Quantum Theory in Real Hilbert Space II
(Addenda and Errats)

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Abstract: A more concise demonstration of the necessity of the thermodynamic signature of the metric $g^{\mu\nu}$ and of the pseudo-chronous character of $\breve{J}$ (corresponding to $i = \sqrt{-1}$) is given (§ 4 of I[1]). Furthermore, the anti-unitary operator (Annex 3 of I) is wrong. A more concise demonstration in terms of observables is given, showing the perfect correspondence between RHS (Real Hilbert Space) and CHS (Complex Hilbert Space).

A more conoise demonstration for the *thermodynamic signature* and the *pseudo-chronous character of* $\breve{J}$ is:

§ 4 bis. The Thermodynamic Signature of $g^{\mu\nu}$ and the Pseudo-Chronous Character of $J = \breve{J}$

For a *local scalar observable,* we have the identity ($\alpha\,\beta \ldots = 12 \ldots n$):

$$'F('x) = F(x) = F(L^{-1}\,'x) = O^{-1}\,F('x)\,O\,,$$

$$L \leftarrow O\colon\ 'x = L\,x = \{'x'^{\alpha} = L'^{\alpha}{}_{\alpha}(x^{\alpha} + L^{\alpha})\}\,,$$

$$g'^{\mu\,'\nu} = L'^{\mu}{}_{\mu}\,L'^{\nu}{}_{\nu}\,g^{\mu\,\nu}. \tag{4 bis. 1}$$

We form the *bilocal observable*

$$\breve{G}(x\,y) = \breve{J}\,[F(x),\,F(y)] \tag{4 bis. 2}$$

appearing in the UP:

$$\langle \Delta F(x)^2\rangle_{\Psi}\,\langle \Delta F(y)^2\rangle_{\Psi} \geqslant \frac{1}{4}\,\langle \breve{G}(x\,y)\rangle_{\Psi}^2. \tag{4 bis. 3}$$

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The hypothesis, that an *L-invariant vacuum state* Ψ^0 exists

$$\langle \breve{G}(x\,y)\rangle_{\Psi^0} = \breve{f}(x\,y) = - \breve{f}(y\,x) \tag{4 bis. 4}$$

requires that

$$'\breve{f}('x\,'y) = c(L)\,\breve{f}(x\,y) = c(L)\,\breve{f}(L^{-1}\,'x\,L^{-1}\,'y) \tag{4 bis. 5}$$

is the same function in every frame, i.e.

$$'\breve{f}('x\,'y) = \breve{f}('x\,'y) \tag{4 bis. 6}$$

because *vacuum is a frame-invariant concept*.
$c(L)$ is a (real) *number*, satisfying the representation condition

$$c(L_2)\,C(L_1) = c(L_2\,L_1)\,. \tag{4 bis. 7}$$

From the identity, following from (4 bis. 5) and (4 bis. 6)

$$c(L)\,\breve{f}(L^{-1}\,'x\,L^{-1}\,'y) = \breve{f}('x\,'y) = - \breve{f}('y\,'x) \tag{4 bis. 8}$$

follows, that $\breve{f}(x y)$ is, except for the factor $c(L)$, an invariant anti-symmetric function. Such a function exists certainly, if we choose a metric

$$\text{signat}\,(g^{\alpha\beta}) = \pm\,(11\,\ldots\,1 - 1)\,. \tag{4 bis. 9}$$

We call $\{x^i\} = \vec{x}$ $(ik\ldots = 12\ldots d,\ d = n - 1)$ *space* and $x^n = t$ *time* and define

$$\breve{f}(x\,y) = \text{sig}\,(x^n - y^n)\,\cdot f((x-y)^2) = - \breve{f}(y\,x) \tag{4 bis. 10}$$

$$f(z^2) = 0 \quad \text{for} \quad z = x - y = \text{spatial.}$$

We have therefore

$$\begin{aligned} c(L)\,\breve{f}(L^{-1}\,'x\,L^{-1}\,'y) &= c(L)\,\text{sig}\,\big(L^{-1\,n}{}_{'n}('x'^n - 'y'^n)\big)\cdot f(('x - 'x)^2) \\ &= c(L)\,\text{sig}(L'^n{}_n)\cdot \text{sig}\,('x'^n - 'y'^n)\cdot f(('x - 'y)^2) \\ &= c(L)\,\text{sig}(L'^n{}_n)\cdot \breve{f}('x\,'y) \equiv \breve{f}('x\,'y)\,, \end{aligned} \tag{4 bis. 11}$$

i.e.

