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Theory of Scattering of Identical Particles

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Abstract. The problem of a satisfactory formulation of the rigorous non-relativistic time-dependent scattering theory for systems with 2 identical particles is solved.

1. Introduction

It is well-known that the scattering problem in non-relativistic quantum mechanics for systems with identical particles needs a special investigation. The state functions must be (anti-) symmetric with respect to the exchange of each pair of identical particles, if they obey (Fermi-)Bose-statistics.

This problem was already investigated by MOTT and MASSEY[1]) on the basis of the non-rigorous description of the stationary-state approach of scattering. On the other hand, the recent rigorous, time-dependent approach to the theory of scattering[2])[3]) does not properly take into account the case of identical particles.

In this paper we give a satisfactory treatment of scattering systems with 2 identical particles**), using a many-component description, proposed by E. CORINALDESI[7]) and reformulating the time-dependent scattering theory developed by J. M. JAUCH[2])[3]) (2nd sect.). In the 3rd sect. we give two representative examples of scattering systems and discuss the formulae for the cross-section.

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2. Formalism for the Description of Identical Particles

Let the scattering system be described by the unitary group of operators, v_t, which is given by the Hamilton operator of the total system, $\mathcal{H}$, through

$$v_t = e^{-i\mathcal{H}t} \tag{2.1}$$

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**) The extension to more than 2 identical particles is straightforward.

and acting in the Hilbert space $\mathfrak{H}$ of states, in the representation of the coordinates $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots$ (or the momenta $\boldsymbol{p}_1, \boldsymbol{p}_2, \dots$ alternatively) of the occurring particles 1, 2, ...

For sake of simplicity, we consider in the following a system with exactly *2 identical particles* (among, possibly others, which are non-identical to each other). We denote them by the indices 1 and 2, and do not specify the rest.

Thus, the states of our system are described by square-integrable functions ψ (12).

v_t is defined on all of $\mathfrak{H}$. To give a correct description of our system, however, we must take only correctly (anti-) symmetrized functions in the arguments 1 and 2 in the case of 2 (fermions) bosons.

The following method allows a complete reformulation of the time-dependent scattering theory*).

In the following, we shall always use ε for

$$\varepsilon = \begin{cases} -1 & \text{for fermions} \\ +1 & \text{for bosons}. \end{cases} \tag{2.2}$$

Now, consider the direct sum of Hilbert spaces

$$\mathfrak{H}^{(2)} = \mathfrak{H} \oplus \mathfrak{H}\,, \tag{2.3}$$

$\mathfrak{H}$ being given in the realization of functions ψ (12) (see above). (2.3) may be represented alternatively as a direct product

$$\mathfrak{H}^{(2)} = V^{(2)} \otimes \mathfrak{H}\,, \tag{2.4}$$

$V^{(2)}$ is the 2-dimensional vector-space. The elements of $\mathfrak{H}^{(2)}$ are of the form

$$\begin{pmatrix} \psi_1(12) \\ \psi_2(12) \end{pmatrix} \quad (\psi_i \in \mathfrak{H}\,,\ i = 1, 2)\,. \tag{2.5}$$

The operator on $\mathfrak{H}^{(2)}$:

$$\mathfrak{P} = \frac{1}{2}\,(\mathbf{1} \otimes E + \tau_1 \otimes P)\,, \tag{2.6}$$

where $\mathbf{1}$ and E are the unity operators in $V^{(2)}$ and $\mathfrak{H}$ respectively, τ_1 is

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.7}$$

and P is defined on $\mathfrak{H}$ by

$$(P\,\psi)\ (12) = \psi(21) \quad \forall\, \psi \in \mathfrak{H}\,, \tag{2.8}$$

is a projection operator because of

$$\mathfrak{P}^2 = \mathfrak{P} = \mathfrak{P}^*\,, \tag{2.9}$$

which is easy to verify. The adjoint operator is defined in the usual way, based on the

*) We do not formulate the case of parastatistics, although it can be treated with this method, too.

definition of the scalar product given below by eq. (2.13). We shall restrict ourselves to the subspace $\mathcal{H}$ of $\mathfrak{H}^{(2)}$ given by the projection operator P

$$\mathfrak{H} = \{\Psi \mid \Psi = \mathfrak{P}\,\psi,\ \psi \in \mathfrak{H}^{(2)}\}\,. \tag{2.10}$$

From (2.6) and (2.10) it follows that the elements of $\mathcal{H}$ are all of the form

$$\Psi\,(12) = \begin{pmatrix} \psi\,(12) \\ \psi\,(21) \end{pmatrix} \qquad \psi \in \mathfrak{H} \tag{2.11}$$

or, more concisely

$$\Psi = \begin{pmatrix} \psi \\ P\,\psi \end{pmatrix} \qquad \psi \in \mathfrak{H}\,, \tag{2.12}$$

P is given by (2.8).

