12}}, \underset{\vee}{g_{34}}$ are real quantities which can be expressed as functions of invariants of the field. One can take the diagonal terms in (13) to be the

[1]) The manifold on which the field-variables are defined is assumed to be topologically equivalent to the Euclidean 4-space. The property of regularity then means: there exists a system of coordinates (x^i), covering the whole manifold, such that when expressed in this coordinate-system, $g^{ik}\,(x^i)$ are regular functions.

'signature' of the non-symmetric field. Now, a necessary condition for carrying out the transformation to the 'canonical' form (13) is: det $(g_{\underline{ik}}) \neq 0$. However, in the theory of the non-symmetric field, one wants to avoid conditions which apply to parts of the total tensor. Instead, one reads the condition (12) as applying to the total g_{ik} tensor. In addition, we require that the field variables Γ^s_{ik} are finite and uniquely determined at each point in terms of the g_{ik} and their first derivatives; from this we can *deduce* that *det* $(g_{ik}) = 0$. Hence one has a well-defined space-time signature at each point [5].

b) Restriction on the antisymmetric field. From the (local) 'canonical' form of the g_{ik} matrix, we must clearly have $|g_{\underset{\vee}{34}}| < 1$, in order to prevent the determinant of g_{ik} from vanishing. This means that the invariants of the antisymmetric field cannot be arbitrarily given [5].

c) Vanishing of mass for static fields. In the gravitational theory we have the Einstein-Pauli theorem for static fields, which states that if the field is everywhere regular, satisfies the field-equations, and behaves at large distances like a Schwarzschild solution, then its mass must vanish. In the proof, Gauss' theorem is applied to a divergence which is known to vanish in the static field. Since the field is assumed regular, the volume-integral over the divergence can be converted into a surface-integral; the boundary conditions are inserted, and it is found that the integral is proportional to the 'mass' of the Schwarzschild solution. On the other hand, this integral vanishes, since its integrand is everywhere zero.

The proof can be carried out almost as readily in the non-symmetric field theory. Equation (8) in which $\mathfrak{T}^s_a$ is defined shows that $\mathfrak{T}^4_4 = 0$ in a static field. On the other hand, equation (9) gives us $\mathfrak{T}^4_4$ as the divergence of some function of the field-variables. From here on the proof is formally the same as in the gravitational theory [6].

d) Are static fields locally Euclidean? Lichnerowicz [7] has shown that this is the case for the gravitational field theory. He makes use of theorems about elliptic operators: $F.V \equiv g^{ij} V_{,ij} + a^i V_{,i}$ (g^{ij} is a definite-negative form); if $F.V$ is known to be non-negative in a given domain, then V cannot attain a minimal value within the domain, without reducing to a constant. Now, the gravitational field-equations, in the time-independent case, can be put into a form where the theorems apply. To do this, one has to express the Γ^s_{ik} explicitly in terms of the g_{ik}. In addition, one assumes that the solution behaves asymptotically like a Schwarzschild particle – in particular[1]): $g_{44} \sim 1 - m/r \leq 1$ so that a *regular* g_{44} must attain its lowest value at some point in space. In the non-symmetric

[1]) Here a tacit assumption is brought in: the Schwarzschild constant m (teh 'mass') must be positive.

field-theory, however, the expression of Γ^{s}_{ik} through the g_{ik} is so complicated that it may not be possible to establish whether or not $F.g_{44}$ is everywhere non-negative. It would be very desirable to provide a proof of this theorem which does not depend on the explicit substitution of Γ by g; such a proof could then be extended to the non-symmetric theory [8].