$$c(L) = \text{sig}(L'^n{}_n)\,. \tag{4 bis. 12}$$

This means, that $\breve{f}(x y)$ is a *pseudo-chronous number, caracterising the vacuum*. Therefore $\breve{G}(x y)$ must be a *pseudochronous observable* i.e.

$$'\breve{G}('x\,'y) = \mathrm{sig}(L'^{n}{}_{n})\,\breve{G}(x\,y) = \mathrm{sig}(L'^{n}{}_{n})\,\breve{G}(L^{-1}\,'x\,L^{-1}\,'y) =$$

$$= O^{-1}\,\breve{G}('x\,'y)\,O = (O^{-1}\,\breve{J}\,O)\,[O^{-1}\,F('x)\,O, O^{-1}\,F('y)\,O] =$$

$$= (O^{-1}\,\breve{J}\,O)\,['F('x), 'F('y)] = (O^{-1}\,\breve{J}\,O)\,[F(x), F(y)] =$$

$$= (O^{-1}\,\breve{J}\,O\,\breve{J}^{-1})\,\breve{J}[F(x), F(y)] = (O^{-1}\,\breve{J}\,O\,\breve{J}^{-1})\,\breve{G}(x\,y) \tag{4 bis. 13}$$

where we have made use of (4 bis. 1). Comparing the second and the last member, we have $O^{-1}\,\breve{J}\,O\,\breve{J}^{-1} = \mathrm{sig}(L'^{n}{}_{n}).1$

$$O^{-1}\,\breve{J}\,O = \mathrm{sig}(L'^{n}{}_{n})\,\breve{J} \equiv '\breve{J} \tag{4 bis. 14}$$

which is equation (4.14) of I. $\breve{J}$ is thus a *pseudochronous operator*. Writing $L_{(\mathrm{ochr})} \leftarrow O_{(\mathrm{ochr})}$ and $L_{(\mathrm{pchr})} \leftarrow O_{(\mathrm{pchr})}$ for *ortho-chronous* ($\mathrm{sig}(L'^{n}{}_{n}) = +1$) and *pseudo-chronous* ($\mathrm{sig}(L'^{n}{}_{n}) = -1$) *Lorentz-transformations*, we have

$$[O_{(\mathrm{ochr})}, \breve{J}] = [O_{(\mathrm{ochr})}, \breve{J}]_{-} = 0\,, \tag{4 bis. 15_}$$

$$(O_{(\mathrm{pchr})}, \breve{J}) = [O_{(\mathrm{pchr})}, \breve{J}]_{+} = 0\,. \tag{4 bis. 15+}$$

We are left to show, that (4 bis. 9) are the only two possible signatures of $g^{\mu\nu}$. To show this, let us assume a metric, in which $z = \{z^{\alpha}\}$ has $d = n - r$ *space components* and $r \leqslant 1/2\,n$ *time components* $\vec{t} = \{t^{a}\} = \{z^{a}\}$ ($i\,k\ldots = 12\ldots\,n - r$, $a\,b\ldots = (n - r + 1)\,(n - r + 2)\ldots n$).

