Electromagnetic properties of the nucleon and relativistic electron-proton scattering according to meson theory

Electromagnetic Properties of the Nucleon and Relativistic Electron-Proton Scattering According to Meson Theory

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(14. II. 1958)

Summary. The relativistic cut-off prescription of ARNOUS and HEITLER is used here to calculate the anomalous magnetic moments of the nucleons and the elastic scattering of high energy electrons by protons on the basis of charge-symmetrical pseudoscalar meson theory, using both pseudoscalar and pseudovector couplings. The calculations are carried to the first order in perturbation theory. Good agreement is found with the experimental results of HOFSTÄDTER *et al.* on the electron proton scattering. Further the non-relativistic limit of the anomalous moments has been studied and a discrepancy in the results of the usual non-relativistic meson theory models has been pointed out and treated.

Introduction

It has been felt for a long time now that the present day field theories stand in need of a fundamental change in their structure. For instance, meson theory, inspite of the large amount of work done on it, has had little more than qualitative success. Recently, however, a hope has arisen that the theory can still be used in its present form with moderate success within a certain region of its validity. The limit of validity of the theory is usually expressed in the form of a 'cut-off'. Such a cut-off prescription was proposed some time ago by ARNOUS and HEITLER[1]). This prescription has the advantage that the cut-off is applied in a relativistically invariant way to the three-dimensional momenta of the virtual particles. In the non-relativistic limit this cut-off is equivalent to an extended source. It is, naturally, interesting to study the effect of this cut-off on the various meson-theoretical phenomena. With this in

view, we deal in the present work with some of the electromagnetic properties of the nucleons according to meson theory. Our chief purpose has been to calculate the scattering of high energy electrons by protons, interest in which has grown since its measurement at Stanford by HOFSTADTER, MCALLISTER and CHAMBERS[2]). The measurements have been performed with electron energies ranging up to 550 Mev. These experimenters were able to find nice fits for their data on the basis of phenomenological models describing the proton as having an extended charge and magnetic moment distribution. Since meson theory also leads to some sort of an extended structure for the nucleons, it is worth while studying this scattering on the basis of meson theory. The high energies involved require the theory to be relativistic, which is the case in the present work. Alongside this, the old problem of the anomalous magnetic moments of the nucleons has also been studied. The calculations have been carried to the lowest order in covariant perturbation theory using both pseudoscalar and pseudovector couplings of the mesons to the nucleons.

Without any cut-off, the electron proton scattering was studied meson-theoretically first by ROSENBLUTH[3]), but his results have never been compared with experiments. The anomalous magnetic moments were calculated similarly by many authors[4]). A non-relativistic calculation of the charge and magnetic moment distribution and electron proton scattering on the basis of the static model – recently used by CHEW[5]) – has also been performed by SALZMAN[6]).

In the first section of this paper, we describe briefly the general formalism used. In the second section, we deal with the nucleon in the presence of an external electromagnetic field with a view to obtaining the anomalous magnetic moments. This external electromagnetic field is then replaced in the third section by the electron and its field leading to the electron-proton scattering. In the fourth section we study the non-relativistic limit of our theory for the anomalous moments, as it brings out a very interesting feature of the cut-off procedure used here. It is found that this non-relativistic limit is different from the result obtained from the static model since the latter is unable to take account of the deformation of the source for a moving nucleon – an effect which has, even in the limit of the nucleon at rest, a nonvanishing result. This is because (as shown in section IV) the ambiguous terms to be removed can be identified only when the nucleon moves.

Our calculations show that meson theory, with the cut-off used, leads to very good agreement with the electron-proton scattering experiments at all energies.

Section I

1. *Notations*

The neutron and proton are described by the Dirac spinors ψ_N, ψ_P. As usual, these are considered as the two charge states of a single particle, the nucleon, described by the eight-component spinor ψ. The 'charge-operators' acting on these charge states are defined as:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \tau = \frac{1}{2}(\tau_1 + i\tau_2); \; \tau^\dagger = \frac{1}{2}(\tau_1 - i\tau_2);$$

with the properties,

$$\tau\,\psi_N = \psi_P; \quad \tau\,\psi_P = 0; \quad \tau^\dagger\,\psi_P = \psi_N; \quad \tau^\dagger\,\psi_N = 0.$$

We also define:

$$\tau_P = \frac{1}{2}(1 + \tau_3)\;; \quad \tau_N = \frac{1}{2}(1 - \tau_3)\;;$$

so that

$$\tau_P\,\psi_P = \psi_P\;; \quad \tau_N\,\psi_N = \psi_N\;; \quad \tau_P\,\psi_N = \tau_N\,\psi_P = 0.$$

M denotes the mass of the nucleon.

Throughout we shall use the units for which $c = \hbar = 1$.

The four-dimensional time-space coordinates are:

$$x^0 = t\;; \quad x^1 = x\;; \quad x^2 = y;\; x^3 = z\;,$$

and such a four-vector is written in the text

$$\boldsymbol{x} = (x^0, \vec{x})$$

and in the diagrams with an underlined letter, $\underline{x} = (x^0, \vec{x})$.

The metric $g_{\mu\nu}$ has the components

$$g_{00} = 1\;; \quad g_{ii} = -1\;(i = 1, 2, 3); \quad g_{\mu\nu} = 0\;(\mu \neq \nu).$$

β, α^k are the usual Dirac-operators and we also write

$$\gamma^0 = \beta\;; \quad \gamma^k = \beta\alpha^k\;\;(k = 1, 2, 3)\;;$$

$$\gamma_5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3\;; \quad \bar{\psi} \equiv \psi^\dagger\gamma_0\;;$$

we have then

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}\,I.$$

The product of two four-vectors is written, for example, as follows:

$$(\gamma\cdot\boldsymbol{A}) = \sum_{\mu=0}^{3}\gamma^\mu A_\mu = \sum_{\mu=0}^{3}\gamma_\mu A^\mu = \gamma^0 A^0 - \vec{\gamma}\cdot\vec{A}.$$

The π-mesons are described by a pseudoscalar field. The neutral mesons are described by a real field φ_3 and the charged mesons by a complex field φ with the real and imaginary parts essentially φ_1 and φ_2:

$$\varphi = \frac{1}{\sqrt{2}} (\varphi_1 + i\,\varphi_2)\,; \quad \varphi^\dagger = \frac{1}{\sqrt{2}} (\varphi_1 - i\,\varphi_2).$$

The mass of the meson is denoted by μ.

An external electromagnetic field will be described by the four-potential A^μ.

The electron spinors and operators will be distinguished by an index 'e'.

2. *Interaction Hamiltonian*

Let us put

$$B = (\overline{\psi}\,\gamma_5\,\tau\,\psi)\,; \qquad B^\dagger = (\overline{\psi}\,\gamma_5\,\tau^\dagger\,\psi)\,; \qquad B^{(3)} = (\overline{\psi}\,\gamma_5\,\tau_3\,\psi)\,;$$
$$B_\mu = i\,(\overline{\psi}\,\gamma_5\gamma_\mu\,\tau\,\psi)\,; \quad B^\dagger_\mu = i\,(\overline{\psi}\,\gamma_5\gamma_\mu\,\tau^\dagger\,\psi)\,; \quad B^{(3)}_\mu = i\,(\overline{\psi}\,\gamma_5\gamma_\mu\,\tau_3\,\psi).$$

We have then, in the 'Interaction-representation', the state vector $\Phi(t)$, of the system of the nucleons and mesons in an electromagnetic field, described by the equation

$$i\,\frac{\partial\,\Phi(t)}{\partial\,t} = H_{\mathrm{I}}\,\Phi(t)\,,$$

where

$$H_{\mathrm{I}} = \int \mathfrak{H}_{\mathrm{I}}(\boldsymbol{x})\,d^3x\,;$$

$$\begin{aligned}
\mathfrak{H}_{\mathrm{I}}(\boldsymbol{x}) = {} & GB^{(3)}(\boldsymbol{x})\,\varphi_3(\boldsymbol{x}) + \sqrt{2}\,GB(\boldsymbol{x})\,\varphi(\boldsymbol{x}) + \sqrt{2}\,GB^\dagger(\boldsymbol{x})\,\varphi^\dagger(\boldsymbol{x}) + \\
& + \frac{F}{\mu}\,B^{(3)}_\mu(\boldsymbol{x})\,\frac{\partial\,\varphi_3(\boldsymbol{x})}{\partial\,x_\mu} + \sqrt{2}\,\frac{F}{\mu}\,B^\dagger_\mu(\boldsymbol{x})\,\frac{\partial\,\varphi^\dagger(\boldsymbol{x})}{\partial\,x_\mu} + \\
& + \sqrt{2}\,\frac{F}{\mu}\,B_\mu(\boldsymbol{x})\,\frac{\partial\,\varphi(\boldsymbol{x})}{\partial\,x_\mu} + \\
& + \frac{F^2}{\mu^2}\left\{\frac{1}{2}\,[N_\mu B^{(3)\mu}(\boldsymbol{x})]^2 + 2\,[N_\mu B^{\mu\,\dagger}(\boldsymbol{x})]\,[N_\nu B^\nu(\boldsymbol{x})]\right\} - \\
& - i\,e\left\{\frac{\partial\,\varphi^\dagger(\boldsymbol{x})}{\partial\,x_\mu}\,\varphi(\boldsymbol{x}) - \varphi^\dagger(x)\,\frac{\partial\,\varphi(\boldsymbol{x})}{\partial\,x_\mu}\right\} A_\mu(\boldsymbol{x}) + \\
& + e\,\overline{\psi}(\boldsymbol{x})\,\gamma^\mu\tau_P\,\psi(\boldsymbol{x})\,A_\mu(\boldsymbol{x}) - \\
& - e^2\left\{A_\mu(\boldsymbol{x})\,A^\mu(\boldsymbol{x}) - [N_\mu A^\mu(\boldsymbol{x})]^2\right\}\varphi^\dagger(\boldsymbol{x})\,\varphi(\boldsymbol{x}) + \\
& + i\sqrt{2}\,e\,\frac{F}{\mu}\Big\{B_\mu(\boldsymbol{x})\,A^\mu(\boldsymbol{x})\,\varphi(\boldsymbol{x}) - B^\dagger_\mu(\boldsymbol{x})\,A^\mu(\boldsymbol{x})\,\varphi^\dagger(\boldsymbol{x}) - \\
& \qquad - [N_\mu B^\mu(\boldsymbol{x})]\,[N_\nu A^\nu(\boldsymbol{x})]\,\varphi(\boldsymbol{x}) + \\
& \qquad + [N_\mu B^{\mu\,\dagger}(\boldsymbol{x})]\,[N_\nu A^\nu(\boldsymbol{x})]\,\varphi^\dagger(\boldsymbol{x})\Big\}. \qquad (\mathrm{I}-1)
\end{aligned}$$

N_μ is the unit normal to the space-like surface at $\mathbf{x}$. Here we may drop the 'normal-dependent' terms and use Mathews' rules for the calculation of the S-matrix*). G is the pseudoscalar coupling constant and F the pseudovector coupling constant. We may put either of them equal to zero, depending on the coupling we wish to use. The theory used is throughout charge-symmetrical. e is the charge of the proton.

The commutation rules are, as usual:

$$[\varphi_3(\mathbf{x}), \varphi_3(\mathbf{x}')] = [\varphi(\mathbf{x}), \varphi^\dagger(\mathbf{x}')] = i\,\Delta(\mathbf{x}-\mathbf{x}'),$$

where

$$\Delta(\mathbf{x}) = \frac{-i}{(2\pi)^3} \int_{-\infty}^{+\infty} d^4 k\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \delta(\mathbf{k}^2-\mu^2)\, \varepsilon(\mathbf{k}),$$

$$\varepsilon(\mathbf{k}) = \frac{k_0}{|k_0|}$$

and

$$[\psi_\alpha(\mathbf{x}), \bar{\psi}_\beta(\mathbf{x}')]_+ = -i\, S_{\alpha\beta}(\mathbf{x}-\mathbf{x}')$$

where

$$S(\mathbf{x}-\mathbf{x}') = -\left(i\gamma_\mu \frac{\partial}{\partial x_\mu} + M\right) \Delta(\mathbf{x}-\mathbf{x}'; M),$$

$\Delta(\mathbf{x}; M)$ being the same Δ-function as above excepting that μ is replaced by M.

In case we use a quantised electromagnetic field we shall also have the commutation relation

$$[A_\mu(\mathbf{x}), A_\nu(\mathbf{x}')] = -i\, g_{\mu\nu} D(\mathbf{x}-\mathbf{x}')$$

where $D(\mathbf{x}-\mathbf{x}')$ is equal to $\Delta(\mathbf{x}-\mathbf{x}')$ with $\mu = 0$.

3. *S-matrix*

It is defined by

$$\Phi(\infty) = S\,\Phi(-\infty).$$

$$S = 1 + \sum_{n=1}^{\infty} S_n;$$

$$S_n = \frac{(-i)^n}{n!} \int d^4 x_1 \int d^4 x_2 \int \cdots \int d^4 x_n$$
$$P\left(\mathfrak{H}_I(\mathbf{x}_1)\, \mathfrak{H}_I(\mathbf{x}_2) \cdots \mathfrak{H}_I(\mathbf{x}_n)\right).$$

P denotes the chronological operator of Dyson.

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*) See P. T. Mathews: Phys. Rev. *76*, 684 *L* (1949); Phys. Rev. *76*, 1419 *L* (1949)

We have appended the above notations and definitions for the sake of clarity and completeness. We shall assume hence-forth the methods of writing the matrix elements for the various processes, as they are all very widely known and discussed.

4. *The cut-off procedure*

We shall apply the cut-off to the three-dimensional momenta of the virtual mesons absorbed and emitted by a nucleon. This cut-off is invariant in the sense that it is equivalent to applying it to the invariant variable of integration z given by

$$z^2 = \left\{ \sqrt{M^2 + (\vec{p} - \vec{k})^2} + \sqrt{\mu^2 + \vec{k}^2} \right\}^2 - \vec{p}^{\,2}, \qquad (\mathrm{I} - 2)$$

where $\vec{p}$ is the momentum of the nucleon and $\vec{k}$ that of the meson. Integrals, which by their physical nature are invariant, can be expressed as integrals over z only instead of the variable k. When the nucleon is at rest, this cut-off in z is equivalent to cutting off $\vec{k}$ symmetrically to a radius K_0. If $\vec{p}$ is not equal to zero then the above sphere of integration for $\vec{k}$ is transformed by a Lorentz-deformation, which is expressed by the invariance of z.

