

**Zeitschrift:** Helvetica Physica Acta  
**Band:** 39 (1966)  
**Heft:** 5

**Artikel:** On the spectra of Schrödinger multiparticle hamiltonians  
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**DOI:** <https://doi.org/10.5169/seals-113698>

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# On the Spectra of Schrödinger Multiparticle Hamiltonians

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(23. V. 66)

*Abstract.* For two-body potentials which are locally square-integrable and vanish (arbitrarily slowly) at infinity, the spectrum of the  $N$ -particle Hamiltonian in the center-of-mass frame is shown to have the form which is commonly expected: it consists of a continuum  $\sigma_c$  (generated by states in which the system is broken up into independent parts, and characterized, therefore, in terms of the Hamiltonians of these parts), and, in the complement of  $\sigma_c$ , of eigenvalues only, which are of finite multiplicity and can accumulate at most at the lower end of  $\sigma_c$ . Some properties of the corresponding bound-state wave-functions are derived, and a problem is posed concerning the generalization of Faddeev's equations to  $N > 3$ .

## 1. Introduction

For the purposes of time-independent scattering theory, S. WEINBERG [1]<sup>1)</sup> and C. VAN WINTER [2] have independently derived a functional equation for the  $N$ -particle Green's function which is superior, in many respects, to the usual resolvent equations. In the present paper, we want to apply this functional equation to the simpler task of discussing the bound states below the continuum. Some results in this direction have also been obtained by C. VAN WINTER [2, 3], but since our objective here is not scattering theory, we can manage with less restrictive assumptions on the potentials. Essentially, we deal with potentials which are locally square-integrable and vanish arbitrarily slowly at infinity (for the precise hypothesis see Theorem 2). The existence of the continuum, which can be inferred from time-dependent scattering theory in the case of short-range forces, is established as a consequence of the spatial cluster-decomposition properties of the Hamiltonian.

## 2. The Spectrum of $H$

Using customary notation, we first recall some properties of the  $N$ -particle Hamiltonian

$$H = \sum_{i=1}^N \frac{P_i^2}{2m_i} + \sum_{\alpha} V_{\alpha}(x_{\alpha}) = H_0 + V \quad (1)$$

on the Hilbertspace  $\mathfrak{H} = L^2(R^{3N})$ . For simplicity, we only treat the case of two-body forces.  $\alpha$  labels the  $\binom{N}{2}$  pairs of different particles, and  $x_{\alpha}$  denotes the relative coordinates of the pair  $\alpha$ . The potentials  $V_{\alpha}$  are always supposed to be real.

<sup>1)</sup> Numbers in brackets refer to References, page 462.

*Theorem 1* (KATO [4])

For any  $\alpha$ , let

$$V_\alpha(\cdot) \in L^2(R^3) + L^\infty(R^3)^2. \quad (2)$$

Then  $D(V_\alpha) \supset D(H_0)$  and, for any  $a > 0$ , there exists  $b < \infty$  such that

$$\|V_\alpha u\| \leq a \|H_0 u\| + b \|u\| \quad (3)$$

for all  $\alpha$  and all  $u \in D(H_0)$ .  $H_0 + V$ , defined on  $D(H_0)$ , is self-adjoint and bounded from below.

If the potentials  $V_\alpha(x_\alpha)$  vanish as  $|x_\alpha| \rightarrow \infty$ , we expect that the  $N$ -particle system can break up into parts which are independent, if they are far separated from each other. To formulate this, let  $D_k = (C_1 \dots C_k)$  be a partition of the set of particles  $(1 \dots N)$  into  $k$  subsets (clusters), and let  $I_{D_k}$  be the sum of all two-body potentials which link particles in different clusters. We define

$$H_{D_k} = H - I_{D_k}, \quad (4)$$

so that the Hamiltonian  $H_{D_k}$  describes the system of noninteracting clusters  $C_1 \dots C_k$ . Let  $u$  be a state of the  $N$ -particle system in which the clusters are far separated from each other, then  $H u = H_{D_k} u$ , approximately. Therefore, we expect