We have to build a function $\breve{f}(z) = \breve{f}(\vec{x}, \vec{t})$ which is *invariant* under *homogeneous continuous transformations* $L_{(\mathrm{cont})}$ (= *proper Lorentz-transformations*) and *changes sign* under *reflections* ($'x = (PT)\,x = -x$). Thus, we look for a vector $z_0 = \{z_0^{\alpha}\}$, which changes sign under the transformation $L = PT$ but does not change sign under any continuous transformation. If $r > 1$, it is evident, at least for *space-like* and *time-like vectors* z, that no such vector z_0 exists, because every vector $z = (\vec{t}, \vec{z})$ can be transformed into $-z$ by rotations in $\vec{z}$-space and $\vec{t}$-space. For null-vectors ($g_{\alpha\beta}\,z^{\alpha}\,z^{\beta} = 0$) we use the theorem[2]), that in n dimensional space time with $r \leqslant 1/2\,n$, there exist exactly r two-by-two orthogonal linearly independent null vectors z. Therefore, for $r > 1$, a rotation of this r-dimensional 'frame', transformes z into $-z$ (for instance z on the 'frame'). Thus only in the case $r = 1$, time-like vectors and null vectors (on the light cone) exist, which change signe under $L = PT$.

The text of I is correct, if we write, on page 748, under b):

'We pose...' instead of 'We try, posing...' and leave out, on p. 749, the text following (A-3.20). (A-3.20) is the correct formula, the text following (A-3.20) is erroneous. However, a more logical deduction, using only observables $F^X = F^{XT} = F^{\alpha\beta\ldots}(xy\ldots)$ $(X = \{\alpha\,\beta\ldots\ xy\ldots\})$ shall be given below in:

Annex 3bis. Unitary ($\widehat{U}$) and Anti-Unitary ($\widehat{V}$) Operators in CHS

We rewrite the operator identity of (I 3.9) in RHS

$$F'^X_{'a\,'b} = L'^X{}_X\, O_{'a\,a}\, F_{a\,b}\, O^T_{b\,'b}; \qquad F^X_{a\,b} = F^X_{(a\,b)}\,. \tag{A-3 bis. 1}$$

For:

a) Orthochronous Lorentz Transformations

We decompose the RHS (a-space) in the direct product of a two-dimensional ((r) (i)-space) and a $\omega_C = (1/2)\,\omega_R$ dimensional p-space, using the two-dimensional matrices (in (r) (i)-space)

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A-3 bis. 2}$$

On account of (4 bis. 15_), we have ($O = O_{(\text{ochr})}$):

$$\begin{aligned} F'^X_{'a\,'b} &= 1 \times F'^X_{(r)\,'p\,'q} + j \times F'^X_{(i)\,'p\,'q} \\ &= L'^X{}_X\,(1 \times O_{(r)\,'p\,p} + j \times O_{(i)\,'p\,p})\,(1 \times F^X_{(r)\,p\,q} + j \times F^X_{(i)\,p\,q}) \cdot \\ &\quad \cdot (1 \times O^T_{(r)\,q\,'q} - j \times O^T_{(i)\,q\,'q}\,, \end{aligned}$$

$$F^X_{(r)\,p\,q} = F^X_{(r)\,(p\,q)}\,; \quad F^X_{(i)\,p\,q} = F^X_{(i)\,[p\,q]} \tag{A-3 bis. 3}$$

formula, which, may be rewritten in CHS, using the correspondence RHS $\rightleftarrows$ CHS, $A \rightleftarrows \widehat{A}$ given in (I A-2.3) $\rightleftarrows$ (I A-2.5) in the form

$$\widehat{F}'^X_{'p\,'q} = L'^X{}_X\, \widehat{O}_{'p\,p}\, \widehat{F}^X_{p\,q}\, \widehat{O}^\dagger_{q\,'q} = L'^X{}_X\, \widehat{U}_{'p\,p}\, \widehat{F}^X_{p\,q}\, \widehat{U}^\dagger_{q\,'q} \tag{A-3 bis. 4}$$

with:

$$\widehat{F}^\dagger = \widehat{F} = \widehat{F}^{T*} = \widehat{F}^{*T}\,,$$

$$\widehat{U} = \widehat{O}; \quad \widehat{U}^\dagger\,\widehat{U} = \widehat{U}\,\widehat{U}^\dagger = 1; \quad \widehat{U}^\dagger = \widehat{U}^{-1} \tag{A-3 bis. 5}$$