Of course, it should be noted that the cut-off is not here meant for (nor is it capable of) suppressing any ambiguities due to divergences of the field theory. These must always be first removed by the usual physical considerations of covariance, renormalization, etc. and then the resulting integrals are to be cut-off, whether finite or infinite.

Section II

The nucleon in an external electromagnetic field

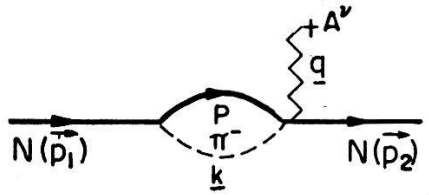
To obtain the anomalous magnetic moments of the neutron and proton we consider their scattering by a slowly varying external electromagnetic field described by the four-potential:

$$\left.\begin{aligned} & A^\nu(\mathbf{x}) = \int a^\nu(\boldsymbol{q})\, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\, d^4 q \\ \text{with} \quad & \frac{\partial A_\nu}{\partial x_\nu} = 0\,; \quad i \cdot e,\ \boldsymbol{q}\cdot\boldsymbol{a}(\boldsymbol{q}) = 0. \end{aligned}\right\} \qquad -(\mathrm{II} - 1)$$

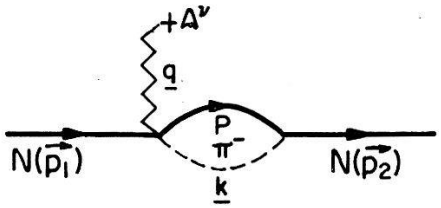
For simplicity, we consider briefly the case of the neutron as it will bring out all the essential features necessary for our further discussions. (For more details about this see FRIED (ref. [4].) We limit ourselves to

the lowest order S-matrix elements, proportional to $e\,F^2$. The pseudovector coupling is used and we shall indicate below how the pseudoscalar coupling is studied alongside.

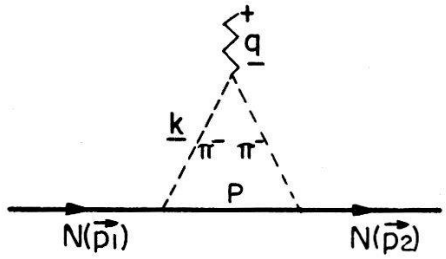
Below are shown the various Feynman-diagrams which contribute to our process:



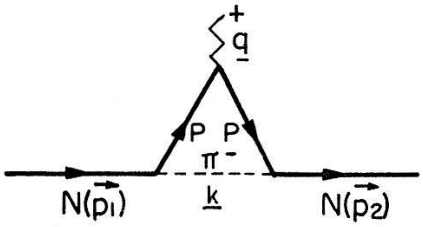
(1) $M_2^{(1)}$



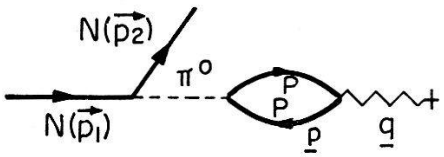
(2) $M_2^{(2)}$



(3) $M_3^{(1)}$



(4) $M_3^{(2)}$



(5) $M_3^{(3)}$

Here $\vec{p}_1$ is the initial neutron momentum and $\vec{p}_2$ the final. Let u_1 and u_2 represent the corresponding Dirac spinors. Writing

$$E\left(\vec{p}\right) = \sqrt{p^2 + M^2}\,,$$

$$\boldsymbol{q} = \boldsymbol{p}_2 - \boldsymbol{p}_1\,,$$

and

$$\Lambda \equiv e\left(\frac{F}{\mu}\right)^2 \frac{1}{(2\,\pi)^3} \sqrt{\frac{M^2}{E\left(\vec{p}_1\right) E\left(\vec{p}_2\right)}}\,,$$

*

the matrix elements corresponding to the above five diagrams are respectively:

$$M_2^{(1)} = -2\Lambda \int d^4k \frac{\bar{u}_2 (\gamma\cdot \boldsymbol{a}(\boldsymbol{q}))\{\gamma\cdot(-\boldsymbol{p}_1+\boldsymbol{k})+M\}(\gamma\cdot\boldsymbol{k})\, u_1}{[(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2]}, \tag{II – 2}$$

$$M_2^{(2)} = -2\Lambda \int d^4k \frac{\bar{u}_2 (\gamma\cdot \boldsymbol{k})\{\gamma\cdot(-\boldsymbol{p}_2+\boldsymbol{k})+M\}(\gamma\cdot\boldsymbol{a}(\boldsymbol{q}))\, u_1}{[(\boldsymbol{p}_2-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2]}, \tag{II – 3}$$

$$M_3^{(1)} = 2\Lambda \int d^4k \frac{\bar{u}_2\{\gamma\cdot(\boldsymbol{q}+\boldsymbol{k})\}\{\gamma\cdot(-\boldsymbol{p}_1+\boldsymbol{k})+M\}(\gamma\cdot\boldsymbol{k})\, u_1}{[(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2][(\boldsymbol{k}+\boldsymbol{q})^2-\mu^2]}\cdot \boldsymbol{a}(\boldsymbol{q})\cdot(2\boldsymbol{k}+\boldsymbol{q}), \tag{II – 4}$$

$$M_3^{(2)} = 2\Lambda \int d^4k \frac{\bar{u}_2(\gamma\cdot\boldsymbol{k})\{\gamma\cdot(-\boldsymbol{p}_2+\boldsymbol{k})+M\}(\boldsymbol{a}(\boldsymbol{q})\cdot\gamma)\{\gamma\cdot(-\boldsymbol{p}_1+\boldsymbol{k})+M\}(\gamma\cdot\boldsymbol{k})\, u_1}{[(\boldsymbol{p}_2-\boldsymbol{k})^2-M^2][(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2]}, \tag{II – 5}$$

$$M_3^{(3)} = -\Lambda \int d^4p\big(\bar{u}_2\gamma_5(\gamma\cdot\boldsymbol{q})\, u_1\big)\frac{\mathrm{Spur}\,[\gamma_5(\gamma\cdot\boldsymbol{q})(\gamma\cdot\boldsymbol{p})(\gamma\cdot\boldsymbol{a}(\boldsymbol{q}))\{\gamma\cdot(\boldsymbol{p}+\boldsymbol{q})\}]}{[\boldsymbol{p}^2-M^2][(\boldsymbol{p}+\boldsymbol{q})^2-M^2][\boldsymbol{q}^2-\mu^2]}. \tag{II – 6}$$

The spur occurring in $M_3^{(3)}$ is equal to zero. Thus we have

$$M_3^{(3)} \equiv 0.$$

In the above integrals, the k_0 variable is to be integrated first with the prescription that M and μ in the denominators carry a small negative imaginary part which goes to zero after the integration.

Using the properties of Dirac equation and some simplifications, we may combine these matrix elements thus:

$$M_{\mathrm{I}} \equiv M_3^{(1)} = 2\Lambda\,(2M)^2 \int d^4k \frac{(\bar{u}_2\gamma\cdot\boldsymbol{k}\, u_1)\{\boldsymbol{a}(\boldsymbol{q})\cdot(2\boldsymbol{k}+q)\}}{[(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2][(\boldsymbol{k}+\boldsymbol{q})^2-\mu^2]} + $$
$$+ 2\Lambda \int d^4k \frac{(\bar{u}_2(\gamma\cdot\boldsymbol{k}+2M)\, u_1)(2\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{a}(\boldsymbol{q})}{[\boldsymbol{k}^2-\mu^2][(\boldsymbol{k}+\boldsymbol{q})^2-\mu^2]}, \tag{II – 7}$$

$$M_{\mathrm{II}} \equiv M_3^{(2)} + M_2^{(1)} + M_2^{(2)} =$$
$$= 2\Lambda\,(2M)^2 \int d^4k \frac{\bar{u}_2(\gamma\cdot\boldsymbol{k})(\gamma\cdot\boldsymbol{a}(\boldsymbol{q}))(\gamma\cdot\boldsymbol{k})\, u_1}{[(\boldsymbol{p}_2-\boldsymbol{k})^2-M^2][(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2]} -$$
$$-2\Lambda \int d^4k \frac{\bar{u}_2\gamma\cdot\boldsymbol{a}(\boldsymbol{q})\, u_1}{[\boldsymbol{k}^2-\mu^2]}. \tag{II – 8}$$

In these we shall put

$$\boldsymbol{q}\cdot\boldsymbol{a}(\boldsymbol{q}) = 0.$$

The first terms in M_{I} and M_{II} are exactly what one gets for the pseudoscalar coupling if one puts $(2MF/\mu) = G$. The second terms are the extra effect due to the pseudovector coupling. M_{I} is then the so-called meson-current contribution and M_{II} the nucleon-current contribution to the scattering, as is clear from the diagrams (3) and (4).

Further evaluations and simplifications are done as usual. An example is given below.

Thus in the first term of M_{II} we are interested in an integral of the form

$$\mathfrak{J} = \int d^4 k \frac{k_\mu k_\nu}{[\boldsymbol{k}^2 - 2\,\boldsymbol{p}_1 \cdot \boldsymbol{k} + \Delta_1]\,[\boldsymbol{k}^2 - 2\,\boldsymbol{p}_2 \cdot \boldsymbol{k} + \Delta_1]\,[\boldsymbol{k}^2 + \Delta_2]},$$

where we use

$$\boldsymbol{p}_1^2 - M^2 = 0 = \boldsymbol{p}_2^2 - M^2,$$

and so have

$$\Delta_1 = 0, \Delta_2 = -\mu^2.$$

To combine the denominators we use the Feynman formulae:

$$\frac{1}{a\,b} = \int_0^1 d\,x \frac{1}{[a\,x + b\,(1-x)]^2}$$

$$\frac{1}{a\,b\,c} = \int_0^1 d\,x \int_0^1 2\,y \cdot d\,y \frac{1}{[a\,x\,y + b\,y\,(1-x) + c\,(1-y)]^3}.$$

Putting:

$$a = \boldsymbol{k}^2 - 2\,\boldsymbol{p}_1 \cdot \boldsymbol{k} + \Delta_1, \quad b = \boldsymbol{k}^2 - 2\,\boldsymbol{p}_2 \cdot \boldsymbol{k} + \Delta_1, \quad c = \boldsymbol{k}^2 + \Delta_2$$

and

$$\boldsymbol{p}_x = \boldsymbol{p}_1\,x + \boldsymbol{p}_2\,(1-x),$$

we have

$$\mathfrak{J} = \int d^4 k \int_0^1 d\,x \int_0^1 2\,y\,d\,y \frac{k_\mu\,k_\nu}{[\boldsymbol{k}^2 - 2\,\boldsymbol{p}_x \cdot \boldsymbol{k}\,y + \Delta_1\,y + \Delta_2\,(1-y)]^3}.$$

To make the denominator an even function of the integration variable $\boldsymbol{k}$ we put

$$\boldsymbol{k} - \boldsymbol{p}_x\,y = \boldsymbol{k}'.$$

Note that to obtain the magnetic moments we go in the end to the case where the nucleon is at rest, i.e. $\vec{p}_1 = \vec{p}_2 = 0$. Thus the above transformation will not change the cut-off limits. But in the case of an actual nucleon scattering (as is the case in section III) we may have $\vec{p}_1 = 0$ but $\vec{p}_2$ is not equal to zero. Then the cut-off limits are to be transformed suitably, or, as is done by us, we go back to the original $\vec{k}$ from $\vec{k}'$ before applying the cut-off. Our integral is now:

$$\mathfrak{J} = \int d^4 k' \int_0^1 d\,x \int_0^1 2\,y\,d\,y \frac{k_\mu'\,k_\nu' + (p_x\,y)_\mu\,(p_x\,y)_\nu + k_\mu'\,(p_x\,y)_\nu + k_\nu'\,(p_x\,y)_\mu}{(\boldsymbol{k}'^2 + \Delta)^3},$$

where

$$\Delta = -\,\boldsymbol{p}_x^2\,y^2 + \Delta_1\,y + \Delta_2\,(1-y).$$

A great simplification is now achieved by using (according to FEYNMAN) the equations:

$$\int d^4k' \cdot (\text{an odd number of factors } k'_\mu)\, f(\boldsymbol{k}'^2) = 0\,, \qquad \text{(II} - 9)$$

$$\int d^4k'\, k'_\mu k'_\nu\, f(\boldsymbol{k}'^2) = \int d^4k'\, \frac{1}{4}\,(\boldsymbol{k}'^2 g_{\mu\nu})\, f(\boldsymbol{k}'^2)\,.$$

This last equation could also be written, for two four-vectors A_μ, B_μ, thus:

$$\int d^4k'\, (\boldsymbol{k}'\cdot\boldsymbol{A})\,(\boldsymbol{k}'\cdot\boldsymbol{B})\, f(\boldsymbol{k}'^2) = (\boldsymbol{A}\cdot\boldsymbol{B}) \int d^4k' \left(\frac{1}{4}\,\boldsymbol{k}'^2\right) f(\boldsymbol{k}'^2)\,. \qquad \text{(II} - 10)$$

The justification of this equation (which is important for us exactly for our divergent and ambiguous integrals) is obvious when the limits of integration are infinite. That it is justified also for our invariant cut-off follows in this manner. Since the above integral is to be relativistically invariant and since after the integrations we at most introduce only an invariant cut-off *constant*, the only form it can have is

$$\text{Inv. constant} \times (\boldsymbol{A}\cdot\boldsymbol{B}).$$

And this form clearly requires that

$$\int d^4k'\, k_0'^2\, f(\boldsymbol{k}'^2) = -\int d^4k'\, k_1'^2\, f(\boldsymbol{k}'^2) = -\int d^4k'\, k_2^2\, f(\boldsymbol{k}'^2) =$$
$$= -\int d^4k'\, k_3'^2\, f(\boldsymbol{k}'^2) = \frac{1}{4}\int d^4k'\, \boldsymbol{k}'^2\, f(\boldsymbol{k}'^2)\,.$$

