Borchers' Classes and Duality Theorem

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Recently some attention has been focused on the so called 'duality theorem'[1]. In order to formulate a version of this theorem, we first introduce the following notations:

If B is an open space-time domain, then B' denotes the set of all space-time points which are space-like with respect to B. A space-time domain B for which $B = B''$ will be called a 'diamond'. Corresponding to every given field theory, there will be von Neumann algebras $R(B)$ generated by field operators of the open space-time domains B[2][3].

The 'duality theorem' states that if B is a diamond and $R(B)$ denotes the von Neumann algebra (associated with B) of a *local* field theory then

$$R(B') = R'(B) . \qquad (1)^{**})$$

This result has been proved for free-Bose fields[1]. Unfortunately it has not yet been possible to deduce the duality theorem from the usual postulates of Quantum Field Theory. Nevertheless we shall assume, in this note, that the duality theorem is true and point out some easily deducible, yet amusing consequences.

Proposition 1

Let $R(B)$ denote the von Neumann algebra (associated with the domain B) of a local field theory. Further let the 'duality theorem' be true for this field. Let $R_1(B)$ and $R_2(B)$ denote two other fields which are not necessarily local and for which the duality theorem is not necessarily true. If the fields R_1 and R_2 are local with respect to the field R then

$$R_1(B_1) \subseteq R_2'(B_2)$$

for every pair of domains B_1 and B_2 which are totally space-like with respect to each other (i. e. R_1 and R_2 are local with respect to each other).

Before proving this proposition, it may be noted that it is a slight paraphrase of a well-known theorem of Borchers[4]) and is basic for the theory of 'equivalence classes' (Borchers classes) of fields.

Proposition 1 may be proved with the help of the following

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**) It is known that relation 1 is not true (even in the case of free Bose field) for an *arbitrary* open domain B (cf. ref. [1])).

Lemma 1:

If B is a 'diamond' then

$$R_1(B) \subseteq R(B) \quad \text{and} \quad R_2(B) \subseteq R(B) .$$

Proof of Lemma 1

Since the field R_1 is local with respect to the field R, we have

$$R_1(B) \subseteq R'(B') . \tag{2}$$

Since the duality theorem is assumed to be true for the field $R(B)$ we have

$$R(B') = R'(B)$$

thus

$$R'(B') = R''(B) = R(B) \supseteq R_1(B) \tag{3}$$

One can similarly prove that

$$R_2(B) \subseteq R(B) .$$

The proof of proposition 1 may now be accomplished by noting that if B_1 and B_2 are any two domains, totally space-like with respect to each other, then there always exist 'diamonds' D_1 and D_2 such that

$$B_1 \subseteq D_1 ; \; B_2 \subseteq D_2$$

and D_1 and D_2 are totally space-like with respect to each other. We now have

$$\left.\begin{array}{l} R_1(B_1) \subseteq R_1(D_1) \subseteq R(D_1) ; \\ \text{and} \quad R_2(B_2) \subseteq R_2(D_2) \subseteq R(D_2) . \end{array}\right\} \tag{4}$$

Since D_1 and D_2 are totally space-like with respect to each other and since the field R is local, one has the relation

$$R(D_1) \subseteq R'(D_2) . \tag{5}$$

From (4) and (5) we obtain

$$R_1(B_1) \subseteq R(D_1) \subseteq R'(D_2) \subseteq R_2'(B_2) .$$

We now mention the following

Corollary to proposition 1

Let R be a local field and R_1 a field which is local with respect to R. If the duality theorem is true for both the fields then $R(B) = R_1(B)$ for every 'diamond' B.

Proof of the corollary

We have seen that the assumption that the field R satisfies the duality theorem implies that $R_1(B) \subseteq R(B)$ (Lemma 1). Since R_1 is also assumed to satisfy the duality theorem one obtains similarly $R(B) \subseteq R_1(B)$; hence $R(B) = R_1(B)$.

This corollary may be also considered as the algebraical expression of a result obtained by Epstein[5]) in the case of free fields.

From Lemma 1, it also follows that

$$R_1(\infty) \subseteq R(\infty) ; \; R_2(\infty) \subseteq R(\infty) \tag{6}$$

where ∞ means here the entire space-time. The usual proof of BORCHERS' theorem[4]) assume that $R(\infty)$ is irreducible but one can ask the question whether the condition (6) is not sufficient. Indeed we have the

Proposition 2

Let $A(x)$ be a local field with the algebra $R(\infty)$. If $B(x)$ and $C(x)$ are two fields (with corresponding algebras $R_1(\infty)$ und $R_2(\infty)$ respectively) which are local with respect to $A(x)$ and if $R_1(\infty)$ as well as $R_2(\infty)$ are subalgebras of $R(\infty)$ then $B(x)$ and $C(x)$ are local with respect to each other.

Proof of proposition 2

Proposition 2 is proved by BORCHERS[4]), with the additional assumption that $R(\infty)$ is irreducible. In order to prove proposition 2 without the irreducibility assumption we only have to note that the commutant $R'(\infty)$ of $R(\infty)$ is Abelian[2])[6]).

Since the algebra $R'(\infty)$ is contained in the algebra generated by the superselection observables[2]), it is reasonable to assume that operators in $R'(\infty)$ have discrete spectra. Hence the Hilbert space $\mathfrak{H}$ can be decomposed into a direct sum $\sum_{\oplus k} \mathfrak{H}_k$ and the algebra $R(\infty)$ into a direct sum $\sum_{\oplus k} R_k(\infty)$ where $R_k(\infty)$ denotes the algebra of all bounded operators in $\mathfrak{H}_k$. Since $R_{\frac{1}{2}}(\infty) \subseteq R(\infty)$ we obtain $R'(\infty) \subseteq R'_{\frac{1}{2}}(\infty)$.

Therefore the subspaces $\mathfrak{H}_k$ will also reduce the algebras $R_{\frac{1}{2}}(\infty)$ and hence $R_{\frac{1}{2}}(\infty)$ can be written as the direct sum

$$R_{\frac{1}{2}}(\infty) = \sum_{\oplus k} R^k_{\frac{1}{2}}(\infty) ,$$

with $R^k_{\frac{1}{2}}(\infty) \subseteq R^k(\infty)$. Thus if we confine ourselves to a subspace $\mathfrak{H}_k$ we have an irreducible field R_k. Since the fields R^k_1 and R^k_2 are local with respect to R_k, we conclude that R^k_1 and R^k_2 are local with respect to each other for all k. Hence the fields R_1 and R_2 which are the direct sum $\sum_{\oplus} R^k_1$ and $\sum_{\oplus} R^k_2$ respectively are also local with respect to each other.

N. B. In the proof of Proposition 2 we have assumed, for the sake of convenience, that the superselection observables have only discrete spectra. This assumption does not however seem to be necessary and one could carry through the proof by considering direct integral representation[7]) (instead of direct sum) of the field.

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