where $\widehat{A}^*$ is the *complex conjugate operator* ($(\widehat{A}\widehat{B})^* = \widehat{A}^*\widehat{B}^*$) and $\widehat{A}^\dagger = \widehat{A}^{*T} = \widehat{A}^{T*}$ is the *hermitian conjugate operator* ($(\widehat{A}\widehat{B})^\dagger = \widehat{B}^\dagger\widehat{A}^\dagger$) in CHS. Explicitly written, and omitting the indices $pq\ldots = 12\ldots\omega_C$ referring to a frame in ω_C-dimensional CHS, the identity

$$\widehat{F}^{\prime\alpha\prime\beta\ldots}({}'x\,{}'y\ldots) = c(L)\ L^{\prime\alpha}{}_{\alpha}\, L^{\prime\beta}{}_{\beta}\ldots\ \widehat{U}\ F^{\alpha\beta\ldots}\ (L^{-1}\,{}'x\ L^{-1}\,{}'y\ldots)\ \widehat{U}^{-1}$$

holds, with $c(L) = 1$, $\mathrm{sig}\,(\det(L^{\prime i}{}_{i}))$ and $\mathrm{sig}\,(\det(L^{\prime\alpha}{}_{\alpha}))$ for *ortho* $(\widehat{F})$, *pseudo-chronous* $(\breve{\widehat{F}})$, *pseudochorous* $(\overset{\frown}{\widehat{F}})$ *and pseudo*-$(\mathring{\widehat{F}})$ observables. The *transformed operators* are given by

$$\boxed{\begin{aligned}{}'\widehat{F}^{\prime\alpha\prime\beta\ldots}\ ({}'x\,{}'y\ldots) &= c(L)\ L^{\prime\alpha}_{(\mathrm{ochr})\,\alpha}\ L^{\prime\beta}_{(\mathrm{ochr})\,\beta}\ldots\ \widehat{F}^{\alpha\beta\ldots}\ (L^{-1}\,{}'x\ L^{-1}\,{}'y\ldots)\\ &= \widehat{U}^{-1}\ F^{\prime\alpha\prime\beta\ldots}\ ({}'x\,{}'y\ldots)\ \widehat{U}\,. \qquad \text{(A-3 bis. 6)}\end{aligned}}$$

They lead to the identity, for expectation values:

$$\langle\widehat{\Psi},\ {}'\widehat{F}^{\alpha\ldots}\ ({}'x\ldots)\ \widehat{\Psi}\rangle = c(L)\ L^{\prime\alpha}{}_{\alpha}\ldots\ \langle\widehat{\Psi},\ \widehat{F}^{\alpha\ldots}\ (L^{-1}\,{}'x\ldots)\ \widehat{\Psi}\rangle$$

$$= \langle{}'\widehat{\Psi},\ \widehat{F}^{\prime\alpha\ldots}\ ({}'x\ldots)\ {}'\widehat{\Psi}\rangle \qquad \text{(A-3 bis. 7)}$$

with

$${}'\widehat{\Psi} = \widehat{U}\ \Psi;\qquad L_{(\mathrm{ochr})} \leftarrow \widehat{U}_{(\mathrm{ochr})}\,. \qquad \text{(A-3 bis. 8)}$$

b) *Pseudo-Chronous Lorentz-Transformations*

We need an operator

$$K = k\times 1;\quad K^T = K = K^{-1};\quad K^2 = 1 \qquad \text{(A-3 bis. 8)}$$

which transformes

$${}'\breve{J} = K^{-1}\,\breve{J}\,K = -\,\breve{J};\quad \breve{J} = j\times 1\,. \qquad \text{(A-3 bis. 9) *)}$$

Such an operator can be given in terms of the pseudoquaternions (I A-4.8). We chose, for exemple, the two-dimensional matrix in (r) (i)-space

$$k = k^T = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};\quad k^2 = 1,\ k^T = k,\ (k, j) = 0\,. \qquad \text{(A-3 bis. 10)}$$

) In CHS $K \rightleftarrows \widehat{K} : \widehat{K}\,i\,\widehat{K} = -\,i$ (A-3 bis 9).