It is true that when the integral is diverging, the above form does not necessarily follow from every explicit calculation (as is usually the case of the well-known ambiguous integrals of field theory); but then invariance and therefore the above from has to be demanded as an additional postulate. Thus our integral reduces to

$$\mathfrak{J} = \int d^4k' \int_0^1 dx \int_0^1 2\,y\,dy\, \frac{\frac{1}{4}\boldsymbol{k}'^2 g_{\mu\nu} + (p_x y)_\mu (p_x y)_\nu}{(\boldsymbol{k}'^2 + \Delta)^3}\,.$$

Using the Dirac equation and the properties of γ-matrices we obtain the matrix elements in the following form (we write $M_{\mathrm{I}}^{(1)}$, $M_{\mathrm{I}}^{(2)}$ for the first and second terms of M_{I} respectively):

$$M_{\mathrm{I}}^{(1)} = 2\Lambda\,(2\,M)^2\,\overline{u}_2 \left\{\tfrac{1}{2} M B_2 [(\gamma\cdot\boldsymbol{a}), (\gamma\cdot\boldsymbol{q})] + 2\,M^2 B_2\,(\gamma\cdot\boldsymbol{a}) + K_2(\gamma\cdot\boldsymbol{a})\right\} u_1,$$

$$M_{\mathrm{II}}^{(1)} = 2\Lambda\,(2\,M)^2\,\overline{u}_2 \{M\,(A + B_1)\,[(\gamma\cdot\boldsymbol{a}), (\gamma\cdot\boldsymbol{q})] + A\,\boldsymbol{q}^2\,(\gamma\cdot\boldsymbol{a}) +$$
$$+ (2\,M^2 A + 2\,M^2 B_1 - K_1)\,(\gamma\cdot\boldsymbol{a})\}\, u_1,$$

$$M_{\mathrm{I}}^{(2)} = 2\Lambda\,(2\,M)^2\,\overline{u}_2 \left\{\frac{K_3}{8\,M^2}\,(\gamma\cdot\boldsymbol{a})\right\} u_1,$$

$$M_{\mathrm{II}}^{(2)} = -\,2\Lambda\,(2\,M)^2\,\overline{u}_2 \left\{\frac{1}{4\,M^2}\,K_4\,(\gamma\cdot\boldsymbol{a})\right\} u_1. \qquad \text{(II} - 11)$$

We have put above simply $\boldsymbol{a}$ for $\boldsymbol{a}(\boldsymbol{q})$, and made the following substitutions:

$$\left.\begin{aligned}
A &= 2\int d^4k' \int_0^1 dx \int_0^1 dy \frac{y^3 x(1-x)}{(\boldsymbol{k}'^2+\Delta)^3}, \\
B_1 &= 2\int d^4k' \int_0^1 dx \int_0^1 dy \frac{y^3 x^2}{(\boldsymbol{k}'^2+\Delta)^3}, \\
K_1 &= \int_0^1 dx \int_0^1 dy \cdot y \int d^4k' \frac{\boldsymbol{k}'^2}{(\boldsymbol{k}'^2+\Delta)^3}, \\
B_2 &= 2\int d^4k'' \int_0^1 dx \int_0^1 dy \frac{y^3 x^2}{(\boldsymbol{k}''^2+D)^3}, \\
K_2 &= \int_0^1 dx \int_0^1 dy \cdot y \int d^4k'' \frac{\boldsymbol{k}''^2}{(\boldsymbol{k}''^2+D)^3}, \\
K_3 &= \int_0^1 dx \int d^4k''' \frac{\boldsymbol{k}'''^2}{(\boldsymbol{k}'''^2+\delta)^2}, \\
K_4 &= \int d^4k \frac{1}{\boldsymbol{k}^2-\mu^2};
\end{aligned}\right\} \qquad \text{(II — 12)}$$

$$\left.\begin{aligned}
\Delta &= -\mu^2(1-y) - M^2y^2 + y^2x(1-x)\,\boldsymbol{q}^2, \\
D &= -M^2x^2y^2 - \mu^2(1-xy) + y(1-y)(1-x)\,\boldsymbol{q}^2, \\
\delta &= -\mu^2 + \boldsymbol{q}^2 x(1-x);
\end{aligned}\right\} \qquad \text{(II — 13)}$$

$$\left.\begin{aligned}
\boldsymbol{k}' &= \boldsymbol{k} - \{\boldsymbol{p}_1 x + \boldsymbol{p}_2(1-x)\}\,y, \\
\boldsymbol{k}'' &= \boldsymbol{k} - \{\boldsymbol{p}_1 x - \boldsymbol{q}(1-x)\}\,y, \\
\boldsymbol{k}''' &= \boldsymbol{k} + \boldsymbol{q}x.
\end{aligned}\right\} \qquad \text{(II — 14)}$$

The total matrix element is thus given by

$$\begin{aligned}
M_s = M_{\mathrm{I}} + M_{\mathrm{II}} = 2\Lambda(2M)^2 \cdot \\
\cdot \bar{u}_2 \{P[(\boldsymbol{\gamma}\cdot\boldsymbol{a}), (\boldsymbol{\gamma}\cdot\boldsymbol{q})] + Q(\boldsymbol{\gamma}\cdot\boldsymbol{a}) + R\,\boldsymbol{q}^2(\boldsymbol{\gamma}\cdot\boldsymbol{a})\}\,u_1,
\end{aligned} \qquad \text{(II — 15)}$$

where we have

$$\left.\begin{aligned}
P &= M(A + B_1 + \tfrac{1}{2}B_2), \\
Q &= 2M^2A + 2M^2B_1 + 2M^2B_2 - K_1 + K_2 + \frac{1}{8M^2}K_3 - \frac{1}{4M^2}K_4 \\
R &= A.
\end{aligned}\right\} \qquad \text{(II — 16)}$$

P, Q, R are all integrals involving $\boldsymbol{q}^2$ as a parameter. To determine the magnetic moment we are interested in fields which vary slowly so that we may take $\boldsymbol{q}^2$ as very small and expand in powers of $\boldsymbol{q}^2$ and retain terms at most linear in $\boldsymbol{q}$.

Thus writing, e.g.,

$$A\,(0) = [A\,(\boldsymbol{q}^2)]_{\boldsymbol{q}^2=0}$$

we have

$$M_s = 2\Lambda(2\,M)^2\;\bar{u}_2\{Q(0)\;(\boldsymbol{\gamma}\cdot\boldsymbol{\alpha}) + P(0)\;[(\boldsymbol{\gamma}\cdot\boldsymbol{\alpha}),\,(\boldsymbol{\gamma}\cdot\boldsymbol{q})] + \cdots\}\,u_1\,.$$

Using

$$a^\nu(\boldsymbol{q}) = \frac{1}{(2\pi)^4}\int A^\nu(\boldsymbol{x})\,e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\,d^4x\;,$$

$$\sigma^{\mu\nu} = \frac{i}{2}\,(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\;,$$

$$\mathfrak{F}^{\mu\nu} = \frac{\partial A\mu}{\partial x_\nu} - \frac{\partial A\nu}{\partial x_\mu},\ \text{etc.}\,,$$

we have

$$M_s = 2\,\Lambda\,(2\,M)^2\frac{1}{(2\pi)^4}\int d^4x\,e^{i\boldsymbol{p}_2\cdot\boldsymbol{x}}$$

$$\bar{u}_2\{Q\,(0)\,\gamma_\mu A^\mu + P\,(0)\,\sigma_{\mu\nu}\,\mathfrak{F}^{\mu\nu} + \cdots\}\;u_1\,\bar{e}^{i\boldsymbol{p}_1\cdot\boldsymbol{x}}. \qquad \text{(II} - 17)$$

Ambiguities

We note first that $P(0)$ is a convergent expression. Care must however be taken in the treatment of $Q(0)$. In $Q(0)$, there occur differences of $K_1\,(0)$, $K_2\,(0)$, $K_3\,(0)$ and $K_4\,(0)$ which are all divergent integrals. Hence $Q(0)$ has an ambiguous value and we may obtain any value we like for it by suitably choosing the method of integration and transformation of variables. The only way of resolving the ambiguity lies in taking recourse to some physical arguments. We have one available in the form of the charge conservation of the nucleon. Since the neutron has to have a charge equal to zero (and the proton equal to e) we must demand that $Q(0)$, which as the coefficient of the $\gamma_\mu A^\mu$-term represents a charge of the neutron, has the value zero. In other words, we simply have to subtract out the $\gamma_\mu A^\mu$-terms independent of $\boldsymbol{q}$. Using this prescription we get

$$M_s = 2\,\Lambda\,(2\,M)^2\frac{1}{(2\,\pi)^4}\int d^4x\,e^{i\boldsymbol{p}_2\cdot\boldsymbol{x}}\;\bar{u}_2\,[P(0) + \sigma_{\mu\nu}\,\mathfrak{F}^{\mu\nu} + \cdots]\;u_1\,\bar{e}^{i\boldsymbol{p}_1\cdot\boldsymbol{x}}.$$

The next higher term (proportional to $\boldsymbol{q}^2$) which represents a contribution to the 'neutron-electron interaction' still diverges for pseudovector coupling but is convergent for pseudoscalar coupling. This divergence in

the case of pseudovector coupling is now to be handled with our cut-off and considered as an observable effect. The higher terms would be needed only for an actual scattering but not for the magnetic moment.

If we now specialise to the case where the external field is a static magnetic field $\vec{H}$, then

$$P(0)\,\sigma_{\mu\nu}\,\mathfrak{F}^{\mu\nu} \longrightarrow P(0)\,\vec{\sigma}\cdot\vec{H}.$$

Going over to the two-component Pauli-equation we are able then to interpret it as a magnetic moment effect. Thus the anomalous magnetic moment of the neutron is

$$\mu_N = -\frac{i\,e}{4\,\pi^4}\left(\frac{2\,MF}{\mu}\right)^2 P(0)\ .$$

Since $P(0)$ is the same for the two couplings, we get the same value for μ_N, provided we assume the equivalence of G and $(2\,MF/\mu)$. We shall often put for shortness

$$G = \frac{2\,MF}{\mu}\ .$$

$$\mu_N = -\frac{i\,e}{4\,\pi^4}\,G^2\,P(0)\ .$$

It is to be noticed that the neutron has been 'brought to rest' after splitting the $\sigma_{\mu\nu}\,\mathfrak{F}^{\mu\nu}$ term. Thus the cut-off to be applied to the integrals in $P(0)$ is simply the spherical cut-off to the radius K_0.

By an exactly similar treatment we get for the anomalous magnetic moment of the proton the value

$$\mu_P = -\frac{i\,e}{4\,\pi^4}\,G^2\,P_P(0)$$

where

$$P_P(0) = \tfrac{1}{2}\,M\left[A\,(0) + B_1(0) - B_2\,(0)\right]. \qquad \text{(II} - 18)$$

We may put our results (in units of $e/2\,M$) now in the following form:

$$\left.\begin{aligned} \mu_N &= -\frac{G^2}{4\,\pi^2}\,\chi_N\,, \\ \mu_P &= \frac{G^2}{4\,\pi^2}\,\chi_P\,, \end{aligned}\right\} \qquad \text{(II} - 19)$$

where

$$\chi_N = \chi_{\text{meson}} + \chi_{\text{nucleon}}\,,$$

$$\chi_P = \chi_{\text{meson}} - \frac{1}{2}\,\chi_{\text{nucleon}}\,.$$

(χ_{meson}) is the contribution from the meson current and (χ_{nucleon}) that due to the nucleon current. After the necessary integrations we have:

$$\chi_{\text{meson}} = \frac{1}{2}\frac{ie}{\pi^2} M B_2(0)$$

$$= \frac{K_0}{M}\frac{1}{\beta^2}\left[\beta^2 - \frac{\eta^4}{4} - \frac{3}{2}\eta^2\beta^2 + \frac{1}{8}\eta^6\right] \times$$

$$\times \left\{ \frac{1-\frac{\eta^2}{2}}{\sqrt{1+\frac{K_0^2}{M^2}}} + \frac{\frac{\eta^2}{2}}{\sqrt{\eta^2+\frac{K_0^2}{M^2}}} \right\} + \frac{K_0}{M}\left(\eta^2 + \beta^2 - \frac{3}{4}\eta^4\right) \times$$

$$\times \left\{ \frac{1}{\sqrt{1+\frac{K_0^2}{M^2}}} - \frac{1}{\sqrt{\eta^2+\frac{K_0^2}{M^2}}} \right\} + \left(\eta^2 - \frac{1}{2}\eta^4\right) \times$$

$$\times \log\left\{ \frac{\sqrt{1+\frac{K_0^2}{M^2}} - \frac{K_0}{M}}{\sqrt{\eta^2+\frac{K_0^2}{M^2}} - \frac{K_0}{M}} \cdot \frac{\sqrt{\eta^2+\frac{K_0^2}{M^2}} + \frac{K_0}{M}}{\sqrt{1+\frac{K_0^2}{M^2}} + \frac{K_0}{M}} \right\} -$$

$$- \frac{1}{2\alpha}\left[2\eta^2 - 4\eta^4 + \eta^6\right] \times$$

$$\times \cos^{-1}\left\{ \frac{\frac{\alpha^2}{\eta}\sqrt{1+\frac{K_0^2}{M^2}} \cdot \sqrt{\eta^2+\frac{K_0^2}{M^2}} - \frac{K_0^2}{M^2}\cdot\frac{\eta}{2}\left(1-\frac{\eta^2}{2}\right)}{\beta^2} \right\},$$

and

$$\chi_{\text{nucleon}} = \frac{ie}{\pi^2} M\left[A(0) + B_1(0)\right]$$

$$= \frac{K_0}{M}\frac{1}{\beta^2}\left[\frac{3}{2}\eta^2\beta^2 - \frac{1}{8}\eta^6\right] \times$$

$$\times \left\{ \frac{1-\frac{\eta^2}{2}}{\sqrt{1+\frac{K_0^2}{M^2}}} + \frac{\frac{\eta^2}{2}}{\sqrt{\eta^2+\frac{K_0^2}{M^2}}} \right\} - \frac{K_0}{M}\left(\beta^2 - \frac{3}{4}\eta^4\right) \times$$

$$\times \left\{ \frac{1}{\sqrt{1+\frac{K_0^2}{M^2}}} - \frac{1}{\sqrt{\eta+{}^2\frac{K_0^2}{M^2}}} \right\} - \frac{1}{2}(\eta^2 - \eta^4) \times$$

$$\times \log\left\{ \frac{\sqrt{1+\frac{K_0^2}{M^2}} - \frac{K_0}{M}}{\sqrt{\eta^2+\frac{K_0^2}{M^2}} - \frac{K_0}{M}} \cdot \frac{\sqrt{\eta^2+\frac{K_0^2}{M^2}} + \frac{K_0}{M}}{\sqrt{1+\frac{K_0^2}{M^2}} + \frac{K_0}{M}} \right\} - \frac{1}{2\alpha}(3\eta^4 - \eta^6) \times$$

$$\times \cos^{-1}\left\{ \frac{\frac{\alpha^2}{\eta}\sqrt{1+\frac{K_0^2}{M^2}}\sqrt{\eta^2+\frac{K_0^2}{M^2}} - \frac{K_0^2}{M^2}\frac{\eta}{2}\left(1-\frac{\eta^2}{2}\right)}{\beta^2} \right\}. \qquad \text{(II — 20)}$$

Here we have put

$$\left.\begin{aligned} \eta &= \frac{\mu}{M} \\ \alpha^2 &= \eta^2 - \frac{\eta^4}{4} \\ \beta^2 &= \alpha^2 + \frac{K_0^2}{M^2} \end{aligned}\right\} . \qquad \text{(II} - 21)$$

In the limit $K_0 = \infty$, the well-known LUTTINGER formulae are obtained.