The scalar product in $\mathcal{H}$ is defined by

$$(\Psi, \Phi)_{\mathcal{H}} = (\psi, \varphi)_{\mathfrak{H}} + (P\,\psi, P\,\varphi)_{\mathfrak{H}} = 2\,(\psi, \varphi)_{\mathfrak{H}} \tag{2.13}$$

for all $\Psi = \binom{P\psi}{\psi}$, $\Phi = \binom{P\varphi}{\varphi}$, i.e. it is related (up to a factor) to the scalar product in $\mathfrak{H}$.

It follows that it has all properties of an ordinary scalar product.

The scattering system is now described by elements of the space $\mathcal{H}$; the unitary group V_t describing the time-evolution of the system is simply given by

$$V_t = \mathbf{1} \otimes v_t\,, \tag{2.14}$$

v_t defined in (2.1). Now, because particles 1 and 2 are indistinguishable, it follows that

$$[P, v_t] = 0 \qquad \forall\, t\,. \tag{2.15}$$

By (2.14) we have

$$[\mathfrak{P}, V_t] = 0 \qquad \forall\, t\,. \tag{2.16}$$

The operators $\mathbf{1} \times P$ and $\tau_1 \times E$ are identical on the space $\mathcal{H}$ since

$$\begin{aligned} (\mathbf{1} \otimes P)\ \Psi &= \begin{pmatrix} P\,\psi \\ \psi \end{pmatrix}, \\ (\tau_1 \otimes E)\,\Psi &= \begin{pmatrix} P\,\psi \\ \psi \end{pmatrix} \\ \text{for all} \quad \Psi &= \begin{pmatrix} \psi \\ P\,\psi \end{pmatrix} \in \mathfrak{H}\,. \end{aligned} \tag{2.17}$$

This will be used later.

To define now a scattering system we impose the following conditions:

1. *The asymptotic condition*

The strong limits

$$\underset{t \to t_{ex}}{s\text{-lim}}\ V_t^*\, U_t^{(\alpha)}\, \Phi^{(\alpha)} = \Phi_{ex}^{(\alpha)}\,, \tag{2.18}$$

should exist for some unitary groups $U_t^{(\alpha)}$ ($\alpha = 1, 2, \ldots ; -\infty \leqq t \leqq \infty$) of operators and for some $\Phi^{(\alpha)} \in \mathcal{H}$ for both $t_{ex} = -\infty$ and $t_{ex} = +\infty$.

The $U_t^{(\alpha)}$ have to fulfil the following requirements:

(α) They are diagonal in the vector space $V^{(2)}$.

(β) The infinitesimal generators of the groups $U_t^{(\alpha)}$ all have a purely continuous spectrum.

(γ)
$$[\mathfrak{P}, U_t^{(\alpha)}] = 0 \,, \tag{2.19}$$
$$[U_t^{(\alpha)}, U_{t'}^{(\beta)}] = 0 \,, \tag{2.20}$$

for all t, t', α, β.

(δ) The equation

$$\left.\begin{array}{c} U_t^{(\alpha)} \Phi = U_t^{(\beta)} \Phi \,, \\ \text{for all } t \text{ and } \alpha \neq \beta \text{ implies } \Phi = 0 \,. \end{array}\right\} \tag{2.21}$$

Condition 1 (δ) ensures us that the groups $U_t^{(\alpha)}$ and $U_t^{(\beta)}$ for $\alpha \neq \beta$ describe 'essentially different' channels. It implies the orthogonality of channels in the sense of equation (2.25).

The asymptotic condition (2.18) defines the (linear) wave operators $\Omega_{ex}^{(\alpha)}$:

$$\Omega_{ex}^{(\alpha)} \Phi^{(\alpha)} = \Phi_{ex}^{(\alpha)} \,. \tag{2.22}$$

The $\Omega_{ex}^{(\alpha)}$ are linear partial isometries and can hence be extended to the whole space $\mathcal{H}$. (The proof is identical to the proof given in [2]) and [3]).) Similarly we define the adjoint operators $\Omega_{ex}^{(\alpha)*}$ by

$$(\Phi, \Omega_{ex}^{(\alpha)} \Psi) = (\Omega_{ex}^{(\alpha)*} \Phi, \Psi), \quad \forall \Phi, \Psi \in \mathfrak{H} \,. \tag{2.23}$$

From the linear isometry property of the $\Omega_{ex}^{(\alpha)}$ it follows that

$$F_{ex}^{(\alpha)} = \Omega_{ex}^{(\alpha)} \Omega_{ex}^{(\alpha)*} \tag{2.24}$$

are projection operators on the range $R_{ex}^{(\alpha)}$ of $\Omega_{ex}^{(\alpha)}$. From condition 1 (δ) it follows:

$$F_{ex}^{(\alpha)} F_{ex}^{(\beta)} = 0 \tag{2.25}$$

for $\alpha \neq \beta$ and both cases of "*ex*". For the proof see again [3]).

