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Autor: O'Raifeartaigh, L.

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The Dirac Matrices and the Signature of the Metric Tensor

by **L. O'Raifeartaigh**
Institut für Theoretische Physik der Universität Zürich

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Summary. A problem mentioned by JAUCH and RÖHRLICH (1955, p. 430) in connexion with the Dirac matrices is solved. The problem concerns certain properties of the matrices of charge-conjugation and time reversal. The reason for the difficulty of this problem becomes clear. It is due to the fact that the properties in question are dependant on the signature of the metric tensor $g_{\mu\nu}$. On account of this latter result, the connexion between the Dirac matrices and the signature of the metric is investigated further and the different special representations of the Dirac matrices (i.e. Hermitian, real etc.) which can exist for each possible signature are found.

Introduction

In carrying out the charge conjugation and time reversal transformations in quantum field theory one has occasion to define the two matrices C and D by

$$\gamma_\mu^* = C\,\gamma_\mu\,C^{-1}, \qquad \begin{pmatrix} -\gamma_0^* \\ \gamma_i^* \end{pmatrix} = D \begin{pmatrix} \gamma_0 \\ \gamma_i \end{pmatrix} D^{-1}, \tag{1}$$

where * denotes complex, not Hermitian, conjugate, and γ_μ are the usual Dirac matrices satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2\,g_{\mu\nu}\,, \tag{2}$$

$g_{ii} = 1$, $g_{00} = -1$ (JAUCH and RÖHRLICH (1955), pp. 90, 94). For space inversion, one has occasion to consider a matrix S defined by

$$\begin{pmatrix} \gamma_0 \\ -\gamma_i \end{pmatrix} = S \begin{pmatrix} \gamma_0 \\ \gamma_i \end{pmatrix} S^{-1} \tag{3}$$

but it is obvious that $S = \gamma_0$, so that the matrix S requires no further consideration.

It follows immediately from (1) that

$$C^*\,C = cI\,, \quad D^*\,D = dI\,, \tag{4}$$

where c and d are numbers (I is the unit matrix), and it follows from (4) that

$$c = c^*\,, \quad d = d^*\,, \tag{5}$$

so that they are also *real* numbers. What does not follow so easily from (1) and (4) is that

$$c > 0\,, \quad d < 0\,, \tag{6}$$

although these relations are absolutely vital for the invariance of the theory under the transformations in question*).

As far as we know, the only proof of (6) existing consists in showing that (6) is independent of the representation of the γ_μ's, and then showing that (6) is valid for some explicit representation (JAUCH and RÖHRLICH (1955), pp. 90, 430, especially, p. 430). This sort of proof is rather unsatisfactory, particularly since this is the only point in field theory in which it is necessary to use an explicit representation. It seems to us, therefore, worthwhile to give a rather simple proof of (6) which is representation-independent. This proof is given in 1.

The question might well be asked why this particular problem presents so much difficulty. This question is discussed and the answer to it is given in 2. The answer is that the relations (6) depend, in fact, on the signature of the metric $g_{\mu\nu}$ in (2). To facilitate discussions concerning the signature of the metric, a metric tensor M is then defined in 2 and this turns out to have some very simple and interesting properties.

The general connexion between the signature of the metric and the representations of the γ-matrices is discussed in 3. It has seemed to us that the most practical and useful way of examining this question is to establish the various special representations of the γ's (i. e. Hermitian, real etc. representations) which can exist for each signature. Throughout the paper we consider only irreducible (i.e. 4×4) representations of the γ's.

1. Proof of equation (6)

The first step in the proof of (6) is the rather obvious one of showing that the signs of c and d are interrelated, so that, at any rate, $cd < 0$. This is shown as follows. By inserting in (1) one can easily verify that if C satisfies the first equation of (1), then

$$D = \alpha\, C\, \gamma_0\, \gamma_5 \tag{7}$$

satisfies the second (α any number). Hence, by a well-known theorem (JAUCH and RÖHRLICH (1955), p. 425), D is uniquely determined in terms of C (up to a constant factor). But then

*) Note the although C and D are defined only up to a constant factor, the relations (4), (5) are invariant under a change of this factor. In fact, if we change D, for example, to $D' = \alpha\, D$, α any number, we find that $d' = (\alpha^* \alpha)\, d$ so that the sign of d remains unchanged. The change in the *magnitude* of d under such a transformation can be used to make $|d| = 1$, thus determining D to within a phase factor. Similarly for C.

$$d = D^* D = \alpha^* \alpha C^* \gamma_0^* \gamma_5^* C \gamma_0 \gamma_5 = \alpha^* \alpha C^* C \gamma_0 \gamma_5 \gamma_0 \gamma_5 =$$

$$= - \alpha^* \alpha C^* C \gamma_0^2 \gamma_5^2 = (-1)^3 \alpha^* \alpha C^* C = - \alpha^* \alpha c \tag{8}$$

as required.