The operator $O_{(\mathrm{pchr})} = O$ can now be given in terms of

$$O = O'\,K\,; \quad O^T = O^{-1} = K^T\,O'^T = K\,O'^T\,; \quad [O',\,\breve{J}] = 0\,,$$
$$O'^T\,O' = O'\,O'^T = 1 \qquad \text{(A-3 bis. 11)}$$

and the identity (A-3 bis. 1) can be written, using the decomposition

$$\begin{aligned}F'^X_{'a'b} &= 1 \times F'^X_{(r)'p'q} + j \times F'^X_{(i)'p'q} \\ &= L'^X{}_X\,(1 \times O'_{(r)\,p'p} + j \times O'_{(i)'p\,p})\,(k \times 1)\,(1 \times F^X_{(r)\,p\,q} + j \times F^X_{(i)\,p\,q})\,\cdot \\ &\quad \cdot (k \times 1)\,(1 \times O'^T_{(r)\,q'q} - j \times O'^T_{(i)\,q'q}) \\ &= L'^X{}_X\,(1 \times O'_{(r)'p\,p} + j \times O'_{(i)'p\,p})\,(1 \times F^X_{(r)\,p\,q} - j \times F^X_{(i)\,p\,q})\,\cdot \\ &\quad \cdot (1 \times O'^T_{(r)\,q'q} - j \times O'^T_{(i)'q\,q})\,. \qquad \text{(A-3 bis. 12)}\end{aligned}$$

We have the correspondence RHS $\rightleftarrows$ CHS.

$$1 \times F^X_{(r)\,p\,q} - j \times F^X_{(i)\,p\,q} \rightleftarrows F^X_{p\,q} - i\,F^X_{p\,q} = \widehat{F}^{X*}_{p\,q}\,. \qquad \text{(A-3 bis. 13)}$$

Now, using again the RHS $\rightleftarrows$ CHS correspondence (I A-2.3) $\rightleftarrows$ (I A-2.5), we may write (A-3 bis. 12) in CHS, omitting again the p-space indices,

$$\begin{aligned}\widehat{F}'^X &= F'^X_{(r)} + i\,F'^X_{(i)} = L'^X{}_X\,\widehat{O}'\,\widehat{F}^{X*}\,\widehat{O}'^{-1} \\ &= L'^X{}_X\,\widehat{U}'\,\widehat{F}^{X*}\,\widehat{U}'^{-1}\,; \quad \widehat{U}'^{-1} = U'^{\dagger} \qquad \text{(A-3 bis. 14)}\end{aligned}$$

or, explicitly

$$\widehat{F}^{'\alpha'\beta\ldots}\,('x\,'y\ldots) = c(L)\,L'^{\alpha}{}_{\alpha}\,L'^{\beta}{}_{\beta}\ldots\widehat{U}'\,\widehat{F}^{\alpha\beta\ldots *}\,(L^{-1}\,'x\,L^{-1}\,'y\ldots)\,\widehat{U}'^{-1}\,. \qquad \text{(A-3 bis. 15)}$$

The transformed operator is defined by

$$\boxed{\begin{aligned}{}'\widehat{F}'^{\alpha\ldots}\,('x\ldots) &= c(L_{(\mathrm{pchr})})\,L'^{\alpha}_{(\mathrm{pchr})\,\alpha}\ldots\widehat{F}^{\alpha\ldots}\,(L^{-1}\,'x\ldots) \\ &= \widehat{U}'^{-1*}\,\widehat{F}^{\alpha\ldots *}\,('x\ldots)\,\widehat{U}'^{*}\,. \qquad \text{(A-3 bis. 16)}\end{aligned}}$$