2. *Completeness relations*

The ranges $R_{ex}^{(\alpha)}$ of $\Omega_{ex}^{(\alpha)}$ satisfy the relations

$$\overline{\{R_+^{(\alpha)}\}} = \overline{\{R_-^{(\alpha)}\}} = N \,, \tag{2.26}$$

where

$$N = V^{(2)} \otimes \mathfrak{N} \tag{2.27}$$

is the continuum subspace of V_t. From (2.25) and (2.26) it follows

$$\sum_\alpha F_+^{(\alpha)} = \sum_\alpha F_-^{(\alpha)} = E_N \tag{2.28}$$

where E_N is the projection operator on N.

Note: In the case of only one solution U_t of (2.18) with $R_{ex} = N$ we call the system a *simple scattering system*; if more solutions are present, it is called a *multi-channel system*.

The S-operator is now defined by

$$S = \sum_{\alpha} \Omega_{+}^{(\alpha)} \, \Omega_{-}^{(\alpha)\,*} \,. \tag{2.29}$$

The sum extends over all distinct channels. In the case of infinitely many channels the sum converges in the strong topology on all of $\mathcal{H}$. Moreover, it is independent of the order of summation. Finally, the S-operator is unitary on N

$$S^* \, S = S \, S^* = E_N \tag{2.30}$$

(cf. [3])).

Up to here, we have developed this formalism quite analogous to the ordinary 1-component one, given in [2]) and [3]). To find now a correspondence between both descriptions, we define the projection operator

$$\pi_\varepsilon = \frac{1}{2} \left[\mathbf{1} \otimes E + \varepsilon \, \tau_1 \otimes E \right] . \tag{2.31}$$

The relations

$$\pi_\varepsilon^2 = \pi_\varepsilon = \pi_\varepsilon^* \,, \tag{2.32}$$

follow immediately (and hence π_ε is a projection). The subspace of π_ε we call

$$\mathcal{H}_\varepsilon = \{ \pi_\varepsilon \Phi \mid \Phi \in \mathcal{H} \} \,. \tag{2.33}$$

In the one-component description the states of the physical system are described by all the elements $\varphi \in \mathfrak{H}$ if the particles 1 and 2 are *non-identical*, and by all $\varphi^{(\varepsilon)} \in \mathfrak{H}$, where

$$P \, \varphi^{(\varepsilon)} = \varepsilon \, \varphi^{(\varepsilon)} \tag{2.34}$$

if the particles 1 and 2 are *identical*.

Theorem 1. The relation (2.11)

$$\Phi = \begin{pmatrix} \varphi \\ P \, \varphi \end{pmatrix},$$

gives a one-to-one correspondence between the elements of the spaces $\mathcal{H}$ and $\mathfrak{H}$ on the one hand, and between the elements of the spaces $\mathcal{H}_\varepsilon$ and $\mathfrak{H}_\varepsilon$ on the other hand. Here, $\mathcal{H}_\varepsilon$ is given by (2.33) and $\mathfrak{H}_\varepsilon$ by

$$\mathfrak{H}_\varepsilon = \{ \varphi^{(\varepsilon)} \mid P \, \varphi^{(\varepsilon)} = \varepsilon \, \varphi^{(\varepsilon)} \in \mathfrak{H} \} \,. \tag{2.35}$$

Proof. By (2.11), to every $\varphi \in \mathfrak{H}$ there corresponds one element $\Phi \in \mathcal{H}$. From $\varphi \neq \varphi'$ it follows

$$\begin{pmatrix} \varphi \\ P \, \varphi \end{pmatrix} - \begin{pmatrix} \varphi^1 \\ P \, \varphi^1 \end{pmatrix} = \begin{pmatrix} \varphi - \varphi^1 \\ P \, (\varphi - \varphi^1) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

hence to different φ correspond different Φ, and vice-versa. Furthermore, from $\Phi \in \mathcal{H}_\varepsilon$ it follows:

$$\Phi = \pi_\varepsilon \Phi = \frac{1}{2} \begin{pmatrix} \varphi + \varepsilon P \varphi \\ P \varphi + \varepsilon \varphi \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \otimes (\varphi + \varepsilon P \varphi) = \frac{1}{2} \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \otimes \varphi^{(\varepsilon)} . \tag{2.36}$$

Hence every element in $\mathcal{H}_\varepsilon$ is of the form (2.36), where $\varphi^{(\varepsilon)} \in \mathfrak{H}_\varepsilon$, and to two different $\varphi_1^{(\varepsilon)} \neq \varphi_2^{(\varepsilon)}$ in $\mathfrak{H}_\varepsilon$ correspond two different $\Phi \in \mathfrak{H}_\varepsilon$:

$$\Phi_i = \frac{1}{2} \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \otimes \varphi_i^{(\varepsilon)} . \tag{2.37}$$

q. e. d.