$$\sigma(H_{D_k}) \subset \sigma(H), \quad (5)$$

where  $\sigma(A)$  denotes the spectrum of  $A$ . From here on, we fix the center-of-mass of the  $N$ -particle system without changing the notation:  $H_0$ ,  $H$ , and  $H_{D_k}$  are now Hamiltonians in the center-of-mass frame, acting on a Hilbertspace  $\mathfrak{H} = L^2(3^{N-3})$ . Except for the explicit form (1) of  $H$ , everything said so far applies word for word in this modified situation. If governed by the Hamiltonian  $H_{D_k}$ , the clusters  $C_1 \dots C_k$  ( $k \geq 2$ ) can still move freely relative to each other, hence  $\sigma(H_{D_k})$  is still continuous, extending from some real number  $\varepsilon_{D_k}$  to  $+\infty$ . Defining

$$\sigma_c = \bigcup_{D_k, k \geq 2} \sigma(H_{D_k}) = [\min_{D_k, k \geq 2} \varepsilon_{D_k}, +\infty), \quad (6)$$

we can now state our main result, which will be proved in Sections 4–6:

*Theorem 2*

For any  $\alpha$ , let

$$V_\alpha(\cdot) \in L^2(R^3) + L^\infty(R^3), \quad (7)$$

such that the  $L^\infty$ -component of  $V_\alpha$  can be chosen arbitrarily small, in the sense of the  $L^\infty$ -norm. Then  $\sigma_c \subset \sigma(H)$ , and the part of  $\sigma(H)$  in the complement of  $\sigma_c$  consists of eigenvalues only, which are of finite multiplicity and can accumulate at most at the lower end of  $\sigma_c$ .

<sup>2)</sup>  $f \in L^p + L^q$  means that there exists an  $f_p \in L^p$  and an  $f_q \in L^q$  such that  $f(x) = f_p(x) + f_q(x)$  almost everywhere.

The hypothesis of this theorem is satisfied, for example, if, for any  $\alpha$ ,

$$V_\alpha(\cdot) \in L^2(R^3) + L^p(R^3), \quad 2 \leq p < \infty,$$

or if  $V_\alpha(x_\alpha)$  is locally square-integrable and vanishes (arbitrarily slowly) as  $|x_\alpha| \rightarrow \infty$ . In particular, this covers the case of Coulomb-interactions.

Among other things, Theorem 2 provides a basis for perturbation theory. The usual perturbation formalism for bound states has been justified to a large extent, notably by F. RELICH [5] and T. KATO [6], but the underlying assumption is always that one has to deal with an isolated eigenvalue of finite multiplicity.

### 3. Properties of Bound-State Wave-Functions

In this section, we want to apply Theorem 2 to obtain some information about the behavior of bound-state wave-functions at infinity (in configuration space).

In the center-of-mass frame, we describe the positions of the  $N$  particles by relative coordinates  $x_1 \dots x_m$ ,  $m = 3N - 3$ , which are taken as linear combinations of the cartesian coordinates of the  $N$  particles. For any multiindex  $n = (n_1 \dots n_m)$ ,  $n_i$  integer  $\geq 0$ , let  $x^n$  denote the operator of multiplication by the monomial  $x_1^{n_1} \dots x_m^{n_m}$ , defined on all elements of  $L^2(R^m)$  for which this product is again in  $L^2(R^m)$ , and define

$$D_n = \bigcap_{\substack{k \leq n \\ l \leq |n| - |k|}} D(x^k H^l),$$

where  $k \leq n$  means  $k_i \leq n_i$ ,  $i = 1 \dots m$ , and  $|n| = n_1 + n_2 + \dots + n_m$ . On  $D_n$  we introduce the norm

$$\|u\|_n = \sup_{\substack{k \leq n \\ l \leq |n| - |k|}} \|x^k H^l u\|. \quad (8)$$

Since  $H^l$  and  $x^k$  are closed,  $D_n$ , normed by  $\|\cdot\|_n$ , is complete.