The identity for expectation values takes the form

$$\begin{aligned}\langle\widehat{\Psi},\,'\widehat{F}'^{\alpha\ldots}\,('x\ldots)\,\widehat{\Psi}\rangle &= c(L)\,L'^{\alpha}{}_{\alpha\ldots}\,\langle\widehat{\Psi},\,\widehat{F}^{\alpha\ldots}\,(L^{-1}\,'x\ldots)\,\widehat{\Psi}\rangle \\ &= \langle'\widehat{\Psi},\,\widehat{F}'^{\alpha\ldots}\,('x\ldots)\,'\widehat{\Psi}\rangle\,. \qquad \text{(A-3 bis. 17)}\end{aligned}$$

Defining the anti-unitary operator $\widehat{V}$ by

$$'\widehat{\Psi} = \widehat{V}\,\widehat{\Psi} = \widehat{U}\,\widehat{\Psi}^* \to {}'\Psi_{'p} = U'_{'pp}\,\Psi^*_p\,; \quad L_{(\mathrm{pchr})} \leftarrow \widehat{V}\,, \qquad \text{(A-3 bis. 18)*)}$$

we have

$$\langle '\widehat{\Psi},\, \widehat{F}'^{\alpha\cdots}('x\ldots)\,'\widehat{\Psi}\rangle = c(L)\,L'^{\alpha}{}_{\alpha}\ldots \langle\widehat{\Psi},\, \widehat{F}^{\alpha\cdots}(L^{-1}\,'x\ldots)\,\widehat{\Psi}\rangle =$$

$$= \widehat{\Psi}_p\,\widehat{U}'^{T*}_{p'p}\,\widehat{F}'^{\alpha\cdots}_{'p'q}('x)\,\widehat{U}'_{'qq}\,\Psi^*_q = \widehat{\Psi}^*_q\,\widehat{U}'^{T}_{q'q}\,\widehat{F}'^{\alpha\ldots T}_{'q'p}('x)\,\widehat{U}'^*_{'pp}\,\widehat{\Psi}_p =$$

$$= \langle\widehat{\Psi},\, \widehat{U}'^{\dagger *}\,\widehat{F}'^{\alpha\ldots T}('x)\,\widehat{U}'^*\,\widehat{\Psi}\rangle\,. \qquad \text{(A-3 bis. 19)}$$

By definition, $F^{\alpha\cdots}(x\ldots)$ is an observable. Therefore it follows, from (A-3 bis. 5), that in (A-3 bis. 19), we have

$$\widehat{F}'^{\alpha\ldots T}('x\ldots) = \widehat{F}'^{\alpha\ldots *}('x\ldots) \qquad \text{(A-3 bis. 20)}$$

and, from (A-3 bis. 16) ($\widehat{U}'^{\dagger} = U'^{-1}$)

$$\widehat{U}'^{\dagger *}\,\widehat{F}'^{\alpha\ldots T}('x\ldots)\,\widehat{U}'^* = c(L)\,L'^{\alpha}{}_{\alpha}\ldots\widehat{F}^{\alpha\cdots}(L^{-1}\,'x\ldots)\,. \quad \text{(A-3 bis. 21)}$$

Thus, formula (A-3 bis. 19) is identical with the second equation (A-3 bis. 17) and with (A-3 bis. 16).

In order to show the physical significance of (A-3 bis. 16) let us consider the *'real' scalar free field***) considered as an observable:

$$w(x) = w^T(x)\,;\ \widehat{w}(x) = \widehat{w}^{\dagger}(x)\,, \qquad \text{(A-3 bis. 22)}$$

expanded in plane waves (signat $(g^{\alpha\beta}) = (11\ldots 1-1)$).

$$\widehat{w}(x) = (2)^{-1/2}\,(2\,\pi)^{-d/2}\int d\sigma(\breve{k})\,(\widehat{a}(\breve{k})\,e^{i(\breve{k},x)} + \widehat{a}^{\dagger}(\breve{k})\,e^{-i(\breve{k},x)})\,,$$

$$\breve{k}^2 + M^2 = 0\,; \quad \breve{k}^n > |M|\,,$$

$$d\sigma(\breve{k}) = (\breve{k}^n)^{-1}\,d^d\,\breve{k}\,; \quad d^d\,\breve{k} = \prod_{i=1}^{i\ d} d\breve{k}^i\,, \quad \text{(A-3 bis. 23)***)}$$

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*) $'\widehat{\Psi} = \widehat{U}'\,\widehat{K}\,\Psi = \widehat{U}'\,\widehat{\Psi}^*$.