We now examine the asymptotic condition (2.18). Because of (2.13):

$$|| \Phi ||^2_{\mathcal{H}} = || \varphi ||^2_{\mathfrak{H}} + || P \varphi ||^2_{\mathfrak{H}} = 2 \, || \varphi ||^2_{\mathfrak{H}}$$

we obtain:

$$|| V_t^* \, U_t^{(\alpha)} \Phi^{(\alpha)} - \Phi_{ex}^{(\alpha)} \, ||^2_{\mathcal{H}} = || \, v_t^* \, u_{1t}^{(\alpha)} \, \varphi^{(\alpha)} - \varphi_{ex}^{(\alpha)} \, ||^2_{\mathfrak{H}} + || \, v_t^* \, u_{2t}^{(\alpha)} \, P \varphi^{(\alpha)} - P \varphi_{ex}^{(\alpha)} \, ||^2_{\mathfrak{H}} \, . \tag{2.38}$$

Because of (2.15) and condition 1 (α) we find:

$$U_t^{(\alpha)} = \begin{pmatrix} u_{1t}^{(\alpha)} & 0 \\ 0 & u_{2t}^{(\alpha)} \end{pmatrix}, \quad u_{2t}^{(\alpha)} = P \, u_{1t}^{(\alpha)} \, P^{-1} \, . \tag{2.39}$$

Hence both summands on the r.h.s. of Equation (2.38) are equal. It follows that the asymptotic condition (2.18) implies and is implied by the ordinary asymptotic condition in the 1-component space $\mathfrak{H}$.

Furthermore, from

$$|| \, \Psi_t \, ||^2_{\mathcal{H}} = || \, \pi_\varepsilon \, \Psi_t \, ||^2_{\mathcal{H}} + || \, (\mathbf{1} - \pi_\varepsilon) \, \psi_t \, ||^2_{\mathcal{H}} \, , \tag{2.40}$$

which is true for any $\Psi_t \in \mathcal{H}$, especially for

$$\Psi_t = V_t^* \, U_t^{(\alpha)} \, \Phi^{(\alpha)} - \Phi_{ex}^{(\alpha)}$$

it follows that the asymptotic condition (2.18) (and hence the 1-component one, too) imply together with (2.40):

$$\lim_{t \to t_{ex}} \, || \, \pi_\varepsilon \, (V_t^* \, U_t^{(\alpha)} \, \Phi^{(\alpha)} - \Phi_{ex}^{(\alpha)} \, || = 0 \, , \tag{2.41}$$

which can be written in one component as

$$\underset{t \to t_{ex}}{s\text{-lim}} \, v_t^* \, (1 + \varepsilon \, P) \, (u_{1t}^{(\alpha)} \, \varphi^{(\alpha)}) = \varphi_{ex}^{(\alpha)\,(\varepsilon)} \, . \tag{2.42}$$

Condition (2.42) was already proposed by BRENIG and R. HAAG[4]) as a modified condition in the case of identical particles. This proves our second theorem:

Theorem 2. The asymptotic condition (2.18) in the space $\mathcal{H}$ is equivalent to the one-component one in the space $\mathfrak{H}$. Restricted on the subspace $\mathcal{H}_\varepsilon$, condition (2.18) is equivalent to condition (2.42).

The S-operator is defined by (2.29) on the whole space $\mathcal{H}$. But in the case of identical particles, only the subspace $\mathcal{H}_\varepsilon$ is related to the physical system. Hence we must take the restriction $S^{(\varepsilon)}$ of S on $\mathcal{H}_\varepsilon$, v.i.z.

$$S^{(\varepsilon)} = \pi_\varepsilon \, S \, \pi_\varepsilon \, . \tag{2.43}$$

The following theorem justifies now our method.

(*Main*)-*Theorem 3.* The S-operator defined by (2.29) commutes with π_ε

$$[S, \pi_\varepsilon] = 0 \, . \tag{2.44}$$

The restriction $S^{(\varepsilon)}$ of S on $\mathcal{H}_\varepsilon$ is unitary in the continuum space to the Hamiltonian H, restricted to $\mathcal{H}_\varepsilon$.

Proof. To any unitary group $U_t^{(\alpha)}$ satisfying the asymptotic condition (2.18) there correspond by (2.16) and (2.17) another one, $U_t^{(\beta)}$ given by

$$U_t^{(\beta)} = (\tau_1 \otimes E) \, U_t^{(\alpha)} \, (\tau_1 \otimes E) \, , \tag{2.45}$$

satisfying condition (2.18) too.