#### Theorem 3 [7]

Under the hypothesis of Theorem 1,  $D_n$  is invariant under  $e^{-iHt}$ ,  $-\infty < t < +\infty$ . For any  $u \in D_n$ ,  $e^{-iHt} u$  is continuous in  $t$  in the sense of the norm  $\|\cdot\|_n$ , and there exists a constant  $c_n$  such that

$$\|e^{-iHt} u\|_n \leq c_n (1 + |t|)^{|n|} \|u\|_n.$$

Now let  $\varrho$  be a compact isolated subset of  $\sigma(H)$ , and let  $P$  denote the spectral projection corresponding to  $\varrho$ :  $P = \int \varrho dE(\lambda)$ , where  $H = \int \lambda dE(\lambda)$  is the spectral representation of  $H$ .

#### Lemma 1

- (a) For any  $n$ ,  $D_n$  is invariant under  $P$ :  $P D_n \subset D_n$ .
- (b) If  $P$  is of finite rank and  $D_n$  dense in  $\mathfrak{H}$ , then the range of  $P$  is contained in  $D_n$ :  $P \mathfrak{H} \subset D_n$ .

Proof:

(a) By hypothesis, there exists a  $C^\infty$ -function  $f(\lambda)$  of compact support such that

$$\begin{aligned} f(\lambda) &= 1 \text{ for } \lambda \in \varrho, \\ f(\lambda) &= 0 \text{ for } \lambda \notin \varrho, \quad \lambda \in \sigma(H). \end{aligned}$$

Its Fourier-Transform

$$\tilde{f}(t) = (2\pi)^{-1} \int_{-\infty}^{+\infty} d\lambda e^{i\lambda t} f(\lambda)$$

belongs to the space  $\mathfrak{S}$  of rapidly decreasing functions, and, for any  $u \in \mathfrak{H}$ , the projection  $P u$  can be expressed by

$$P u = \int_{-\infty}^{+\infty} dt e^{-iHt} \tilde{f}(t) u.$$

Now let  $u \in D_n$ . Then, by Theorem 3, the integrand is a function of  $t$  with values in  $D_n$ , and, in the sense of the norm (8), this function is continuous in  $t$  and vanishes faster than any inverse power of  $t$  as  $|t| \rightarrow \infty$ . Since  $D_n$  is complete, this implies that the integral (as the limit of Riemann sums) converges in  $D_n$ . Hence  $u \in D_n$  implies  $P u \in D_n$ .

(b) If  $P$  is of finite rank and  $D_n$  dense in  $\mathfrak{H}$ , we have

$$P \mathfrak{H} = P \overline{D_n} \subset \overline{P D_n} = P D_n \subset D_n.$$

Combining this lemma with Theorem 2, we obtain

#### Theorem 4

Under the hypothesis of Theorem 2, the eigenvectors of  $H$  corresponding to eigenvalues in the complement of  $\sigma_c$ , belong to  $D(x^n)$  – for any multi-index  $n$  for which  $D_n$  is dense in  $\mathfrak{H}$ .

For  $|n| = 1$ ,  $D_n$  is dense in  $\mathfrak{H}$ . It is somewhat annoying that, under the hypothesis of Theorem 1, we are not able to assert the same for arbitrary  $n$ . However, if, for example, each  $V_\alpha(x_\alpha)$  is a  $C^\infty$ -function on an open set of  $R^3$  whose complement has measure 0, then  $\overline{D_n} = \mathfrak{H}$  for all  $n$  (7). Thus, in all the cases commonly considered in physics, the bound states below the continuum belong to  $D(x^n)$  for all  $n$ , or, roughly speaking, the bound-state wave-functions decrease faster than any inverse power of  $|x|$  as  $|x| \rightarrow \infty$ . If the potentials  $V_\alpha$  are  $C^\infty$ -functions on all of  $R^3$ , vanishing for  $|x_\alpha| \rightarrow \infty$  and having bounded derivatives to all orders, the wave-functions of the bound states belong to  $\mathfrak{S}(R^m)$  [7].