Below we give a table for our numerical results. The value of η used in this table is $0 \cdot 15$.

K_0	χ_N	χ_P	χ_N/χ_P
M	0·53	0·13	4·1
$2/3\ M$	0·38	0·12	3·2
$1/2\ M$	0·278	0·104	2·67
$1/4\ M$	0·098	0·050	1·96

For $K_0 = \infty$, $\frac{\chi_N}{\chi_P} \simeq 8$. The experimental value for $\frac{\chi_N}{\chi_P}$ is about $1 \cdot 08$.

The ratio of the magnitude of the anomalous moment of the neutron to that of the proton drops from the value of about 8, obtained without any cut-off, to the value of about 3 with a cut-off of $2/3\ M$. The large value of (χ_N/χ_P) is due, as is well known, to the considerable contribution from the nucleon current. The cut-off suppresses it a good deal, thus improving the situation, but it does not suffice to reach complete agreement. This, of course, is impossible in a first order calculation. Phenomenological considerations of the type first made by FRÖHLICH and HEITLER*) using the concept of a probability of dissociation of a nucleon into nucleon plus *one* meson show already that with the observed value of μ/M the magnetic moments can never both be fitted accurately to the observed values. The coupling constant (also then the probability of dissociation) drops out in the ratio $|\mu_N/\mu_P|$, and the value for this ratio is in fact about $1 \cdot 4$ (for $\mu/M = 0 \cdot 15$). Value of the order $1 \cdot 4$ is obtained by us, for this ratio, as the lowest value in the limit $K_0 \to 0$. This shows that we need also the two-meson states for exact fitting.

We may thus expect that the inclusion of higher orders would lead to much more satisfactory results.

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*) H. FRÖHLICH and W. HEITLER, Nature *141*, 37 (1938).

Section III

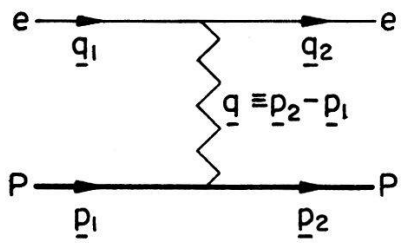
Electron-proton scattering

For this problem we replace the external electromagnetic field of the last section by a quantised electromagnetic field and besides we have to add to the interaction Hamiltonian, (I – 1), a term equal to

$$- e\, \overline{\psi}^{(e)}_{(\boldsymbol{x})}\, \gamma^{\mu\,(e)}\, \psi^{(e)}_{(\boldsymbol{x})}\, A_\mu(\boldsymbol{x})$$

describing the interaction of the quantised electron-field with the electromagnetic field, through which it interacts with the proton.

Then the lowest order matrix element for electron-proton scattering, proportional to e^2, is represented by the following Feynman diagram:



(1) M_2^P

The corresponding matrix element is

$$M_2^P = - ie^2 \frac{1}{(2\pi)^2} \sqrt{\frac{M^2}{E(p_1)\,E(p_2)}} \sqrt{\frac{m^2}{E_e(q_1)\,E_e(q_2)}}\, (\overline{u}_2 \gamma^\mu u_1)\, (\overline{\omega}_2 \gamma^{(e)}_\mu \omega_1) \frac{1}{\boldsymbol{q}^2}$$

$$\delta^{(4)}(\boldsymbol{q}_2 - \boldsymbol{q}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_1)\,, \qquad \text{(III – 1)}$$

where $\boldsymbol{q}_2, \boldsymbol{q}_1$ are the final and initial electron four-momenta and ω_2 and ω_1 the corresponding Dirac spinors. m is the electron mass, $E_e\,(q)$ is the electron energy corresponding to momentum $\vec{q}$. The rest of the notation is as in section II (the neutron there being of course here replaced by the proton). Putting

$$\chi = e^2 \frac{1}{(2\pi)^6} \sqrt{\frac{M^2}{E(p_1)\,E(p_2)}} \sqrt{\frac{m^2}{E_e(q_1)\,E_e(q_2)}} \frac{1}{\boldsymbol{q}^2}\, \delta^4(\boldsymbol{q}_2 - \boldsymbol{q}_1 + \boldsymbol{p}_2 - \boldsymbol{p}_1) \qquad \text{(III – 2)}$$

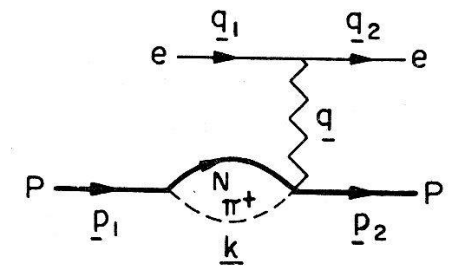
and

$$j_\mu \equiv (\overline{\omega}_2 \gamma^{(e)}_\mu \omega_1)\,, \qquad \text{(III – 3)}$$

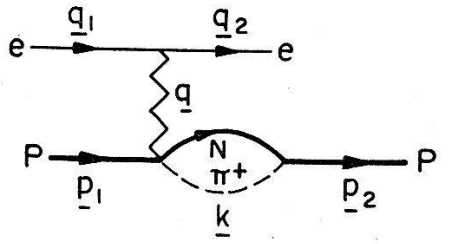
we have

$$M_2^P = - i\,(2\pi)^4\, \chi\, j_\mu\, (\overline{u}_2 \gamma^\mu u_1)\,. \qquad \text{(III – 4)}$$

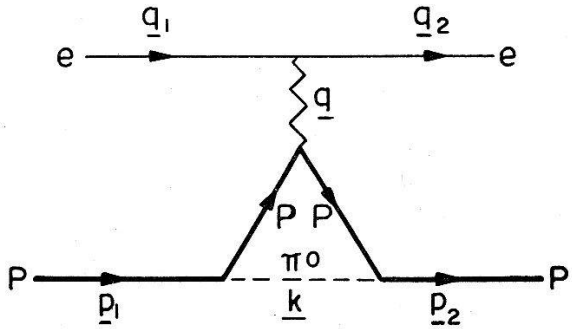
The next higher order matrix elements, proportional to e^2F^2, are represented now by the following Feynman diagrams:



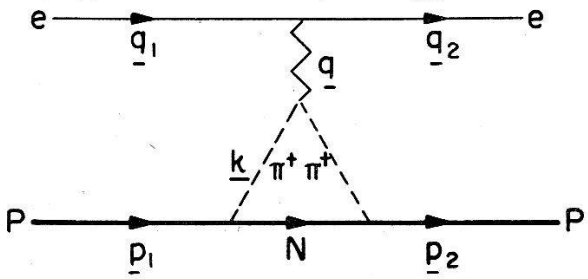
(2) $M_3^{P(1)}$



(3) $M_3^{P(2)}$

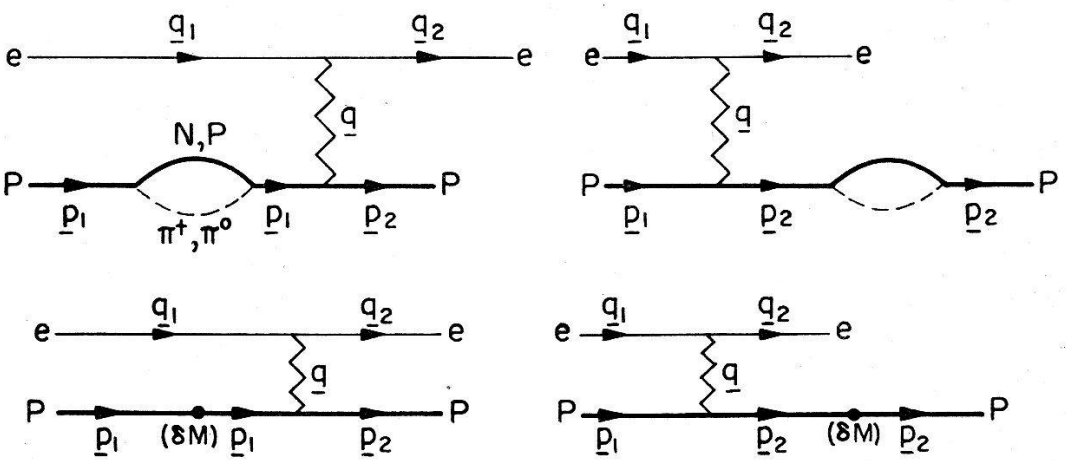


(4) $M_4^{P(1)}$



(5) $M_4^{P(2)}$

Besides these diagrams, we also have four more diagrams due to nucleon mass renormalisation and self-energy of the type shown below:



The matrix elements for all these diagrams have the form

$$M^P \sim \text{const.}\ \chi j^\nu M_\nu$$

where M_ν is essentially the contribution due to the proton-meson part of the diagram and χj^ν that due to the electron-photon part. The part of M_ν, which does not vanish for $\boldsymbol{q}=0$, is of the form $\bar{u}_2 \gamma_\nu u_1$ (as will be seen below). As explained under 'Ambiguities' in section II, this part (which is again ambiguous) must be subtracted out to maintain the proton charge at the value e–which value has already been taken into account in the lowest order diagram (1). Thus to obtain the result we are calculating we need not consider the above so-called renormalisation diagrams as the proton-meson parts of their matrix elements are independent of $\boldsymbol{q}$ and so will not contribute after this subtraction.

We give now the matrix elements corresponding to the diagrams (2) to (5). They are, respectively

$$M_3^{P(1)} = 2\left(\frac{F}{\mu}\right)^2 \chi j^\nu \int d^4k \frac{\bar{u}_2 \gamma_\nu \{\boldsymbol{\gamma}\cdot(-\boldsymbol{p}_1+\boldsymbol{k})+M\}(\boldsymbol{\gamma}\cdot\boldsymbol{k})\, u_1}{[(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2]},$$

$$M_3^{P(2)} = 2\left(\frac{F}{\mu}\right)^2 \chi j^\nu \int d^4k \frac{\bar{u}_2 (\boldsymbol{\gamma}\cdot\boldsymbol{k})\{\boldsymbol{\gamma}\cdot(-\boldsymbol{p}_2+\boldsymbol{k})+M\}\gamma_\nu u_1}{[(\boldsymbol{p}_2-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2]},$$

$$M_4^{P(1)} = \left(\frac{F}{\mu}\right)^2 \chi j^\nu \int d^4k \frac{\bar{u}_2(\boldsymbol{\gamma}\cdot\boldsymbol{k})\{\boldsymbol{\gamma}\cdot(-\boldsymbol{p}_2+\boldsymbol{k})+M\}\gamma_\nu\{\boldsymbol{\gamma}\cdot(-\boldsymbol{p}_1+\boldsymbol{k})+M\}(\boldsymbol{\gamma}\cdot\boldsymbol{k})\, u_1}{[(\boldsymbol{p}_2-\boldsymbol{k})^2-M^2][(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2]},$$

$$M_4^{P(2)} = -2\left(\frac{F}{\mu}\right)^2 \chi j^\nu \int \frac{\bar{u}_2\{\boldsymbol{\gamma}\cdot(\boldsymbol{k}+\boldsymbol{q})\}\{\boldsymbol{\gamma}\cdot(-\boldsymbol{p}_1+\boldsymbol{k})+M\}(\boldsymbol{\gamma}\cdot\boldsymbol{k})\, u_1(2k_\nu+q_\nu)}{[(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2][(\boldsymbol{k}+\boldsymbol{q})^2-\mu^2]}. \tag{III–5}$$

Excepting for the factors outside the integrals and the replacement of $\boldsymbol{a}(\boldsymbol{q})$ by $\boldsymbol{j}$, these are exactly of the same form as the matrix elements we have already discussed in section II. Thus the further reductions and simplifications are easily read off here from the results of section II. Just like there, here also we may combine these matrix elements to the form:

$$\left.\begin{aligned}
M_{\mathrm{I}}^P &= \left(\frac{2MF}{\mu}\right)^2 \chi j_\nu \int d^4k \frac{\bar{u}_2(\boldsymbol{\gamma}\cdot\boldsymbol{k})\,\gamma^\nu(\boldsymbol{\gamma}\cdot\boldsymbol{k})\, u_1}{[(\boldsymbol{p}_2-\boldsymbol{k})^2-M^2][(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2]},\\
M_{\mathrm{II}}^P &= -2\left(\frac{2MF}{\mu}\right)^2 \chi j_\nu \int d^4k \frac{\bar{u}_2(\boldsymbol{\gamma}\cdot\boldsymbol{k})\, u_1(2k^\nu+q^\nu)}{[(\boldsymbol{p}_1-\boldsymbol{k})^2-M^2][\boldsymbol{k}^2-\mu^2][(\boldsymbol{k}+\boldsymbol{q})^2-\mu^2]},\\
M_{\mathrm{III}}^P &= -2\left(\frac{F}{\mu}\right)^2 \chi j_\nu \int d^4k \frac{\bar{u}_2[(\boldsymbol{\gamma}\cdot\boldsymbol{k})+2M]\, u_1(2k^\nu+q^\nu)}{(\boldsymbol{k}^2-\mu^2)[(\boldsymbol{k}+\boldsymbol{q})^2-\mu^2]},\\
M_{\mathrm{IV}}^P &= \frac{3}{2}[M_3^{P(1)}+M_3^{P(2)}] - \left(\frac{F}{\mu}\right)^2 \chi j_\nu \int d^4k \frac{\bar{u}_2\gamma^\nu u_1}{[\boldsymbol{k}^2-\mu^2]}.
\end{aligned}\right\} \tag{III–6}$$

Here again M_{I}^P, M_{II}^P are equal to the matrix elements one obtains from pseudoscalar coupling provided one writes G for $(2MF/\mu)$. M_{III}^P,

M^P_{IV} arise as extra terms due to pseudovector coupling. Due to these the electron-proton scattering is different for the two couplings.