The second step in the proof consists therefore in proving *either* $c > 0$ or $d < 0$. The question is: which? In (1) the first equation is the more compact one, and so it would appear that $c > 0$ should be the easier relation to prove. But, in fact, this is not the case, and $d < 0$ is much more easily established. The key to the solution is to note that the second equation of (1) can also be written compactly, namely, as

$$\gamma_\mu^{*-1} = D \gamma_\mu D^{-1} \tag{9}$$

and to note that (9) suggests making use of the well-known result (proved below for completeness' sake) that unitary representations of the γ-matrices always exist. (We do not, of course, need to give such a representation explicitly, we need only to know that it exists). For *this* class of representations (9) becomes

$$\gamma_\mu^{\sim} = D \gamma_\mu D^{-1} \tag{10}$$

($\sim$ means transpose). But for *all* representations, PAULI (1936) has defined a matrix B by

$$\gamma_\mu^{\sim} = B \gamma_\mu B^{-1} \tag{11}$$

and deduced that from (11) alone

$$B^{\sim} = b B \tag{12}$$

with $b = -1$.

Hence, for the unitary representations

$$D = \beta B \tag{13}$$

where β is some constant, and so

$$d = D^* D = \beta^* \beta B^* B = - \beta^* \beta B^{*\sim} B = - \beta^* \beta B^+ B < 0 , \tag{14}$$

and since the sign of d is representation independent, $d < 0$ as required.

It remains only to give the usual proof that unitary representations of the γ-matrices exist: Out of the four γ-matrices one forms the usual sixteen linearly independent matrices $(1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu \gamma_5, \gamma_5)$ and also the sixteen matrices $(-1, -\gamma_\mu, -\sigma_{\mu\nu}, -\gamma_\mu \gamma_5, -\gamma_5)$. One sees immediately that these thirty two matrices form a group. Hence, the problem reduces to proving the standard theorem that any representation of a group can be made unitary by a similarity transformation. This is proved as follows. Let G_r, $r = 1 \ldots n$ be n matrices representing a group. Let

$$\Lambda = \sum_{s=1}^{s=n} G_s^+ \, G_s \, . \tag{15}$$

Λ is Hermitian and, therefore, a unitary transformation u exists so that

$$u \, \Lambda \, u^+ = \lambda \tag{16}$$

where λ is a (real) diagonal matrix. Then

$$g_r = \lambda^{1/2} \, u \, G_r \, u^+ \, \lambda^{-1/2} \tag{17}$$

is a unitary representation of the group.

2. Rôle of the signature of the metric: Metric Matrix

In connexion with the proof given in 1 two questions might well be asked:

(1) Although no use is made of an explicit representation of the γ's in the proof, nevertheless use is made of the *existence* of a certain class of representations. Would it not be possible to find a proof which does not make use of even the *existence* of such particular representations (such as PAULI's proof that for the B-matrix, $b = -1$)?

(2) It might be asked why the problem of proving $c > 0$, $d < 0$ provides so much difficulty (when again, the proof that $b = -1$, for example, though very ingenious, is relatively easy)?

These two questions are intimately connected, and both are answered by noting that the relations $c > 0$, $d < 0$ differ from the relation $b = -1$ in one very fundamental way, namely, that whereas the relation $b = -1$ depends only on the group properties of the γ-matrices, the relations $c > 0$, $d < 0$ depend not only on the group properties but also on the signature of the metric tensor used in (2). That is to say, $b = -1$, no matter what signature the metric has, but $c > 0$, for example, only for certain signatures (of which $(+++-)$, of course, is one). To persuade ourselves of this, let us consider the signature $(++++)$. With this signature, the unitary representations of the γ's (which always exist since in the proof of 1 the signature of $g_{\mu\nu}$ played no rôle) is also a Hermitian representation ($\gamma_\mu = \gamma_\mu^{-1} = \gamma_\mu^+$). But then the first equation of (1) becomes for this representation

$$\gamma_\mu^\sim = C \, \gamma_\mu \, C^{-1} \tag{18}$$

so that all the arguments that were applied in 1 to show that $d < 0$ can be applied here to show that $c < 0$. Since $c > 0$ for the signature $(+++-)$

in 1 we see clearly that the sign of c is signature dependent*). This is why it is so difficult to determine the sign of c (question (2)) and why a general proof depending only on the group properties of the γ's is not possible (question (1)).