Hence the corresponding wave operators are related by

$$\Omega_{ex}^{(\beta)} = (\tau_1 \otimes E) \; \Omega_{ex}^{(\alpha)} \, (\tau_1 \otimes E) \, . \tag{2.46}$$

Of course

$$(\tau_1 \otimes E)^{-1} = \tau_1 \otimes E \, .$$

Now, we distinguish 2 cases:

(a) The channels α and β are essentially different by condition ($1\,\delta$). Then, in the sum of the S-operator both summands occur:

$$\Omega_+^{(\alpha)} \, \Omega_-^{(\alpha)\,*} + \Omega_+^{(\beta)} \, \Omega_-^{(\beta)\,*} = \Omega_+^{(\alpha)} \, \Omega_-^{(\alpha)\,*} + (\tau_1 \otimes E) \, \Omega_+^{(\alpha)} \, \Omega_-^{(\alpha)\,*} \, (\tau_1 \otimes E) \tag{2.47}$$

by Equation (2.46).

(b) α and β are *not* essentially different; but then, $U_t^{(\alpha)}$ (and of course $U_t^{(\beta)}$) must commute with $\tau_1 \otimes E$ (because it commutes with $(\mathbf{1} \otimes P)$; cf. (2.17) and (2.19))

It follows:

$$[\pi_\varepsilon, U_t^{(\alpha)}] = 0 \quad \forall t$$

and hence

$$[\pi_\varepsilon, \Omega_{ex}^{(\alpha)}] = 0 \, . \tag{2.48}$$

Now, we write the S-operator (because of the independence of the sum with respect to the order of summation):

$$\begin{aligned} S_{(a)} &= \sum_{\alpha \text{ of case } (a)} [\Omega_+^{(\alpha)} \, \Omega_-^{(\alpha)\,*} + (\tau_1 \otimes E) \, \Omega_+^{(\alpha)} \, \Omega_-^{(\alpha)\,*} \, (\tau_1 \otimes E)] \, , \\ S_{(b)} &= \sum_{\beta \text{ of case } (b)} \Omega_+^{(\beta)} \, \Omega_-^{(\beta)\,*} \\ S &= S_{(a)} + S_{(b)} \, . \end{aligned} \tag{2.49}$$

Every term in $S_{(a)}$, as well as in $S_{(b)}$ commutes with π_ε, hence we have proved:

$$[\pi_\varepsilon \, S] = 0 \, . \tag{2.50}$$

From the unitarity of S (2.30) on N and (2.50) it follows:

$$S^{(\varepsilon)*} S^{(\varepsilon)} = \pi_\varepsilon S^* S \pi_\varepsilon = \pi_\varepsilon \tag{2.51}$$

and

$$S^{(\varepsilon)} S^{(\varepsilon)*} = \pi_\varepsilon S S^* \pi_\varepsilon = \pi_\varepsilon , \tag{2.51}$$

hence $S^{(\varepsilon)}$ in unitary on $N \cap \mathcal{H}_\varepsilon$.
We must finally check that the numerical factors in the expression of the transition matrix elements are correct. This we show by considering 2 examples of potential scattering systems in the following section.

3. 2 Illustrative Examples

(a) First we consider a single channel scattering system of 2 identical particles, described by the self-adjoint (Hamilton-) operator

$$\mathcal{H} = \frac{p_1^2}{2\,m} + \frac{p_2^2}{2\,m} + V . \tag{3.1}$$

The operator V is defined in the $\boldsymbol{x}$-representation by

$$(V \varphi)\,(\boldsymbol{x}_1\,\boldsymbol{x}_2) = V\,(\boldsymbol{x}_1 - \boldsymbol{x}_2)\,\varphi\,(\boldsymbol{x}_1\,\boldsymbol{x}_2) , \tag{3.2}$$

where $V(\xi)$ is square integrable: $(\xi = \boldsymbol{x}_1 - \boldsymbol{x}_2)$

$$\int |\,V(\xi)\,|^2\, d^3\xi < \infty . \tag{3.3}$$

Henceforth, we treat the problem in its centre-of-mass-system. Because of the identity of particles 1 and 2 we have:

$$V\,(-\,\xi) = V(\xi) . \tag{3.4}$$

Then, by a theorem by J. M. Cook[5]) (see, also J. M. Jauch and I. I. Zinnes[6])) the one-component asymptotic condition is satisfied, and, by Theorem 2 the many component one, (2.18) too. Furthermore, for a sufficiently well-behaved function $V(\xi)$ condition 2 (2.26) is satisfied too, for the (elastic) channel given by

$$H_0 = \mathbf{1} \otimes \mathcal{H}_0 , \qquad \mathcal{H}_0 = \frac{p_1^2}{2\,m} + \frac{p_2^2}{2\,m} \tag{3.5}$$

The S-operator is defined by

$$S = \Omega_+\,\Omega_-^* \tag{3.6}$$

which, when restricted to the subspace $\mathcal{H}_\varepsilon$ gives the correct physical information.