Corresponding to the equation

$$\boldsymbol{q}\cdot\boldsymbol{a}(\boldsymbol{q})=0\,,$$

of section II, we have here

$$\chi\,(\boldsymbol{q}\cdot\boldsymbol{j})=0\,.$$

As already mentioned we have to subtract from the proton-meson part of $M^P_{\mathrm{I}\to\mathrm{IV}}$ the same with $\boldsymbol{q}^2\equiv 0$. This is the statement of our 'charge conservation' for the proton. This removes, as before, the ambiguities. To represent this subtraction we shall use the notation

$$A'(\boldsymbol{q}^2)\equiv A\,(\boldsymbol{q}^2)-A\,(\boldsymbol{q}^2=0)\,. \qquad \text{(III}-7)$$

Using the same methods as used in section II for further simplifying the matrix elements we get

$$\left.\begin{aligned}
M^P_{\mathrm{I}} &= \left(\frac{2\,MF}{\mu}\right)^2 \chi\, j_\lambda\, \bar{u}_2\{(2\,M^2A+2\,M^2B_1-K_1)\,\gamma^\lambda+A\,\boldsymbol{q}^2\gamma^\lambda+\\
&\quad + M\,(A+B_1)\,[\gamma^\lambda,(\boldsymbol{\gamma}\cdot\boldsymbol{q})]\}\,u_1\,,\\
M^P_{\mathrm{II}} &= -\,2\left(\frac{2\,MF}{\mu}\right)^2 \chi\, j_\lambda\, \bar{u}_2\left\{\frac{1}{2}\,MB_2[\gamma^\lambda,(\boldsymbol{\gamma}\cdot\boldsymbol{q})]+2\,M^2B_2\gamma^\lambda+\right.\\
&\quad \left.+\,K_2\gamma^\lambda\right\}u_1\,,\\
M^P_{\mathrm{III}} &= -\left(\frac{F}{\mu}\right)^2 \chi\, j_\lambda\, \bar{u}_2\{k_3\gamma^\lambda\}\,u_1\,,\\
M^P_{\mathrm{IV}} &= \left\{3\left(\frac{2\,MF}{\mu}\right)^2\int d^4k\int_0^1 d\,x\,\frac{x}{[\boldsymbol{k}^2-M^2x^2-\mu^2(1-x)]^2}+\right.\\
&\quad \left.+\,5\left(\frac{F}{\mu}\right)^2\int d^4k\,\frac{1}{\boldsymbol{k}^2-\mu^2}\right\}\chi\, j_\lambda\, \bar{u}_2\gamma^\lambda u_1\,.
\end{aligned}\right\} \qquad \text{(III}-8)$$

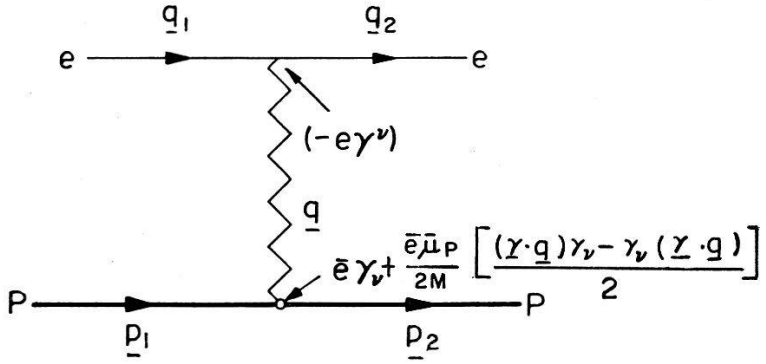
Here the various quantities A, B_1, B_2 etc. are the same as in section (II), equ. (12), (13), (14). Subtracting out the 'ambiguities' we have, for example,

$$M^{P'}_{\mathrm{IV}}=0\,.$$

The total effective matrix element is then

$$\begin{aligned}
M^P &= M_2+M^{P'}_{\mathrm{I}}+M^{P'}_{\mathrm{II}}+M^{P'}_{\mathrm{III}}+M^{P'}_{\mathrm{IV}}=\\
&\quad -\,i\,(2\pi)^4\frac{1}{e}\,\chi\, j^\mu\left(\bar{u}_2\left\{\bar{e}\,\gamma_\mu+\frac{e\,\bar{\mu}_P}{2\,M}\left[\frac{(\boldsymbol{\gamma}\cdot\boldsymbol{q})\,\gamma_\mu-\gamma_\mu\,(\boldsymbol{\gamma}\cdot\boldsymbol{q})}{2}\right]\right\}u_1\right). \qquad \text{(III}-9)
\end{aligned}$$

Schematically we could represent this by the following diagram (the proton-vertex operator being the above effective operator):



The values of $\bar{e}$ and $\bar{\mu}_P$ are:

$$\bar{e} = e\left[1 + \frac{i}{(2\pi)^4}\left(\frac{2\,MF}{\mu}\right)^2\left\{A\,\boldsymbol{q}^2 + 2\,M^2A' + 2\,M^2B_1' - K_1 - \right.\right.$$
$$\left.\left. - 4\,M^2B_2' - 2\,K_2' - \frac{1}{4\,M^2}\,K_3'\right\}\right],$$
$$\bar{e}\,\bar{\mu}_P = -\frac{i\,e}{(2\,\pi)^4}\,.\,4\,M^2\left(\frac{2\,MF}{\mu}\right)^2(A + B_1 - B_2)\,. \qquad \text{(III} - 10)$$

$\bar{e}$ is the effective proton-charge due to the mesonic effects and $\bar{e}\,\bar{\mu}_P$ represents essentially the effective anomalous magnetic-moment distribution. We introduce for convenience the following notation:

$$F_1 = \frac{\bar{e}}{e} = 1 + \frac{i}{(2\,\pi)^4}\left(\frac{2\,MF}{\mu}\right)^2\left[A\,\boldsymbol{q}^2 + 2\,M^2A' + 2\,M^2B_1' - 4\,M^2B_2' - \right.$$
$$\left. - K_1' - 2\,K_2' - \frac{1}{4\,M^2}\,K_3'\right],$$
$$F_2 = \frac{\bar{e}\,\bar{\mu}_P}{e\,\mu_P} = \frac{A + B_1 - B_2}{A\,(0) + B_1\,(0) - B_2\,(0)}, \qquad \text{(III} - 11)$$

where μ_P is our theoretical anomalous magnetic moment of the proton. F_1 can be interpreted as the charge form factor and F_2 as the magnetic moment form factor. They are simply functions of $\boldsymbol{q}^2$ and naturally of the cut-off value K_0. F_2 is the same for both the couplings but F_1 is different in so far as K_3' does not appear in (III – 11) for the case of pseudoscalar coupling.

The differential cross-section for the scattering, from the matrix element (III – 9), has been derived by ROSENBLUTH[3]). It can be written in terms of our F_1 and F_2 in the form

$$\sigma(\theta) = \left(\frac{e^2}{2\,E}\right)^2 \frac{\cot^2\frac{\theta}{2}\,\mathrm{cosec}^2\frac{\theta}{2}}{1 + \frac{2\,E}{M}\sin^2\frac{\theta}{2}}\left[F_1{}^2 - \frac{1}{4}\,\frac{\boldsymbol{q}^2}{M^2}\left\{2\,(F_1 + \mu_P F_2)^2\tan^2\frac{\theta}{2} + \right.\right.$$
$$\left.\left. + \mu_P^2\,F_2^2\right\}\right]. \qquad \text{(III} - 12)$$

In this formula, θ is the angle of scattering, and E the energy of the incident electrons, in the laboratory system, where the proton is initially at rest ($\vec{p}_1 = 0$). The rest-energy of the electron has been neglected here in comparison with its kinetic energy. In terms of E and θ we have

$$\boldsymbol{q}^2 = \frac{-4\,E^2 \sin^2 \frac{\theta}{2}}{1 + \frac{2\,E}{M} \sin^2 \frac{\theta}{2}} \,. \qquad \text{(III} - 13)$$

Since $\vec{p}_1 = 0$, we have

$$\frac{\vec{p}^{\,2}}{M^2} \equiv \frac{\vec{p}_2^{\,2}}{M^2} = \varepsilon^2 \left(1 + \frac{\varepsilon^2}{4}\right),$$

where

$$\varepsilon^2 \equiv -\frac{\boldsymbol{q}^2}{M^2} \,.$$

It should be noted that the ROSENBLUTH formula, (III – 12), was also used by HOFSTÄDTER *et al.*[2]) in their phenomenological interpretation of their experimental data. They then took for μ_P the experimental value and chose for F_1 and F_2 various analytical functions. Here, of course, all these three quantities are derived from meson theory.

Evaluation of F_1, F_2

To evaluate F_1 and F_2, equ. (III – 11), we simply have to evaluate the various integrals occurring there with our cut-off. First we write F_1 and F_2 in the following form

$$F_2 = \frac{\mathfrak{J}_1(\varepsilon^2) + \mathfrak{J}_2(\varepsilon^2) - \mathfrak{J}_3(\varepsilon^2)}{\mathfrak{J}_1(0) + \mathfrak{J}_2(0) - \mathfrak{J}_3(0)} \text{ , for both couplings ;} \qquad \text{(III} - 14)$$

$$F_1 = 1 - \frac{G^2}{16\,\pi^2}\{f_{PS}\} \text{ , for pseudoscalar coupling ;} \qquad \text{(III} - 15)$$

$$F_1 = 1 - \frac{G^2}{16\,\pi^2}\{f_{PV}\} \text{ , } \left(G \equiv \frac{2\,MF}{\mu}\right), \text{ for pseudovector coupling;} \qquad \text{(III} - 16)$$

where

$$f_{PS} = \frac{1}{2}\,\varepsilon^2 \mathfrak{J}_1 - \mathfrak{J}_1' - \mathfrak{J}_2' - 2\,\mathfrak{J}_3' + \mathfrak{J}_7 + 2\,\mathfrak{J}_8 \,, \qquad \text{(III} - 17)$$

$$f_{PV} = f_{PS} + \frac{1}{4}\,\mathfrak{J}_9 \,. \qquad \text{(III} - 18)$$

In these equations, $\mathfrak{J}_1, \mathfrak{J}_2, \ldots, \mathfrak{J}_9$ are the integrals to be evaluated, defined by:

$$\left.\begin{aligned} A &= -\frac{\pi^2 i}{2\,M^2}\mathfrak{J}_1 ;\; B_1 = -\frac{\pi^2 i}{2\,M^2}\mathfrak{J}_2 ;\; B_2 = -\frac{\pi^2 i}{2\,M^2}\mathfrak{J}_3 ; \\ A(0) &= -\frac{\pi^2 i}{2\,M^2}\mathfrak{J}_4 ;\; B_1(0) = -\frac{\pi^2 i}{2\,M^2}\mathfrak{J}_5 ;\; B_2(0) = -\frac{\pi^2 i}{2\,M^2}\mathfrak{J}_6 ; \\ K_1' &= -\pi^2 i\,\mathfrak{J}_7 ;\; K_2' = -\pi^2 i\,\mathfrak{J}_8 ;\; K_3' = -\pi^2 i\,M^2\,\mathfrak{J}_9 . \end{aligned}\right\} \qquad \text{(III} - 19)$$

In terms of these integrals, we have also

$$\mu_P = -\frac{G^2}{8\pi^2}\{\mathfrak{J}_1 + \mathfrak{J}_2 - \mathfrak{J}_3\}_{(\varepsilon^2=0)}. \qquad \text{(III} - 20)$$

As an example, we indicate below the evaluation of A. From (II – 12, 13, 14) we have

$$A = 2\int d^4k' \int_0^1 dx \int_0^1 dy \frac{y^3 x(1-x)}{(\boldsymbol{k}'^2 + \Delta)^3},$$

where

$$\boldsymbol{k}' = \boldsymbol{k} - \{\boldsymbol{p}_1 x + \boldsymbol{p}_2 (1-x) y\},$$

and

$$\Delta = \Delta(\varepsilon^2) = -M^2\{y^2 + \eta^2(1-y) + y^2 x(1-x)\varepsilon^2\}.$$

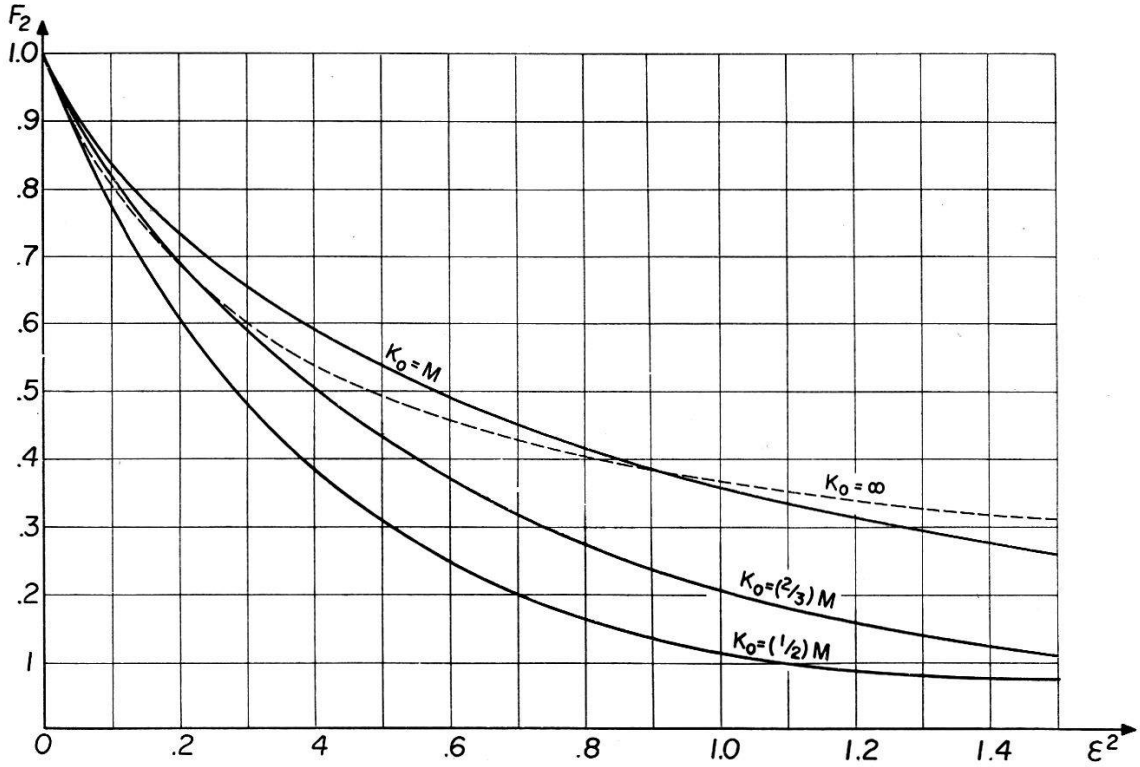


Fig. 1

The magnetic moment from factor F_2 plotted as a function of $\varepsilon^2 (= -\boldsymbol{q}^2/M^2)$ for the cut-off values $K_0 = 1/2\ M,\ 2/3\ M,\ M,\ \infty$.