**) *'Field Quantization in Real Hilbert Space'* will be the object of a forthcoming paper in this journal (referred to as III).

***) The surface integral is to be extended over the *positive shell* of the hyperboloid $\breve{k}^2 + M^2 = 0$. $\breve{k} = \{\breve{k}^{\alpha}\}$ is therefore a *pseudochronous vector* (see III).

where, if the sum over plane waves is made denumerable, $\widehat{a}(\breve{k})$ and $\widehat{a}^{\dagger}(\breve{k})$ are the usual annihilation and creation operators of quanta in a state $\breve{k}$. Using (A-3 bis. 16) for the transformation $L = PT$

$$'x = (PT)\,x = -\,x\,; \quad 'x'^{\alpha} = -\,x'^{\alpha} \qquad \text{(A-3 bis. 24)}$$

we have (A-3 bis. 16):

$$\begin{aligned} '\widehat{w}('x) &= \widehat{w}(x) \equiv \widehat{w}(-'x) \\ &= (2)^{-1/2}\,(2\,\pi)^{-d/2} \int d\sigma(\breve{k})\;\big(\widehat{a}(\breve{k})\;e^{-i\,(\breve{k},\,'x)} + \widehat{a}^{\dagger}(\breve{k})\;e^{i\,(\breve{k},\,'x)}\big) \\ &= \widehat{U}'^{-1*}\;\widehat{w}^{*}('x)\;\widehat{U}'^{*} \\ &= (2)^{-1/2}\,(2\,\pi)^{-d/2} \int d\sigma(\breve{k})\;\big((\widehat{U}'^{*-1}\,\widehat{a}^{*}(\breve{k})\;\widehat{U}'^{*})\;e^{-i\,(\breve{k},\,'x)} + \\ &\quad + \widehat{U}'^{*-1}\,\widehat{a}^{\dagger\,*}(\breve{k})\;\widehat{U}'^{*})\;e^{i\,(\breve{k},\,'x)}\big)\,. \end{aligned} \qquad \text{(A-3 bis. 25)}$$

Thus, we have

$$\widehat{U}'^{-1*}\;\widehat{a}^{*}(\breve{k})\;\widehat{U}'^{*} = \widehat{a}(\breve{k})\,, \qquad \text{(A-3 bis. 26)}$$

$$\widehat{U}'^{-1*}\;\widehat{a}^{\dagger\,*}(k)\;\widehat{U}'^{*} = \widehat{a}^{\dagger}(\breve{k})\,. \qquad \text{(A-3 bis. 26)}$$

As the relations

$$[\widehat{a}(\breve{k}),\,\widehat{a}^{\dagger}(\breve{k}')] = \delta(\breve{k},\,\breve{k}')\,; \quad \delta(\breve{k},\,\breve{k}') = \breve{k}^{n}\,\delta(\vec{k} - \vec{k}')\,, \qquad \text{(A-3 bis. 27)}$$

$$[\widehat{a}(\breve{k}),\,\widehat{a}(\breve{k}')] = 0\,, \qquad \text{(A-3 bis. 28)}$$

are invariant, if we go over to the conjugate complex operators, an unitary matrix $\widehat{U}'^{*}$ exists always, satisfying (A-3 bis. 26). In particular, if the annihilation and creation operators are chosen real, $\widehat{U}'^{*} = 1$ is the unit operator.

References

[1]) E. C. G. Stueckelberg, Helv. Phys. Acta *33*, 727 (1960) to be referred to as I.
[2]) R. Jost and M. Guenin, unpublished.