If, e.g. we are asking for the scattering cross-section for the elastic scattering of a particle from the direction $(\theta_i\,\phi_i)$ into the direction $(\theta_f \phi_f)$ (with respect to a fixed frame of reference) θ, ϕ are the polar angles of the particle momentum), then the answer is given by the formula:

$$\sigma_{i \to f} = \left(\frac{2\,\pi}{k}\right)^2 |\,(\theta_f\,\phi_f\,|\,S^{(\varepsilon)}\,|\,\theta_i\,\phi_i)\,|^2 , \tag{3.7}$$

where the state vectors $(\theta_i\,\phi_i) \in \mathcal{H}_\varepsilon$ and $(\theta_f\,\phi_f) \in \mathcal{H}_\varepsilon$ are both normalized to unity (cf. (2.13)).

From $(\theta\phi) = \Psi \in \mathcal{H}_\varepsilon$ it follows:

$$\Psi = \pi_\varepsilon \Psi = \frac{1}{2} \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \otimes \psi^{(\varepsilon)} ,$$

with

$$\| \psi^{(\varepsilon)} \|_{\mathfrak{H}} = \sqrt{2} \qquad \| \Psi \|_{\mathcal{H}} = 1 .$$

Hence

$$\sigma_{i\to f} = \left(\frac{2\pi}{k}\right)^2 \frac{|\,(\theta_f\,\phi_f)^{(\varepsilon)}\, s\, (\theta_i\,\phi_i)^{(\varepsilon)}\,|^2}{\|(\theta_f\,\phi_f)^{(\varepsilon)}\|^2\, \|(\theta_i\,\phi_i)^{(\varepsilon)}\|^2} \tag{3.8}$$

with $(\theta, \phi)^{(\varepsilon)} = \Psi^{(\varepsilon)}$;

s is the diagonal element of $S^{(\varepsilon)}$ and correspond to the s-operator in the one-component formalism. An explicit calculation of (3.8) yields:

$$\sigma_{i\to f} = \left(\frac{2\pi}{k}\right)^2 \frac{|\,((\theta_f\,\phi_f)\, s\, (\theta_i\,\phi_i)) + \varepsilon\, ((\pi-\theta_f,\, \phi_f+\pi)\, s\, (\theta_i\,\phi_i))\,|^2}{\|(\theta_f\,\phi_f) + \varepsilon\,(\pi-\theta_f,\, \phi_f+\pi)\|^2\, \|(\theta_i\,\phi_i) + \varepsilon\,(\pi-\theta_i,\, \phi_i+\pi)\|^2} , \tag{3.9}$$

$\psi = (\theta, \phi)$ are denoting here one-component non-symmetrized state functions describing a particle with momentum in the direction given by the polar angles θ and ϕ. Representing a state with a sharply peaked momentum around $\boldsymbol{k} = \boldsymbol{k}_0 = (k, \theta, \phi)$ by using the dense set of functions

$$\left.\begin{aligned} &(\theta, \phi) = N(\beta)\, e^{-\beta\,(\boldsymbol{k}-\boldsymbol{k}_0)^2} , \\ &\text{where } N(\beta) \text{ is defined by} \\ &\int |\,(\theta, \phi)\,|^2\, d^3 k = 1 \end{aligned}\right\} \tag{3.10}$$

and great (but finite) β, we get:

$$\| \psi^{(\varepsilon)} \|^2 = \| (\theta, \phi) + \varepsilon\,(\pi-\theta, \phi+\pi) \|^2 = 2\, \| (\theta, \phi) \|^2 + \sigma\,(e^{-2\beta k^2}) . \tag{3.11}$$

The correction term $O\,(e^{-2\beta k^2})$ is for great β as small as desired. Finally we get:

$$\sigma_{i\to f} = \left(\frac{2\pi}{k}\right)^2 |\, f_{if}\,(\theta, \phi) + \varepsilon\, f_{if}\,(\pi-\theta, \phi+\pi)\,|^2 + \sigma\,(e^{-2\beta k^2}) ,$$

with the scattering amplitudes defined by:

$$f_{if}\,(\theta, \phi) = \frac{((\theta_f\,\phi_f)\, s\, (\theta_i\,\phi_i))}{\|(\theta_f\,\phi_f)\|\ \|(\theta_i\,\phi_i)\|} . \tag{3.12}$$

θ and ϕ being the difference angles between $(\theta_i\,\phi_i)$ and $(\theta_f\,\phi_f)$. Equation (3.12) corresponds (for $\beta \to \infty$) to the prescription given by Mott and Massey[1]).