On performing the integration over k_0' (which is equivalent to k_0 since the limits are infinite) by the usual method, we get

$$A = -\frac{3\pi i}{4}\int_0^1 dx \int_0^1 dy \int d^3\vec{k}' \frac{y^3 x(1-x)}{(\vec{k}'^2 - \Delta)^{5/2}}.$$

Now we have to remember that we are working in the laboratory system where $\vec{p}_1 = 0$, and hence $\vec{k}$ is to be cut-off symmetrical to K_0; but here

$$\vec{k}' = \vec{k} - \vec{p}_2 y(1-x) \equiv \vec{k} - \vec{p}\, y(1-x)$$

so we transform the above integral back to the old $\vec{k}$ variable thus:

$$\int d^3\vec{k}' \int \frac{1}{(\vec{k}'^2-\Delta)^{5/2}} = \int d^3\vec{k} \frac{1}{[\vec{k}^2 - 2(\vec{k}\cdot\vec{p})\,y(1-x) + \vec{p}^2\,y^2(1-x)^2 - \Delta]^{5/2}}\,.$$

The integration over the angles (taking $\vec{p}$ as the axis) is easily performed and it gives:

$$\int d^3\vec{k}' \frac{1}{(\vec{k}'^2-\Delta)^{5/2}} = \frac{4\pi}{3}\int_0^{K_0} k^2 d\,k \frac{1}{2\,p\,k\,y\,(1-x)} \left[\frac{1}{\{[k-p\,y\,(1-x)]^2-\Delta\}^{3/2}} - \right.$$
$$\left. - \frac{1}{\{[k+p\,y\,(1-x)]^2-\Delta\}^{3/2}}\right].$$

The integration over k is performed again very easily and we get finally

$$A = -\frac{\pi^2\,i}{2\,M^2}\,\mathfrak{J}_1$$

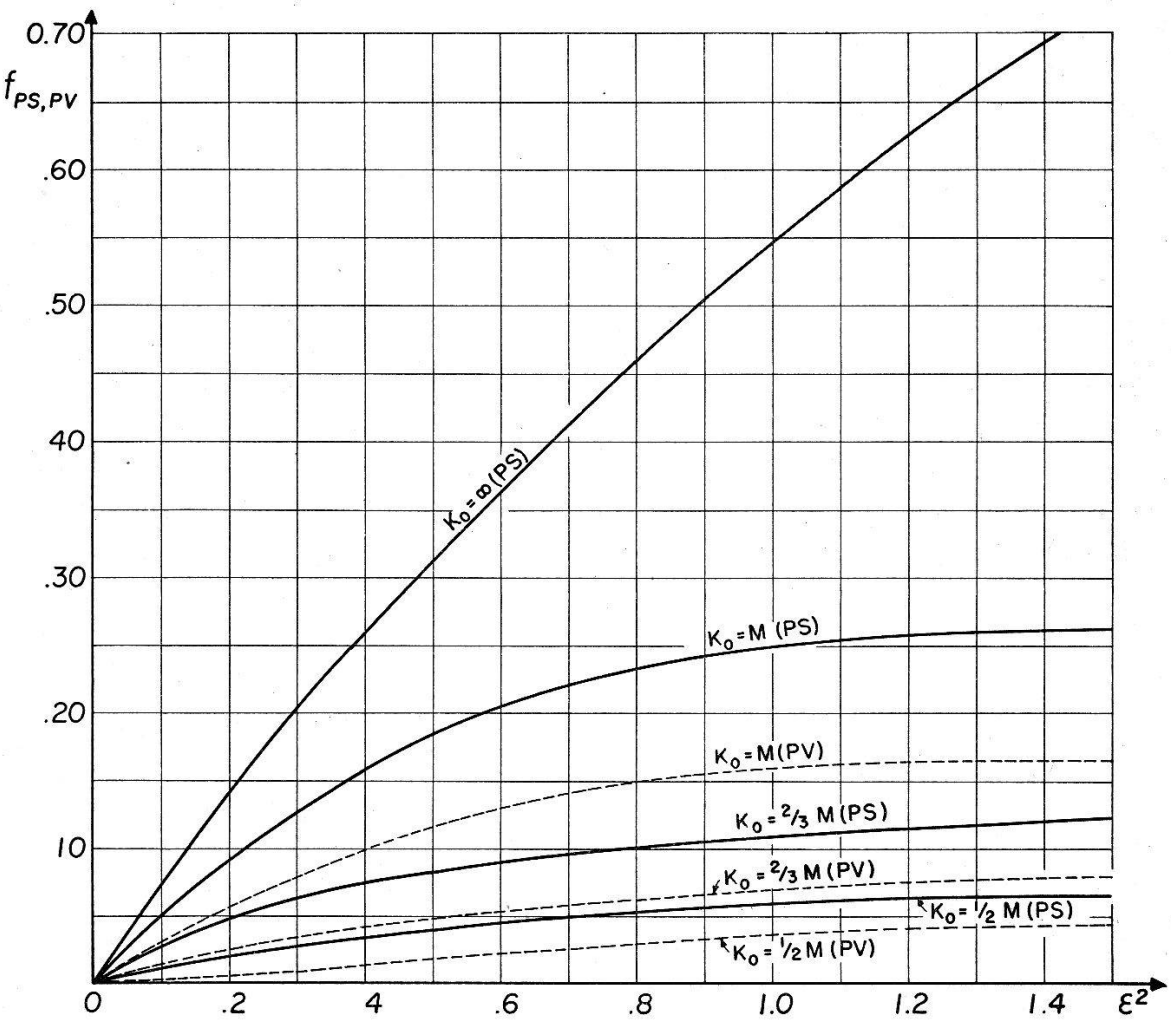


Fig. 2

f_{PS} (continuous curves), f_{PV} (dotted curves) plotted as functions of ε^2 $(=-\boldsymbol{q}^2/M^2)$ for the cut-off values $K_0 = 1/2\,M,\ 2/3\,M,\ M,\ \infty$. f_{PS}, f_{PV} determine, according to equations (III–15, 16), the charge form factor, F_1, for the pseudoscalar and pseudovector couplings respectively.

*

where

$$\mathfrak{J}_1 = \int_0^1 d\,x \int_0^1 d\,y\; f_1(x, y\,; K_0, \varepsilon^2)\ ,$$

with

$$f_1(x, y\,; K_0, \varepsilon^2) = \frac{y^2\,x}{(p/M)} \left\{ \frac{1}{\sqrt{\left[\frac{K_0}{M} + \frac{p}{M}\,y\,(1-x)\right]^2 + y^2 + \eta^2(1-y) + y^2 x\,(1-x)\,\varepsilon^2}} - \right.$$

$$\left. - \frac{1}{\sqrt{\left[\frac{K_0}{M} - \frac{p}{M}\,y\,(1-x)\right]^2 + y^2 + \eta^2\,(1-y) + y^2 x\,(1-x)\,\varepsilon^2}} \right\} +$$

$$+ \frac{y^3\,x\,(1-x)}{[y^2 + \eta^2\,(1-y) + y^2\,x(1-x)\,\varepsilon^2]} \times$$

$$\times \left\{ \frac{\frac{K_0}{M} + \frac{p}{M}\,y\,(1-x)}{\sqrt{\left[\frac{K_0}{M} + \frac{p}{M}\,y\,(1-x)\right]^2 + y^2 + \eta^2\,(1-y) + y^2 x\,(1-x)\,\varepsilon^2}} + \right.$$

$$\left. + \frac{\frac{K_0}{M} - \frac{p}{M}\,y\,(1-x)}{\sqrt{\left[\frac{K_0}{M} - \frac{p}{M}\,y\,(1-x)\right]^2 + y^2 + \eta^2\,(1-y) + y^2 x\,(1-x)\,\varepsilon^2}} \right\}.$$

Now the integrations over the variables x and y remain. These are extremely complicated and so we have performed these numerically for some selected values for the parameter K_0 ($= M,\ 2/3\,M,\ 1/2\,M,\ \infty$) and the parameter ε^2 ($= 0{\cdot}05,\ 0{\cdot}1,\ 0{\cdot}3,\ 0{\cdot}5,\ 0{\cdot}7,\ 1{\cdot}0,\ 1{\cdot}5$). The same has been done for the other integrals, B_1, B_2, K'_1, etc.

The results of this evaluation are shown in the figures (1), (2), where we have plotted F_2, f_{PS}, f_{PV} as functions of the parameter ε^2, for different values of K_0. From these graphs we may read off the values of F_2, f_{PS}, f_{PV} for any value of ε^2 (up to 1·5) as need be.

For K_0 not too large, the difference between f_{PS} and f_{PV} is not large. This leads even to a smaller difference in cross-sections, since F_2 is the same for both the couplings. But when $K_0 \to \infty$, $f_{PV} \to \infty$.

Results – Cross-sections

Now the cross-section as a function of the laboratory angle of scattering, for various values of the initial electron energy, is easily calculated. The results are shown graphically in figures (3) to (6). These scattering curves have been plotted for the different values of K_0 and for differ-

ent choices of the coupling constant. The experimental points shown for comparison, have been taken from the Stanford dissertation of CHAMBERS[2]). We give besides also theoretical curves for $E = 1000$ Mev, at which energy the experiments have not yet been performed.

Discussion of the results Conclusions

The dependence of the curves on the value of K_0, for fixed values of the energy and coupling constant, is easily understood physically. Let

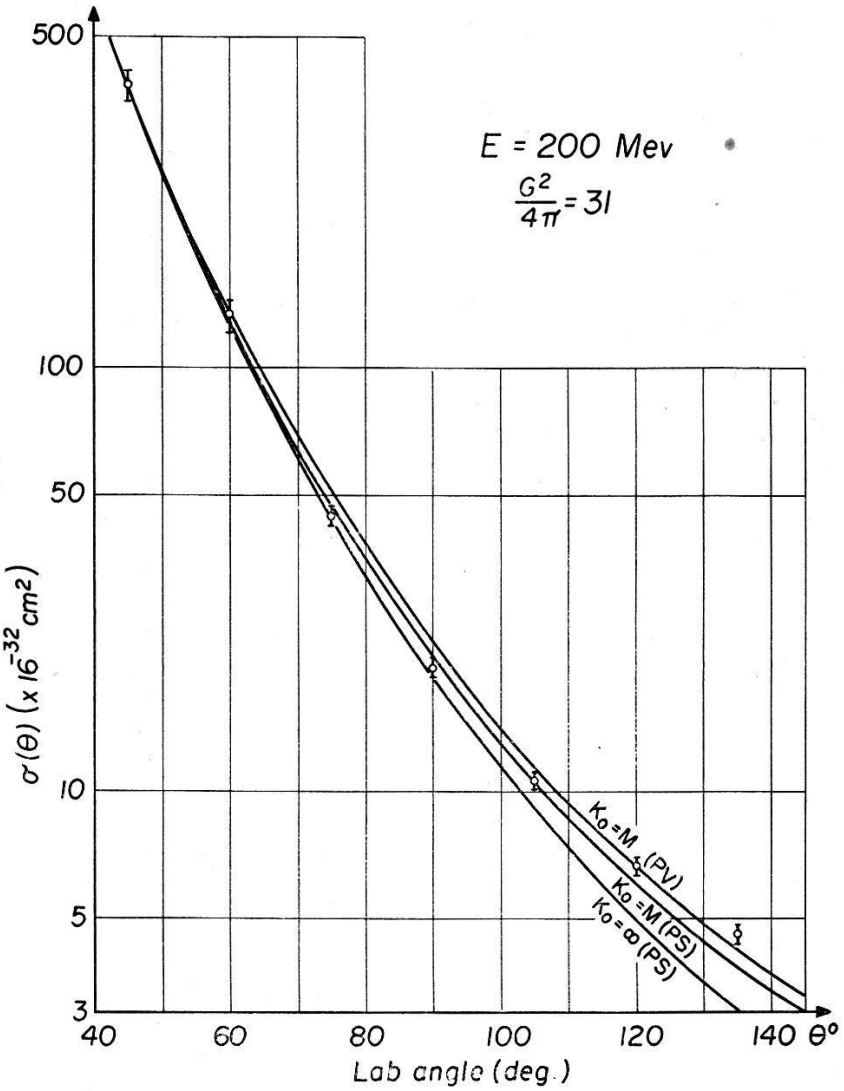


Fig. 3

The differential cross-section for the scattering of 200 Mev electrons plotted as a function of the angle of scattering in the laboratory system. The theoretical curves are drawn for the cut-off value $K_0 = M$ for both pseudoscalar coupling (*ps*) and pseudovector coupling (*pv*). The curve drawn for $K_0 = \infty$ is for pseudoscalar coupling (*ps*); pseudovector coupling gives a divergent result for $K_0 = \infty$. The coupling constant is $G^2/4\pi = 31$. The experimental points shown for comparison are due to CHAMBERS (cf. [2])).