### 4. The Continuous Spectrum

This section is devoted to the proof that  $\sigma_c \subset \sigma(H)$ . The argument given in Section 2 can be made rigorous by aid of the following simple lemma, which we state without proof:

*Lemma 2*

Let  $H$  be a self-adjoint operator on a Hilbertspace  $\mathfrak{H}$ . Then  $\lambda \in \sigma(H)$  if and only if there exists a sequence  $\{u_n\} \subset D(H)$  with  $\|u_n\| = 1$  and  $\lim_{n \rightarrow \infty} \|(\lambda - H) u_n\| = 0$ .

In our case, the states  $u_n$  will be constructed as states in which the clusters of a given decomposition are more and more separated from each other. First, we want to investigate in what sense the interaction between the clusters vanishes in this limit.

Let  $x_1 \dots x_N$  be the cartesian coordinates of the  $N$  particles. In (relative) configuration space  $R^{3N-3}$ , we can choose, for any pair  $i, j$  ( $i \neq j$ ), the  $3N - 3$  independent coordinates  $x = x_i - x_j$  and  $x_h$  ( $h \neq i, j$ ), which shall be denoted by  $(x, x_h)$ . The norm on  $\mathfrak{H} = L^2(R^{3N-3})$  is then given by

$$\|u\|^2 = c_{ij} \int |u(x, x_h)|^2 d^3x \prod_{h \neq i, j} d^3x_h,$$

where  $c_{ij} > 0$  is the modulus of a jacobian depending only on the masses and on the choice of the pair  $i, j$ .

Let  $D_k = (C_1 \dots C_k)$  be a decomposition of  $(1 \dots N)$  into  $k$  clusters ( $k \geq 2$ ). For any set  $s = (s_1 \dots s_k)$  of  $k$  3-vectors we define the operator  $T(s)$  on  $\mathfrak{H}$  by

$$(T(s)u)(x, x_h) = u(x + t_i - t_j, x_h + t_h),$$

where  $t_l = s_m$  for  $l \in C_m$ . In words:  $T(s)$  is the unitary operator which represents a translation of each cluster  $C_m$ , as a whole, by  $s_m$ . Let  $|s| = \min_{n \neq m} |s_n - s_m|$ . As  $|s| \rightarrow \infty$ , the clusters become more and more separated from each other, and the interaction between them vanishes in the following sense:

*Lemma 3*

Under the hypothesis of Theorem 2,

$$\lim_{|s| \rightarrow \infty} \|I_{D_k} T(s) u\| = 0, \quad (9)$$

for all  $u \in D(H_0)$ .

Proof:

We only consider a single term  $V_\alpha$  of  $I_{D_k}$ . For any  $\varepsilon > 0$ , we can choose a splitting  $V_\alpha(x) = V_{\alpha,2}(x) + V_{\alpha,\infty}(x)$  (a.e) such that  $V_{\alpha,p}(\cdot) \in L^p(R^3)$  and that  $\|V_{\alpha,\infty}(\cdot)\|_\infty < \varepsilon$ . Then  $\|V_{\alpha,\infty} T(s) u\| \leq \varepsilon \|u\|$  for all  $s$ , hence it suffices to prove (9) under the hypothesis that  $V_\alpha(\cdot) \in L^2(R^3)$ .  $T(s)$  is unitary and commutes with  $H_0$  on  $D(H_0)$ . Therefore, by (3),

$$\|V_\alpha T(s) u\| \leq a \|H_0 u\| + b \|u\|$$

for all  $s$  and all  $u \in D(H_0)$ . Providing  $D(H_0)$  with the norm  $\|H_0 u\| + \|u\|$ , we see that  $V_\alpha T(s)$  is a bounded operator from  $D(H_0)$  into  $\mathfrak{H}$ , bounded uniformly in  $s$ . It is sufficient, therefore, to prove (9) on a dense set in  $D(H_0)$ , for example on  $\mathfrak{S}(R^{3N-3})$ . We can even restrict ourselves to states  $u$  of the form