(b) Another example, showing a multi-channel scattering system, is given by 3 interacting particles 1, 2, 3; let particles 1 and 2 be identical. We take a Hamilton-operator of the form

$$\mathcal{H} = \frac{p_1^2}{2\,m} + \frac{p_2^2}{2\,m} + \frac{p_3^2}{2\,m_3} + V_{12} + V_{13} + V_{23} , \tag{3.13}$$

with a purely continuous spectrum. In the $\boldsymbol{x}$-representation, the operators V_{ik} are defined by:

$$(V_{ik}\,\varphi)\,(\boldsymbol{x}_1\,\boldsymbol{x}_2\,\boldsymbol{x}_3) = V_{ik}\,(\boldsymbol{x}_i - \boldsymbol{x}_k)\,\varphi\,(\boldsymbol{x}_1\,\boldsymbol{x}_2\,\boldsymbol{x}_3) \tag{3.14}$$

(for $i < k = 1, 2, 3$). Because of the identity of particles 1 and 2 we have

$$V_{13}(\xi) = V_{23}(\xi) \qquad V_{12}(\xi) = V_{21}(\xi)\,, \tag{3.15}$$

where ξ denotes the difference of the corresponding coordinates. Moreover, let the condition

$$\int |\, V_{ik}(\xi)\,|^2\, d^3\xi \leqq 1 \qquad \forall\, i, k = 1, 2, 3\,, \tag{3.16}$$

be satisfied. We assume V_{13} (and by 3.15) V_{23}, too) to allow bound states, i.e. the operator

$$h(\xi) = -\frac{1}{2\,\mu}\,\frac{d^2}{d\xi^2} + V_{13}(\xi) \quad (\mu = m + m_3) \tag{3.17}$$

has, besides the continuum, negative discrete eigenvalues. V_{12} allows no bound states. The last condition can be easily dropped out; we take it for sake of simplicity. For the moment we assume that particles 1 and 2 are 'similar' but not identical. By this we mean that the masses are equal:

$$m_1 = m_2 \qquad \text{see (3.13)}$$

and that (3.15) holds. Then M. N. HACK[6]) has shown that the following channel Hamiltonians (resp. the associated unitary groups $u_t^{(\alpha)} = e^{-i\,\mathcal{H}_\alpha t}$) satisfy the asymptotic condition

$$\begin{aligned}
\mathcal{H}_1 &= \frac{\boldsymbol{p}_1^2}{2\,m} + \frac{\boldsymbol{p}_2^2}{2\,m} + \frac{\boldsymbol{p}_3^2}{2\,m_3} \\
\mathcal{H}_{2\alpha} &= \frac{(\boldsymbol{p}_1+\boldsymbol{p}_3)^2}{2\,(m+m_3)} + \frac{\boldsymbol{p}_2^2}{2\,m} - |\,\varepsilon_\alpha\,| \\
\mathcal{H}_{3\alpha} &= \frac{\boldsymbol{p}_1^2}{2\,m} + \frac{(\boldsymbol{p}_2+\boldsymbol{p}_3)^2}{2\,(m+m_3)} - |\,\varepsilon_\alpha\,|
\end{aligned} \tag{3.18}$$

ε_α are the (negative) discrete eigenvalues of the operator (3.17). Now we prove the

Theorem 4. The unitary groups

$$u_t^{(\beta)} = e^{-i\,\mathcal{H}_\beta\, t}$$

with $\mathcal{H}_\beta$ of the form (3.18) which satisfy the asymptotic condition, satisfy also the 3 requirements: 1.(β), 1.(γ), Equation (2.20) and 1.(δ), (which are the usual requirements in the one-component scattering formalism (cf. J. M. JAUCH[3]))).

Proof. Condition 1. (γ), Equation (2.20), follows directly from the commutation rules;

$$[p_{i\mu}\, p_{j\nu}] = 0 \qquad i, j;\ \mu, \nu = 1, 2, 3. \tag{3.19}$$

(3.19) imply $[\mathcal{H}_\alpha\, \mathcal{H}_\beta] = 0$ and hence

$$[u_t^{(\alpha)}\, u_t^{(\beta)}] = 0 \qquad \forall\, \alpha, \beta, t, t^1$$

Condition 1. (β) follows now from (3.19) and the fact that p and hence p^2 have a purely continuous spectrum, and so have $(p_1 + p_2)^2$ and the sum of the $\lambda\, p^2$; this implies that the $\mathcal{H}_\alpha$ have a continuous spectrum and hence $u_t^{(\alpha)}$. The third condition follows from the equations (in the momentum representation):

$$\left(e^{-it(\mathcal{H}_\alpha - \mathcal{H}_\beta)}\,\varphi\right)(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \left(e^{-it(\mathcal{H}_\alpha(k) - \mathcal{H}_\beta(k))}\right)\varphi\,(\boldsymbol{k}_1\,\boldsymbol{k}_2\,\boldsymbol{k}_3)\,,$$

where $\mathcal{H}_\alpha(k)$ and $\mathcal{H}_\beta(k)$ are the Hamilton-*functions* of k_1, k_2, k_3 corresponding to (3.18). Now:

$$u_t^{(\alpha)}\,\varphi = u_t^{(\beta)}\,\varphi \quad \text{for all } t,$$

implies:

$$e^{-it\left[\mathcal{H}_\alpha(k)\,\mathcal{H}_\beta(k)\right]}\,\varphi\,(\boldsymbol{k}_1\,\boldsymbol{k}_2\,\boldsymbol{k}_3) = \varphi\,(\boldsymbol{k}_1\,\boldsymbol{k}_2\,\boldsymbol{k}_3) \quad \text{for all } t,\ \boldsymbol{k}_1,\ \boldsymbol{k}_2,\ \boldsymbol{k}_3$$

and hence $\mathcal{H}_\alpha(k) = \mathcal{H}_\beta(k)$ as a function of $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$, i.e. $\alpha = \beta$ or

$$\varphi \equiv 0 \qquad \text{q.e.d.}$$

Now again, as in example (a) we assume that the V_{ik} are regular enough such that the completeness relations (2.26) (in the one-component formalism) are satisfied.