Having seen that the signature of the metric plays such an important rôle in questions such as those just considered, we have thought it worthwhile to investigate the influence of the signature on the representations of the γ's in a general way. Practically, we do this as follows. We investigate what sort of special representations of the γ's (Hermitian, real etc.) can exist for each signature. The advantage of this method is that once one knows for any metric which sort of special representations exist, one can determine immediately and without the slightest trouble the signs of all such troublesome constants as b, c, d above. For example, let us suppose that we know (it will be proved below) that for the metric $(+++-)$ *real* representations exist. Then in (1) we could assume that we were using such a real representation so that (1) would read

$$\gamma_\mu = C\,\gamma_\mu\,C^{-1} \tag{19}$$

from which

$$C = \alpha I\,, \quad c = \alpha^* \alpha > 0 \tag{20}$$

follow immediately.

Before going into this question of special representations in detail (which will be done in 3) we find it useful to define here a *metric tensor M* by

$$\gamma^\mu \equiv \gamma_\mu^{-1} = M\,\gamma_\mu\,M^{-1}\,. \tag{21}$$

Non-singular M always exists since γ_μ^{-1} is a possible representation of the γ_μ. This matrix M has a very interesting and useful property, which we shall now describe.

There are sixteen possible signatures for the metric $g_{\mu\nu}$ which may be divided into five classes as follows:

$(++++)$	with 1 member,
$(+++-)$, $(++-+)$ etc.	with 4 members,
$(++--)$, $(+-+-)$ etc.	with 6 members,
$(+---)$, $(-+--)$ etc.	with 4 members,
$(----)$	with 1 member.

(Of course, in physics, only three of these metrics are used extensively $(++++)$, $(+++-)$ and $(---+)$.) The property of M, which we wish

*) Of course, in the case of the metric $(++++)$, $c < 0$, and not $c > 0$, is just what is required by the charge conjugate transformation, since the coordinates x_μ have different reality properties for this signature.

to describe is the following. For the sixteen possible signatures listed, the corresponding sixteen possible M's are

$$1, \quad \gamma_5 \gamma_\mu, \quad \gamma_5 \sigma_{\mu\nu}, \quad \gamma_\mu, \quad \gamma_5 \quad \text{respectively!}$$

(the μ in $\gamma_5 \gamma_\mu$, for example, is equal to that integer for which the minus sign appears in the signature).

A further property of M will be very useful. If B is the Pauli B-matrix given by (11) and (12) then

$$M^\sim = B\, M\, B^{-1}\, (\eta_M) \tag{22}$$

where η_M, a numerical factor, is equal to $(1, -1, -1, 1, 1)$ for the five classes of signatures shown above, in the order listed.

3. Special representations of the γ_μ-matrices

We shall establish in this section the following two theorems.

Theorem I (*metric independent theorem*):

(a) Unitary representations of the γ_μ's exist for all signatures, anti-unitary representations for none.

(b) Symmetric representations exist for no signature, antisymmetric representations exist for no signature.

Proof: (a) The first part of (a) has already been established in 1. To prove the second part, let γ_μ be one of the existing unitary solutions and γ_μ' a supposed anti-unitary solution. Then by the usual well-known theorem, quoted in 1, non-singular S exists such that

$$\gamma'_\mu = S\, \gamma_\mu\, S^{-1}.$$

Taking Hermitian conjugates

$$\gamma'^{+}_\mu = S^{-1+}\, \gamma^+_\mu\, S^+,$$

by hypothesis

$$-\gamma'^{-1}_\mu = S^{-1+}\, \gamma^{-1}_\mu\, S^+,$$

taking the inverse

$$-\gamma'_\mu = S^{-1+}\, \gamma_\mu\, S^+. \tag{23}$$

Hence

$$S\, \gamma_\mu\, S^{-1} = S^{-1+}\, (-\gamma_\mu)\, S^+ = S^{-1+}\, \gamma_5\, \gamma_\mu\, \gamma_5^{-1}\, S^+,$$

so that by the same well-known theorem

$$S = S^{-1+}\, \gamma_5. \tag{25}$$

Hence

$$\text{Tr}\, S^+ S = \text{Tr}\, \gamma_5 = 0\,, \tag{26}$$

which is impossible for $S \neq 0$. Thus (a) is established.