$$u(x, x_h) = f(x) g(x_h),$$

with  $f \in \mathfrak{S}(R^3)$  and  $g \in \mathfrak{S}(R^{3N-6})$ , since the finite linear combinations of such products are dense in  $\mathfrak{S}(R^{3N-3})$ . Choosing the coordinates  $(x, x_h)$  for the pair  $i, j = \alpha$ , we then obtain

$$\|V_{ij} T(s) u\|^2 = c_{ij} \|g\|_2^2 \int d^3x |V_{ij}(x) f(x + t_i - t_j)|^2.$$

Now  $i$  and  $j$  belong to different clusters, so that  $|t_i - t_j| \rightarrow \infty$  as  $|s| \rightarrow \infty$ . Since  $V_\alpha(\cdot) \in L^2(R^3)$  and  $f \in \mathfrak{S}(R^3)$ , the last integral vanishes as  $|s| \rightarrow \infty$ , by the dominated convergence theorem.

It is now a simple matter to show that  $\sigma_c \subset \sigma(H)$ . Let  $\lambda \in \sigma(H_{D_k})$ . By Lemma 2, there exists a sequence  $\{u_n\} \subset D(H_0)$  with  $\|u_n\| = 1$  and

$$\lim_{n \rightarrow \infty} \|(\lambda - H_{D_k}) u_n\| = 0. \quad (10)$$

On the other hand, the translation operators  $T(s)$  corresponding to the decomposition  $D_k$  commute with  $H_{D_k}$  on  $D(H_0)$ , hence

$$\|(\lambda - H) T(s) u_n\| \leq \|(\lambda - H_{D_k}) u_n\| + \|I_{D_k} T(s) u_n\|.$$

By (10) and Lemma 3, the right hand side can be made arbitrarily small by first choosing  $n$  and then  $|s|$  large enough. Since  $T(s) u_n \in D(H_0)$  and  $\|T(s) u_n\| = 1$ , Lemma 2 applies and we conclude that  $\lambda \in \sigma(H)$ .

In the case of short-range forces, time-dependent scattering theory gives a stronger result:

*Theorem 5* (HACK [8])

For any  $\alpha$ , let

$$V_\alpha(\cdot) \in L^2(R^3) + L^p(R^3), \quad 2 \leq p < 3. \quad (11)$$

Then, for any decomposition  $D_k$ , the two strong limits

$$\text{s-lim}_{t \rightarrow \pm\infty} e^{iHt} e^{-iH_{D_k}t} = \Omega_{D_k}^\pm$$

exist on all of  $\mathfrak{H}$ .

(In his proof, HACK assumes that  $V_\alpha(\cdot) \in L^2(R^3)$  and he establishes strong convergence only on a subspace of so-called channel states. It is not difficult, however, to extend his proof so that it covers Theorem 5). It follows that the operators  $\Omega_{D_k}^\pm$  are isometric and satisfy

$$e^{-iHt} \Omega_{D_k}^\pm = \Omega_{D_k}^\pm e^{-iH_{D_k}t},$$

hence the ranges of  $\Omega_{D_k}^\pm$  reduce the group  $e^{-iHt}$  and the parts of  $H$  in these invariant subspaces are unitarily equivalent to  $H_{D_k}$ . This is much more, of course, than  $\sigma_c \subset \sigma(H)$  only.

## 5. The Functional Equation for $G(z)$

The main tool for the proof of the second part of Theorem 2 is the functional equation of S. WEINBERG and C. VAN WINTER for the resolvent  $G(z) = (z - H)^{-1}$ .



In this section, we rederive this equation by analyzing perturbation graphs – a method familiar to physicists, which can easily be justified in the present case.