D. Alternative Theories

Several variations of EINSTEIN's theory have been suggested. I would like to describe two of these very briefly.

a) SCHRÖDINGER's 'purely affine' theory [9] is based on the same principles as EINSTEIN's. However, for his variational function, SCHRÖDINGER chooses $\mathfrak{H}_s \equiv \sqrt{-det\,(R_{ik})}$ (a scalar density!). Thus he does not bring in the additional tensor g^{ik} into the variational procedure. The only quantities to be varied are the Γ^{s}_{ik}. However, SCHRÖDINGER *defines* $\lambda\, \mathfrak{g}_{ik} \equiv \delta\mathfrak{H}/\delta R_{ik}$ (λ being a constant which is inherently $\neq 0$). By so doing, he arrives at a system of differential equations for $\mathfrak{g}$ and Γ, which is found to be not transposition-invariant. For that reason, a change of variables has to be made (from Γ to $^*\Gamma$), which brings the equations into a transposition-invariant form. The final equations are identical with EINSTEIN's equations (11) (the g, Γ representation), except for the appearance of the constant λ, which replaces (11c) by

$$R_{\underline{ik}} = \lambda\, g_{\underline{ik}} \qquad R_{\underset{\smile}{ik},l} + R_{\underset{\smile}{kl},i} + R_{\underset{\smile}{li},k} = \lambda\, (g_{\underset{\smile}{ik},l} + g_{\underset{\smile}{kl},i} + g_{\underset{\smile}{li},k})\,.$$

In SCHRÖDINGER's theory, λ plays the role of a cosmological constant, and is therefore considered as being very small.

It is of interest to note that the change of variables (from Γ to $^*\Gamma$) can be avoided in SCHRÖDINGER's theory just as in EINSTEIN's theory, by using from the beginning the variables U, in terms of which R_{ik} is transposition-invariant.

b) In KURŞUNOĞLU's theory [10], equation (11a) is accepted as a *definition* of the Γ^{s}_{ik}; equation (11b) is also adopted. These equations are shown to lead to 4 relations among the g and Γ, which are identical in form with the generalized BIANCHI identities, except that g_{ik} appears instead of R_{ik}. This suggests a proportionality between the similar relations, which, when carried out, yields the equation system in KURŞUNOĞLU's theory. (11c) is now replaced by

$$R_{\underline{ik}} = -p^2\, (g_{\underline{ik}} - b_{ik}), \quad R_{\underset{\smile}{ik},l} + R_{\underset{\smile}{kl},i} + R_{\underset{\smile}{li},k} = -p^2\, (g_{\underset{\smile}{ik},l} + g_{\underset{\smile}{kl},i} + g_{\underset{\smile}{li},k}).$$

b_{ik} is a symmetric tensor formed from the g_{ik}, but different in general from $g_{\underline{ik}}$. When the antisymmetric field is absent, b_{ik} and $g_{\underline{ik}}$ coincide; hence, the constant p^2 is not a cosmological constant. The equation-system is also derivable from a variational principle, with

$$\mathfrak{H}_K \equiv \mathfrak{g}^{ik} R_{ik} - 2\,p^2 \left\{ [-\det b_{ik}]^{1/2} - [-\det g_{ik}]^{1/2} \right\}.$$

Diskussion – Discussion

Mme A. TONNELAT: Il est possible aussi d'élargir la théorie en supprimant la condition

$$\delta_\varrho\, g^{\underset{\smile}{\mu\varrho}} = 0$$

Pour cela, il suffit de partir d'une densité formée avec un tenseur de RICCI $R_{\mu\nu}\,(L)$ écrit avec une connexion $L^{\varrho}_{\mu\nu}$ dont le vecteur de torsion est nul. Après changement de connexion affine $L^{\varrho}_{\mu\nu} \to \Delta^{\varrho}_{\mu\nu}$, on aboutit finalement à l'équation

$$g^{\underset{+}{\mu}\underset{-}{\nu}}{}_{;\varrho} = 0$$

écrite avec une connexion Δ telle que

$$\delta_\varrho\, g^{\underset{\smile}{\mu\varrho}} = g^{\underline{\mu\varrho}}\, \Delta_\varrho$$

mais $\delta_\varrho\, g^{\underset{\smile}{\mu\varrho}}$ et $\Delta_\varrho = \Delta^{\sigma}_{\underset{\smile}{\varrho\sigma}}$ ne sont dans ce cas pas nuls séparément.

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