To establish (b) we use the procedure used by PAULI to prove that $B^\sim = - B$. We suppose that a symmetric representation of the γ_μ exists. Then it is easy to show that the ten linearly independent matrices $\sigma_{\mu\nu}$ and $\gamma_\mu \gamma_5$ are anti-symmetric, which is impossible since only six linearly independent anti-symmetric 4×4 matrices exist. Similarly the existence of an anti-symmetric representation of the γ_μ would imply the existence of the ten linearly independent anti-symmetric matrices γ_μ, $\sigma_{\mu\nu}$. This completes the proof of theorem I.

Theorem II (metric dependant theorem):

(a) Hermitian representations of the γ_μ's exist only for the signature $(++++)$, anti-Hermitian only for $(----)$.

(b) Real representations exist only for the signatures $(+++-)$ etc. and $(++--)$ etc., pure imaginary only for $(---+)$ etc. and $(++--)$ etc.

Proof: (a) We let γ'_μ be a Hermitian representation of the γ_μ's and let γ_μ be the usual unitary representation. Then with non-singular S

$$\gamma'_\mu = S \gamma_\mu S^{-1}\,.$$

Taking Hermitian conjugates

$$\gamma'^{+}_\mu = S^{-1+} \gamma^+_\mu S^+\,,$$

by hypothesis

$$\gamma'_\mu = S^{-1+} \gamma^{-1}_\mu S^+\,. \tag{27}$$

Hence

$$S \gamma_\mu S^{-1} = S^{-1+} \gamma^{-1}_\mu S^+ \tag{28}$$

or

$$\gamma_\mu S^+ S \gamma_\mu = S^+ S\,, \tag{29}$$

so that

$$\text{Tr}\, S^+ S = \text{Tr}\, \gamma_\mu S^+ S \gamma_\mu = \text{Tr}\, S^+ S \gamma^2_\mu = \gamma^2_\mu\, \text{Tr}\, S^+ S \tag{30}$$

i.e. for all μ

$$\gamma^2_\mu = 1\,, \quad \text{or} \quad g_{\mu\mu} = 1 \tag{31}$$

as required.

The second part of (a) is shown by noting that a representation for any metrics is obtained by multiplying by a factor i any representation of the conjugate metric (i.e. the metric obtained from the first by changing in the signature all the pluses to minuses and vice versa).

(b) the proof of this part of Theorem II is more difficult. Let γ'_μ be a real representation of the γ_μ's and γ_μ a unitary representation. As usual, with non-singular S

$$\left.\begin{array}{l} \gamma'_\mu = S\,\gamma_\mu\,S^{-1}\,, \\ \text{taking complex (not Hermitian) conjugates} \\ \gamma^{1*}_\mu = S^*\,\gamma^*_\mu\,S^{*\,-1}\,, \\ \text{by hypothesis} \\ \gamma'_\mu = S^*\,\gamma^{-1\sim}_\mu\,S^{*\,-1} = S^*\,B\,M\,\gamma_\mu\,M^{-1}\,B^{-1}\,S^{*\,-1} \end{array}\right\} \tag{32}$$

by definition of M (the metric tensor which we now use for the first time) and B.

Hence by the usual theorem

$$B\,M = \alpha\,S^{*\,-1}\,S \tag{33}$$

(where α is a number) so that

$$B^*\,M^*\,B\,M = \alpha^*\,\alpha\,S^{-1}\,S^*\,S^{*-1}\,S = \alpha^*\,\alpha > 0\,. \tag{24}$$

Hence from (22)

$$\eta_M\,M^{*\sim}\,B^*\,B\,M > 0 \tag{35}$$

or from (12)

$$-\,\eta_M\,M^+\,B^+\,B\,M > 0 \tag{36}$$

i.e.

$$\eta_M < 0\,.$$

From (22), however, this is possible *only* for the signatures $(+++-)$ etc. and $(++--)$ etc. It still has to be shown that in these two cases real representations do, in fact, exist. To show this let σ_1 and σ_2 be the two ***real*** (2×2) Pauli spin-matrices. Then

$$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \sigma_1\,\sigma_2 \\ \sigma_1\,\sigma_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{37}$$

and

$$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \sigma_1\,\sigma_2 \\ \sigma_1\,\sigma_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{38}$$

are real representations for the signatures $(+++-)$ etc. and $(++--)$ etc. respectively. Thus the first part of (b) is established. Since, however $(---+)$, etc. and $(++--)$ etc. are metrics conjugate to the two just mentioned, the second part of (b) follows immediately.

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