Let  $G_0(z) = (z - H_0)^{-1}$ . By Theorem 1,  $V_\alpha G_0(z)$  is defined on all of  $\mathfrak{H}$  for  $z \notin \sigma(H_0)$ . For  $\operatorname{Re} z < 0$  we have  $\|G_0(z)\| = |z|^{-1}$  and, by (3),

$$\|V_\alpha G_0(z)\| \leq 2a + \frac{b}{|z|}.$$

Since  $a$  can be chosen arbitrarily small, we conclude that

$$\lim_{\operatorname{Re} z \rightarrow -\infty} \|V_\alpha G_0(z)\| = 0. \quad (12)$$

Let  $M < 0$  be such that  $\|V_\alpha G_0(z)\| < \binom{N}{2}^{-1}$  for all  $\alpha$  if  $\operatorname{Re} z < M$ . Thus, for  $\operatorname{Re} z < M$ , the iteration solution of the resolvent equation

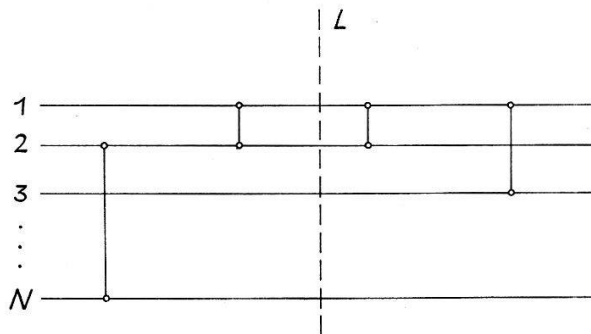
$$G(z) = G_0(z) + G_0(z) V G(z)$$

exists and is given by the series

$$G(z) = \sum_{n=0}^{\infty} \sum_{\alpha_1 \dots \alpha_n} G_0(z) V_{\alpha_1} G_0(z) V_{\alpha_2} \dots G_0(z) V_{\alpha_n} G_0(z), \quad (13)$$

which converges absolutely, in the sense of the operator norm. (This also exhibits that  $H$  is bounded from below). Following S. WEINBERG [1], we define the graph representing the term  $G_0 V_{\alpha_1} G_0 V_{\alpha_2} \dots G_0 V_{\alpha_n} G_0$  of the series (13) to consist of  $N$  horizontal lines („particles”) and  $n$  vertical lines („interactions”) linking just the pairs of particles  $\alpha_1 \dots \alpha_n$  from left to right.

For example, the graph



represents the term  $G_0 V_{2N} G_0 V_{12} G_0 V_{12} G_0 V_{13} G_0$ . Now we turn to a classification of these graphs:

(a) Each graph  $G$  consists of a certain number  $k$  of connected parts ( $1 \leq k \leq N$ ) – only the endpoints of the interactions counting as connections – and thus defines a cluster-decomposition  $D_k(G)$ : two particles belong to the same cluster if their lines belong to the same connected part of  $G$ .

(b)  $D_l \subset D_k$  means that  $D_k$  is obtained by further partitioning the clusters of  $D_l$  ( $l < k$ ). A graph  $G$  is called  $D_l$ -disconnected if  $D_l \subsetneq D_k(G)$ , i.e. if none of the interactions of  $G$  link different clusters of  $D_l$ . Identifying graphs with terms in the series (13), we find, for  $\operatorname{Re} z < M$ ,

$$\sum (\text{all } D_l\text{-disconnected graphs}) = (z - H_{D_l})^{-1} = G_{D_l}(z).$$



(c) Let us cut a graph by a vertical line  $L$  (see Figure), and denote by  $D(L)$  the decomposition defined by the subgraph to the left of  $L$ . First, let  $L$  be to the left of all interactions, then  $D(L) = D_N = (1) (2) \dots (N)$ . If we now shift  $L$  gradually to the right through the whole graph,  $D(L)$  assumes the sequence of values

$$S = (D_N, D_{N-1}, \dots, D_k) \\ D_{i+1} \supset D_i, \quad N \geq k \geq 1, \quad (14)$$

where  $D_k$  is the decomposition corresponding to the whole graph according to (a). In this way, each graph  $G$  uniquely defines a sequence  $S(G)$  of type (14). Conversely, for a given  $S$  of type (14), we define the class  $S$  to consist of all graphs  $G$  with  $S(G) = S$ . Any graph of class  $S$  has the form

$$G_0 \prod_{i=N-1}^k \left[ \left( \begin{array}{c} \text{any interaction linking different} \\ \text{clusters of } D_{i+1} \text{ but not of } D_i \end{array} \right) \left( \begin{array}{c} \text{any } D_i\text{-disconnected} \\ \text{graph} \end{array} \right) \right] \quad (15)$$

where the 'factors' are ordered from left to right as  $i$  decreases.