us, for the sake of visualizing the process, use the non-relativistic terminology of the meson-cloud round the fixed heavy nucleon-core. The charge is distributed between the point core (actually a Dirac particle) and the cloud. Then the cut-off K_0 has the effect that, as it decreases from infinity towards the value zero, the meson-cloud extends more and

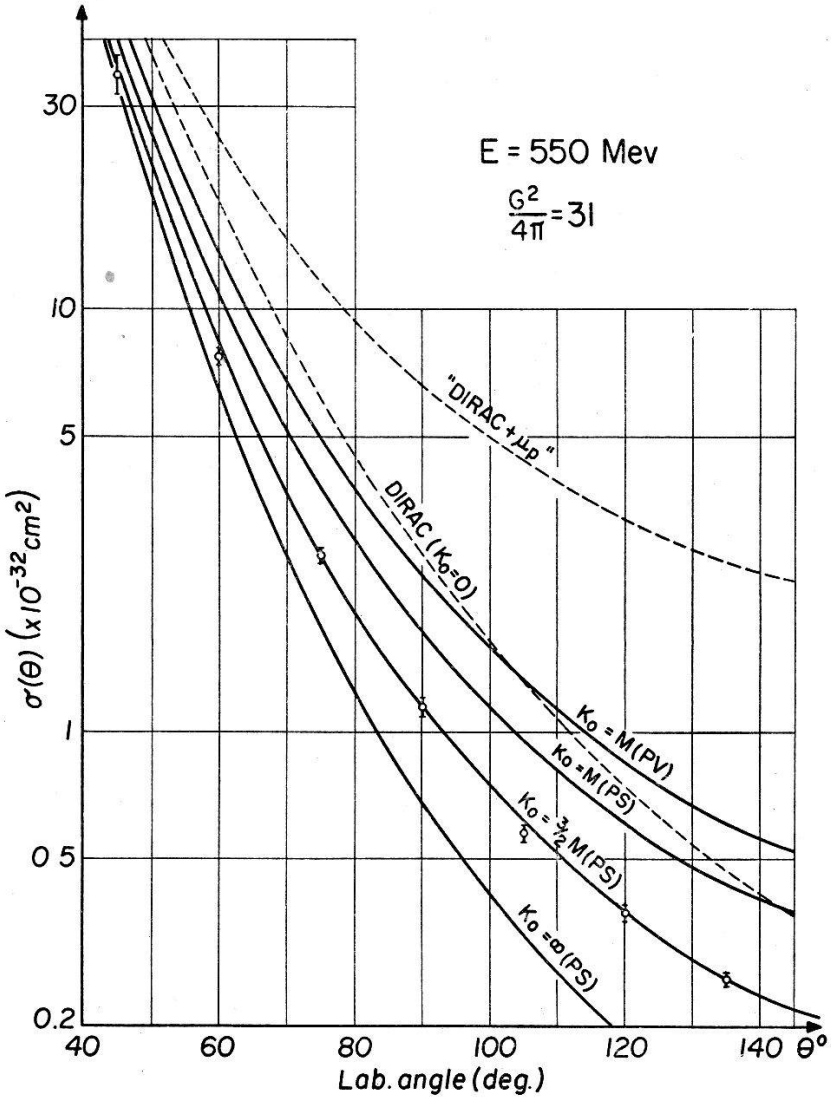


Fig. 4

The differential cross-section for the scattering of 550 Mev electrons plotted as a function of the angle of scattering in the laboratory system. The continous theoretical curves are drawn for the various values of the cut-off constant K_0 indicated. The coupling used is indicated on the curves (*ps* stands for pseudoscalar and *pv* for pseudovector coupling). The coupling constant is $G^2/4\,\pi = 31$. The lower dotted curve indexed 'Dirac' ($K_0 = 0$), is the scattering curve corresponding to the case when the proton is considered as a point Dirac-particle without any mesonic effects. The upper dotted curve marked, 'Dirac$+\mu_P$' represents the scattering by a point Dirac proton along with a point anomalous magnetic moment of 1·78 nuclear magnetons (described by a Pauli term). The experimental points shown for comparison are due to CHAMBERS (cf. [2])).

more, but the total charge on it becomes less and less till the whole of the charge is on the proton core. Similarly, the anomalous magnetic moment of the proton becomes more extended as K_0 decreases, but its total value vanishes for $K_0 = 0$. Thus, for $K_0 = 0$, we have the proton simply as a point Dirac-particle. Thus, as K_0 decreases the curve moves up to be identified, ultimately, with the curve corresponding to the proton as a Dirac particle (the dotted curve indexed–'Dirac'). This is in contrast to the phenomenological treatment of HOFSTÄDTER *et al.*[2]), where a larger radius (in our case smaller K_0) moves the curves down; but, there the total charge is kept fixed on the extended structure. The curve for the case where the proton is considered as a point Dirac particle with a point anomalous magnetic moment of 1·78 nuclear

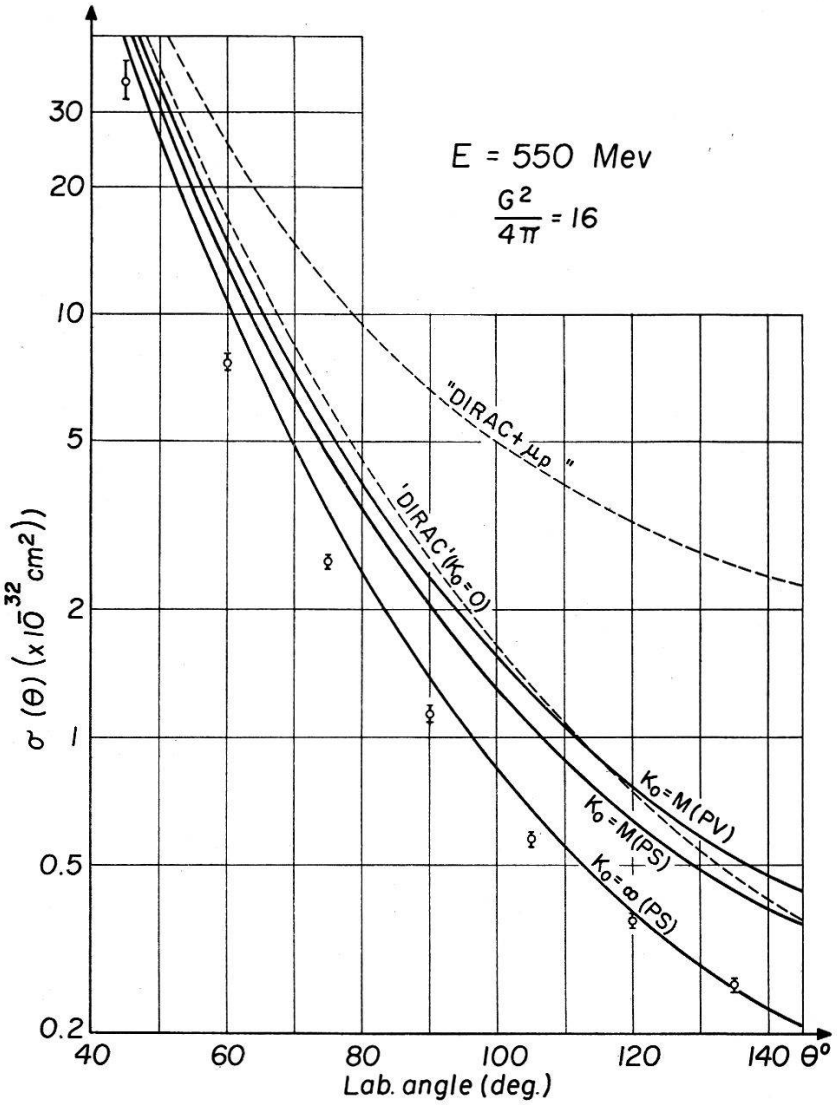


Fig. 5

The differential cross-section for the scattering of 550 Mev electrons plotted as a function of the angle of scattering in the laboratory system. For the indexing and notations see the legend to Fig. 4. The coupling constant used here is $G^2/4\pi = 16$.

magnetons (the dotted curve indexed–'Dirac + μ_P') lies much above all the other curves shown.

Similarly, for a fixed value of K_0 and the electron energy, the variation of the curve with the value of the coupling constant is such that a larger value of the coupling constant implies a more extended structure for the proton as a whole. This is quite clear for when the coupling constant is zero, we have no charge on the meson cloud and the anomalous moment is zero. Both these increase as we increase the value of the coupling constant.

The agreement with experiments is nearly perfect at all energies for the values $G^2/4\pi = 31$ and $K_0 = 3/2\ M$*). For pseudoscalar coupling the

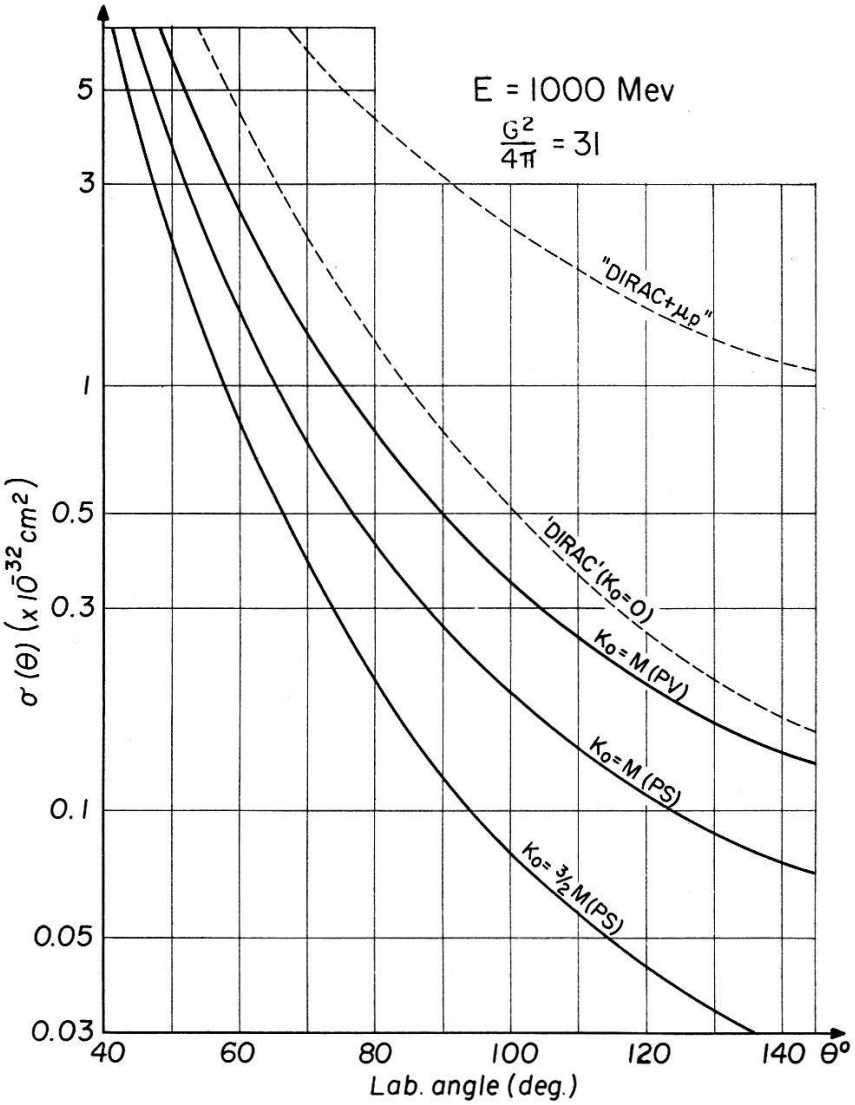


Fig. 6

The differential cross-section for the scattering of 1000 MeV electrons plotted as a function of the angle of scattering in the laboratory system. For the indexing and notations see the legend to Fig. 4. The coupling constant is here given by $G^2/4\pi = 31$. No experimental data is available at this energy so far.

*) The curve for the value $K_0 = 3/2\ M$ has been drawn by interpolation.

choice $K_0 = \infty$ and $G^2/4\pi$ about 16 also gives good agreement. Thus there is a one-parameter choice of the combinations G, K_0 which gives nice agreement for both the couplings (this includes for pseudoscalar coupling also the value $K_0 = \infty$). The constants, for example, $G^2/4\pi = 31$ and $K_0 = 3/2\,M$ agree within a factor 2 with the values derived from other phenomena like pion-nucleon scattering, mass difference of π^0 and π^+. Exact agreement can, of course, not be expected since firstly, our calculations are only to the lowest order in perturbation and secondly, the cut-off theory is nothing more than a crude substitute for the future correct theory. In any case we may conclude that meson theory accounts very well for the electron-proton scattering.

It should be noted that from our calculations, we are not able to decide definitely in favour of either of the two couplings used. Both give good agreement depending on the choice of the cut-off value and the coupling constant. But other phenomena like pion-nucleon scattering[5]) show that pseudovector coupling with cut-off is to be definitely preferred*).