To take correctly into account the identity of the particles 1 and 2, we apply our 2-component formalism. By theorem 2, it follows that the channels

$$H_1 = \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_1 \end{pmatrix} \qquad H_{2\alpha} = \begin{pmatrix} \mathcal{H}_{2\alpha} & 0 \\ 0 & \mathcal{H}_{3\alpha} \end{pmatrix} \qquad H_{3\alpha} = \begin{pmatrix} \mathcal{H}_{3\alpha} & 0 \\ 0 & \mathcal{H}_{2\alpha} \end{pmatrix}, \tag{3.20}$$

satisfy the asymptotic condition (2.18). The index α distinguishes the different channel energies ε_α of which each one gives rise to a distinct channel. By theorem 4 and their diagonality in $V^{(2)}$ the requirements 1 (α), (β), (γ) and (δ) are satisfied for the operators (3.20). Again assuming well-behaved potentials V_{ik} the completeness relations (2.26) are satisfied.

To describe the time-evolution of an asymptotic state, in which one of the two identical particles is bound to the particle 3, and the other is far away from them, we have to take the projection of

$$\Phi_t\,(13;2) = U_t^{(2,\alpha)}\,\Phi\,(13;2)\,, \tag{3.21}$$

on the subspace $\mathcal{H}_\varepsilon$:

$$\Phi_t^{(\varepsilon)}\,(p\,3,\,p) = \pi_\varepsilon\,\Phi_t\,(13;2)\,, \tag{3.22}$$

where p denotes either particle 1 or particle 2. (A more concrete example consists of taking a system of 2 protons and one neutron. The above-described state then corresponds to a free deuteron plus a free proton.)

Let us study now the transition matrix element of the reaction

$$(p\,3,\,p) \to (p\,3,\,p)$$

We denote the corresponding scattering states by Φ_i and Φ_f respectively. With:

$$\Phi_i,\,\Phi_f \in \mathcal{H}_\varepsilon\,,$$

then the matrix element M_{if} is given by:

$$M_{if} = \frac{(\Phi_f\,S^{(\varepsilon)}\,\Phi_i)}{\|\Phi_f\|\ \|\Phi_i\|}\,, \tag{3.23}$$

which can be written as:

$$M_{if} = \frac{\left(\frac{1}{2}\binom{1}{\varepsilon} \otimes \varphi_f^{(\varepsilon)},\ S^{(\varepsilon)}\,\frac{1}{2}\binom{1}{\varepsilon} \otimes \varphi_i^{(\varepsilon)}\right)}{\left\| \frac{1}{2}\binom{1}{\varepsilon}\varphi_f^{(\varepsilon)} \right\| \left\| \frac{1}{2}\binom{1}{\varepsilon}\varphi_i^{(\varepsilon)} \right\|} = \frac{\left(\varphi_f^{(\varepsilon)}\ s\ \varphi_i^{(\varepsilon)}\right)}{\left\| \varphi_f^{(\varepsilon)} \right\| \left\| \varphi_i^{(\varepsilon)} \right\|}$$

$$= \left[(\varphi_f\ s\ \varphi_i) + \varepsilon\,(P\,\varphi_f\ s\ \varphi_i) + \varepsilon\,(\varphi_f\ s\ P\,\varphi_i) + (P\,\varphi_f\ s\ P\,\varphi_i)\right] \frac{1}{\left\| \varphi_f^{(\varepsilon)} \right\| \left\| \varphi_i^{(\varepsilon)} \right\|}\,,$$

or:

$$M_{if} = \frac{(\varphi_f\ s\ \varphi_i) + \varepsilon\,(P\,\varphi_f\ s\ \varphi_i)}{\| \varphi_f \|\ \| \varphi_i \|}\,. \tag{3.24}$$

The last Equation (3.24) holds for the case where the free particle p and the bound state p 3 are infinitely far away from each other such that

$$(\varphi_f\ P\,\varphi_f) = 0\,.$$

P is, as usual, the permutation operator of particles 1 and 2, and the diagonal element of the operator S in $V^{(2)}$. (3.24) shows that the transition matrix element is a sum of a 'direct' term (1st summand) and an 'exchange' term (2nd summand) in full agreement with the prescription by Mott and Massey[1]). The cross-section is again proportional to

$$\sigma_{if} \sim |\, M_{if} \,|^2\,.$$

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