Since the series (13) is absolutely convergent for  $\text{Re } z < M$ , we can rearrange it in the following way: first, we sum over all graphs of a fixed class  $S$ . By (15), this yields

$$G_S(z) = \sum (\text{all graphs of class } S) \\ = G_0(z) \prod_{i=N-1}^k \left[ \left( \begin{array}{c} \text{sum of all potentials linking different} \\ \text{clusters of } D_{i+1} \text{ but not of } D_i \end{array} \right) \left( \begin{array}{c} \text{sum of all } D_i\text{-discon-} \\ \text{nected graphs} \end{array} \right) \right] \\ = G_{D_N}(z) V_{D_N D_{N-1}} G_{D_{N-1}}(z) V_{D_{N-1} D_{N-2}} \dots V_{D_{k+1} D_k} G_{D_k}(z), \quad (16)$$

where  $V_{D_i D_{i-1}} = I_{D_i} - I_{D_{i-1}}$ ,  $I_{D_k}$  being defined by (4).

The remaining finite sum over classes is carried out in two steps: first, we sum over all  $S = (D_N \dots D_k)$  with  $k \geq 2$ . This is the sum of all disconnected graphs and defines the disconnected part  $D(z)$  of  $G(z)$ :

$$D(z) = \sum_{\substack{\text{all } S \\ \text{with } k \geq 2}} G_S(z). \quad (17)$$

Similarly, we obtain the connected part  $C(z)$  by summing over all  $S$  with  $k = 1$  (sum of all connected graphs):

$$C(z) = \sum_{\substack{\text{all } S \\ \text{with } k = 1}} G_S(z). \quad (18)$$

Noting that each term in the last sum ends with a factor  $G_{D_1}(z) = G(z)$ , we finally arrive at

$$G(z) = D(z) + C(z) = D(z) + I(z) G(z), \quad (19)$$

$$I(z) = \sum_{\substack{\text{all } S \\ \text{with } k = 1}} G_{D_N}(z) V_{D_N D_{N-1}} G_{D_{N-1}}(z) \dots G_{D_2}(z) V_{D_2 D_1}. \quad (20)$$

Note that  $D(z)$  and  $I(z)$  are defined for all  $z \notin \sigma_c$ , but that the functional Equation (19) for  $G(z)$  is established, so far, only for  $\text{Re } z < M$ . As it stands,  $I(z)$  is defined on the

intersection of the domains  $D(V_\alpha)$  only. However, it has a unique bounded extension, since (20) implies

$$I(\bar{z})^* = \sum_{\substack{\text{all } S \\ \text{with } k=1}} V_{D_2 D_1} G_{D_2}(z) \dots V_{D_N D_{N-1}} G_{D_N}(z) \quad (21)$$

which, by Theorem 1, is bounded for  $z \notin \sigma_c$ . In the following,  $I(z)$  denotes the unique bounded extension of the operator defined by (20).

*Lemma 4*

(a)  $I(\bar{z})$  and  $D(z)$  are holomorphic in  $z$  for  $z \notin \sigma_c$ .

(b)  $\lim_{\operatorname{Re} z \rightarrow -\infty} \|I(z)\| = 0$ .