Before ending this section some additional remarks about the principle of the cut-off are required. We have applied the cut-off, in this section, to the momentum, $\vec{k}$, of the virtual meson emitted or absorbed by the proton in the initial state, i. e., to the meson-nucleon interaction at the vertex at which the initial proton enters a Feynman diagram. Actually the cut-off must be applied to the *interaction at each vertex*. Consider, as an example, the Feynman diagram number (5) of this section. For the vertex where the initial proton enters the diagram we have the cut-off condition:

$$\left\{\sqrt{(\vec{p}_1-\vec{k})^2+M^2}+\sqrt{\vec{k}^2+\mu^2}\right\}^2-\vec{p}_1^{\,2}\leqslant\left\{\sqrt{K_0^2+M^2}+\sqrt{K_0^2+\mu^2}\right\}^2.$$

This condition is the one used by us. Besides this we have, for the vertex where the final proton leaves the diagram, a condition:

$$\left\{\sqrt{(\vec{p}_1-\vec{k})^2+M^2}+\sqrt{(\vec{k}+\vec{q})^2+\mu^2}\right\}^2-(\vec{p}_1+\vec{q})\leqslant\left\{\sqrt{K_0^2+M^2}+\sqrt{K_0^2+\mu^2}\right\}^2.$$

These two conditions are identical only in the case where $\vec{q}=0$. In the general case ($\vec{q}\neq 0$) we must, naturally, take the superposition of these conditions into account. For this, extremely complicated computations

*) A very common error is prevalent that the γ_5 or ps-coupling is equivalent to the gradient or pv-coupling in the non-relativistic approximation. This, of course, is not at all true for second order processes where the negative energy intermediate states give the largest contribution for the γ_5-coupling, analogously to the case of Thomson-scattering. The negative energy states cannot be omitted for the γ_5-coupling. The good agreement reached by CHEW[15]) for pion-nucleon scattering is due to the using of the gradient (pv) coupling.

would be required. But since the values of q in our calculations are not too large (excepting for large angles and large energies) the one condition used by us is a feasible approximation. Much weight can thus not be attached to the calculated scattering curves for 1000 MEV.

Finally, there are also cut-off conditions to be imposed at the vertices where the photon takes part in the interaction. This condition would chiefly limit the values of q for which the present theory is valid. If for the interactions where a photon takes part, the cut-off value is equal to or larger than our K_0 (for the meson-nucleon system), then the values of q used here lie well within the range of validity of the theory.

Section IV

Non-relativistic limit of the anomalous moments

A very often quoted result for the anomalous magnetic moment of the neutron (also equal but opposite in sign for the proton) according to non-relativistic meson theory calculations is (in units of $e/2\,M$):

$$\mu_N = -\frac{G^2}{4\,\pi^2}\cdot\frac{1}{M}\int_0^{K_0} d\,k\cdot k^2\left(\frac{2}{3}\,\frac{k^2}{\omega^4}\right) \qquad \text{(IV}-1)\ ^{*)}$$

where $$\omega^2 = k^2 + \mu^2$$

and $G = 2\,MF/\mu$ for pseudovector coupling.

Here there is no nucleon current since the nucleon is considered as a fixed heavy extended source.

Let us compare it with the non-relativistic limit of our covariant calculation for μ_N. We have, as shown in section II,

$$\mu_N = -\,2\,i\,\Lambda\,M\,P(0) = -\frac{G^2}{4\,\pi^2}\cdot 3\int_0^{K_0}\frac{d\,k}{M}\cdot\frac{k^2}{M^2}\int_0^1 d\,y\,\frac{y^2}{\{(y-a)^2+b\}^{5/2}}\,,$$

where the integrations over the variables k_0, x and the angles in $\vec{k}$-space have already been performed and

$$a = \frac{1}{2}\,\frac{\mu^2}{M^2}\,;\ b = \frac{k^2}{M^2} + \frac{\mu^2}{M^2} - \frac{1}{4}\,\frac{\mu^4}{M^4}\,.$$

---

*) See, for example, G. SALZMANN: cf. (5), Equ. (18c), where, of course, we must put $v = 1$ and the upper limit equal to K_0 to compare it with the above calculations.

Now to obtain the non-relativistic limit we integrate over the variable y and then take the lowest term in an expansion according to $1/M$. This gives

$$\mu_N = -\frac{G^2}{4\pi^2} \cdot \frac{1}{M} \int_0^{K_0} d\,k \cdot k^2 \left(\frac{\omega^2}{\omega^4}\right). \qquad \text{(IV} - 2)$$

Clearly, the two results (IV – 1) and (IV – 2) do not agree. The Understanding and removal of this discrepancy is simple but noteworthy. A similar discrepancy was also pointed out recently by HEITLER[7]) in another connection. The situation is clarified if we note that in going over to the non-relativistic limit in a consistent way (by expansion in powers of k/M etc.) a cut-off K_0, smaller than M, is automatically implied. Further, this cut-off, which is invariant in a relativistic theory, must be consistently handled when one goes over to the non-relativistic limit. To bring out this point clearly let us look at the form of our matrix element of section II:

$$M_s \sim \bar{u}_2 \{ Q(0)\,(\gamma \cdot \boldsymbol{a}) + P(0)\,[(\gamma \cdot \boldsymbol{a}), (\gamma \cdot \boldsymbol{q})] + \cdots \} u_1$$

$$\text{i.e.} \sim \bar{u}_2 \{ Q(0)\,\gamma_\mu A^\mu + P(0)\,\sigma_{\mu\nu}\,\mathfrak{F}^{\mu\nu} + \cdots \} u_1 .$$

Here $P(0)$ is essentially the anomalous magnetic moment. It is easily identified; whereas the $Q(0)$-term is separated out and dropped for reasons already discussed in section II. Now if we go over to the non-relativistic Pauli two-component case in the usual manner*) we have for the case of a static magnetic field:

$$\gamma \cdot \boldsymbol{A} \longrightarrow \vec{p}_1 \cdot \vec{A} + \frac{1}{2}\vec{\sigma}, \vec{H}$$

$$\sigma_{\mu\nu}\,\mathfrak{F}^{\mu\nu} \longrightarrow \vec{\sigma}, \vec{H} .$$

Thus in a calculation which is non-relativistic from the start we shall have the $\vec{\sigma} \cdot \vec{H}$-terms mixed up from the two separate type of terms, one being the real observable effect and the other an ambiguous term which must be subtracted out, for its value *must* be zero, since the neutron has a charge equal to zero. To obtain the correct result then, we must not let $\vec{p}_1$ be equal to zero, so that we may recognise, with the help of the $\vec{p}_1 \cdot \vec{A}$-term, the $\vec{\sigma} \cdot \vec{H}$-term (with half the same coefficient as the $\vec{p}_1 \cdot \vec{A}$-term) which must be subtracted out, leaving the rest of the $\vec{\sigma} \cdot \vec{H}$-term as the true anomalous moment effect. But since $\vec{p}_1$ is not zero, the cut-

*) See, for example, W. HEITLER: Quantum Theory of Radiation, Third Edition, pp. 109, 110 (1954).

off cannot be applied symmetrically so that, for instance, the following equations

$$\int d^3k\,(\vec{k}\cdot\vec{A})\,f(\vec{k}^2) = 0\ ,$$

$$\int d^3k\,(\vec{k}\cdot\vec{A})\,(\vec{k}\cdot\vec{B})\,f(\vec{k}^2) = \frac{1}{3}\,(\vec{A}\cdot\vec{B})\int d^3k\,\vec{k}^2\,f(\vec{k}^2)\ ,$$

which are always used in the usual non-relativistic calculations, do not hold any longer. In other words, we have to take the 'deformation' of the source for the moving nucleon into account. This can be done for the present case in a very simple way. Let $\vec{k}$ be the virtual meson momentum corresponding to the nucleon with momentum $\vec{p}_1$ and $\vec{k}'$ that corresponding to the case when $\vec{p}_1 = 0$. Then we have for the invariant z, equ. (I – 2),

$$z^2 = \left\{\sqrt{M^2+(\vec{p}_1-\vec{k})^2}+\sqrt{\mu^2+\vec{k}^2}\right\}^2-\vec{p}_1{}^2 = \left\{\sqrt{M^2+\vec{k}'^2}+\sqrt{\mu^2+\vec{k}'^2}\right\}^2.$$

Since we are interested in the non-relativistic conterpart of this equation we compare the two sides after an expansion in powers of $1/M$ and retaining terms only up to $1/M$. Then

$$\omega' \simeq \omega\left[1-\frac{\vec{p}_1\cdot\vec{k}}{M\omega}\right],$$
$$\omega \equiv \sqrt{k^2+\mu^2}\ ,\ \omega' \equiv \sqrt{k'^2+\mu^2}\ . \qquad \text{(IV – 3)}$$

Thus, all the integrals over d^3k which we consider must be transformed to those over d^3k' and then only a symmetrical spherical cut-off to the radius K_0 must be applied.

When this procedure is used consistently the above mentioned discrepancy gets removed and the correct non-relativistic result (IV – 2) and not (VI – 1) is obtained.

We illustrate this by an example. Consider the first terms $M_{\mathrm{I}}^{(1)}$, $M_{\mathrm{II}}^{(1)}$ of equations (II – 7), (II – 8) which correspond to the simpler pseudoscalar coupling. Instead of doing the rest of the calculations convariantly, we consider the case of a static magnetic field,

$$A_0 = 0\ ,\ \vec{A}(\vec{x}) = \int \vec{a}(\vec{q})\,e^{i\vec{q}\cdot\vec{x}}\,d^3q\ ,\ \vec{a}(\vec{q})\cdot\vec{q} = 0\ ,$$

integrate out over k_0, expand in powers of $1/M$, and retain the lowest terms till the one which first contains $\vec{\sigma}$. For the sum we have then

$$M^{(N.R.)} \simeq -4\pi i\Lambda\,(2M)^2\,\delta(q_0)\int\frac{d^3k}{4M\omega^4}$$
$$\left[v_2^\dagger\left\{-\omega+\frac{i}{M}\vec{\sigma}\cdot[\vec{q}\times\vec{k}]+\cdots\right\}v_1\right](\vec{k}\cdot\vec{a}(\vec{q}))\ ,$$

where v_2, v_1 are the Pauli spinors and where we have also dropped all

the other terms which do not enter the effect we are calculating. Here we have now

$$\int \frac{d^3k}{\omega^4} (\vec{k}\cdot\vec{a}) \frac{i}{M} \vec{\sigma}\cdot[\vec{q}\times\vec{k}] \simeq \int \frac{d^3k'}{\omega'^4} \frac{1}{3} k'^2 \frac{i}{M} \vec{\sigma}\cdot[\vec{q}\times\vec{a}] ,$$

the correction due to (IV – 3) being neglected as it will be of an order higher in $1/M$. Now integrating over angles round the axis $\vec{p}_1$:

$$\int \frac{d^3k}{\omega^4} \vec{k}\cdot\vec{a}(-\omega) = -\int d^3k \frac{\vec{k}\cdot\vec{a}}{\omega^3} {}^{*)} = -2\pi \frac{\vec{p}_1\cdot\vec{a}}{p_1} \int_{-1}^{+1} d\cos\theta \int k^2 dk \frac{k\cos\theta}{\omega^3}$$

$$= -2\pi \frac{\vec{p}_1\cdot\vec{a}}{p_1} \int_{-1}^{+1} d\cos\theta \int k'\,dk' \times \left[1 + \frac{p_1\cos\theta}{M}\left(\frac{\omega'}{k'} + \frac{k'}{\omega'}\right)\right] \times$$

$$\times k'^2 \left[1 + 2\frac{\omega'}{k'}\cdot\frac{p_1\cos\theta}{M}\right] \frac{1}{\omega'^3} \left[1 - 3\frac{k'}{\omega'}\frac{p_1\cos\theta}{M}\right] \cos\theta ,$$

using the transformation of the variable (IV – 3). Finally

$$\int \frac{d^3k}{\omega^4} \vec{k}\cdot\vec{a}(-\omega) = -4\pi \frac{(\vec{p}_1\cdot\vec{a})}{M} \int_0^{K_0} dk' \frac{k'^2}{\omega'^4} \left\{\frac{1}{3}k'^2 + \mu^2\right\} .$$

Thus we have

$$M^{(N.R.)} \simeq -(4\pi)^2 i \Lambda M \delta(q_0) \int_0^{K_0} dk' \cdot \frac{k'^2}{\omega'^4} v_2^\dagger \left\{-\frac{\vec{p}_1\cdot\vec{a}}{M}\left(\frac{1}{3}k'^2 + \mu^2\right) + \right.$$

$$\left. + \frac{2}{3}k'^2 \frac{i}{2M} \vec{\sigma}\cdot[\vec{q}\times\vec{a}]\right\} v_1 .$$

Subtracting $\vec{p}_1\cdot\vec{a}$-term and along with it the corresponding $\vec{\sigma}$-term we are left with

$$M^{(N.R.)}_{effective} \simeq -(4\pi)^2 i \Lambda M \delta(q_0) v_2^\dagger \int_0^{K_0} \frac{dk'}{\omega'^4} k'^2 \left\{\frac{1}{3}k'^2 + \mu^2 + \frac{2}{3}k'^2\right\} \frac{i}{2M}$$

$$\vec{\sigma}\cdot[\vec{q}\times\vec{a}] v_1 = -(4\pi)^2 i \Lambda M \delta(q_0) v_2^\dagger \int_0^{K_0} dk' \cdot k'^2 \left(\frac{\omega'^2}{\omega'^4}\right) \frac{i}{2M} \vec{\sigma}\cdot[\vec{q}\times\vec{a}] v_1 .$$

Remembering that the Fourier transform of $i\vec{\sigma}\cdot[\vec{q}\times\vec{a}]$ is our $\vec{\sigma}\cdot\vec{H}$, we have, as seen easily, the anomalous magnetic moment (dropping the accent on k' now)

$$\mu_N = -\frac{G^2}{4\pi^2} \frac{1}{M} \int_0^{K_0} dk \cdot k^2 \left(\frac{\omega^2}{\omega^4}\right) .$$

It should be noted that if we do not treat the $\vec{p}_1\cdot a$-term and the cor-

*) Since the region of integration is not symmetrical, this integral is not zero, as it is usually taken to be.

responding $\vec{\sigma}$-term as done above and let either $\vec{p}_1$ be zero right at the start or perform the faulty symmetrical $\vec{k}$-integration then we get the wrong often-quoted result (IV – 1).

In this sense we may say that the usual non-relativistic models of meson theory are not really non-relativistic limits of a relativistic theory.

Acknowledgements

I wish to express here my deepest gratitude to my respected teacher, Professor W. HEITLER, for his kind guidance throughout this work. To Dr. E. ARNOUS, Dr. CH. TERREAUX, and Mr. G. KARPMAN I am indebted for many useful discussions. For the use of their electronic calculating machine, I am much obliged to the Director, Prof. E. STIEFEL, and the staff of the Institute for Applied Mathematics of the Swiss Federal Institute of Technology. My grateful thanks are due to the Government of India for the award of the Central States Scholarship. The last stage of the work was supported by a grant from the 'Jubiläumsspende' of the University of Zürich, for which I would like to express my sincerest gratitude.

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