Proof:

If  $I(\bar{z})^*$  is holomorphic in  $z$  for  $z \notin \sigma_c$ , so is  $I(z)$ , hence it suffices to prove assertions (a) and (b) for the operator  $V_\alpha G_D(z)$ , where  $\alpha$  is any pair and  $D$  any decomposition of  $(1 \dots N)$  into at least two clusters. Let  $D(H_D)$  be normed by  $\|u\|_D = \|H_D u\| + \|u\|$ . For  $z \notin \sigma_c$ ,  $G_D(z)$  is a bounded operator from  $\mathfrak{H}$  onto  $D(H_D)$ , and holomorphic in  $z$  for  $z \notin \sigma_c$ . By Theorem 1, the norm  $\|u\|_D$  is equivalent to the norm  $\|H_0 u\| + \|u\|$  on  $D(H_0) = D(H_D)$ , hence  $V_\alpha$  is a bounded operator from  $D(H_D)$  into  $\mathfrak{H}$ . This proves (a). To prove (b), we note that an estimate of type (3) also holds if  $H_0$  is replaced by  $H_D$ , thus (b) follows in the same way as (12).

By Lemma 4(a), the functional equation

$$G(z) = D(z) + I(z) G(z), \quad (22)$$

with  $D(z)$  and  $I(z)$  defined by (16) (17) (20), extends by analyticity from  $\operatorname{Re} z < M$  to all  $z \notin \sigma_c$ ,  $z \notin \sigma(H)$ . This is the final step in the derivation based on the perturbation series (13). Of course, (22) can also be obtained from the resolvent equations which link the various resolvents  $G_{D_k}(z)$  [2].

## 6. The Discrete Spectrum

*Lemma 5*

If the potentials  $V_\alpha$  satisfy the hypothesis of Theorem 2, then  $I(z)$  is a compact operator for  $z \notin \sigma_c$ .

Proof:

For each  $\alpha$ , there exists a sequence  $V_{\alpha,n}(\cdot) \in L^2(R^3)$  such that  $V_{\alpha,n}(\cdot) - V_\alpha(\cdot) \in L^\infty(R^3)$  and, for  $n \rightarrow \infty$ ,  $\|V_{\alpha,n}(\cdot) - V_\alpha(\cdot)\|_\infty \rightarrow 0$ . Let  $I_n(z)$  be defined by (20), with the proviso that in the terms  $V_{D_{k+1} D_k}$  (but not in the resolvents  $G_{D_k}$ ) the potentials  $V_\alpha$  are to be replaced by  $V_{\alpha,n}$ . Clearly, for  $z \notin \sigma_c$ ,

$$\lim_{n \rightarrow \infty} \|I_n(z) - I(z)\| = 0.$$

On the other hand, it was shown by C. VAN WINTER [2] and also by the author [9], that each term in the sum (20) is a Hilbert-Schmidt operator, for  $z \notin \sigma_c$ , provided that

all the potentials occurring explicitly (i. e. in the terms  $V_{D_{k+1}D_k}$ ) are square-integrable. Hence  $I(z)$ , being the uniform limit of a sequence of Hilbert-Schmidt operators, is compact for  $z \notin \sigma_c$ .

Combining Lemmas 4 and 5 we conclude (see appendix), that  $(1 - I(z))^{-1}$  is meromorphic in  $z$  for  $z \in \sigma_c$ , and that

$$G(z) = (1 - I(z))^{-1} D(z)$$

for all  $z \notin \sigma_c$  for which  $(1 - I(z))^{-1}$  exists (it might happen that  $G(z)$  exists but not  $(1 - I(z))^{-1}$ . This point is further discussed in Section 7), thus  $G(z)$  too is meromorphic in the complement of  $\sigma_c$ . This proves that the part of  $\sigma(H)$  in the complement of  $\sigma_c$  consists of eigenvalues only, which can accumulate at most at the lower end of  $\sigma_c$ .

To show the finite multiplicity, let  $z_0$  be a pole of  $G(z)$  (eigenvalue of  $H$ ) in the complement of  $\sigma_c$ , and  $P$  the projection onto the corresponding subspace of eigenvectors:

$$P = \lim_{z \rightarrow z_0} (z - z_0) G(z).$$

On the other hand, by (22),

$$(z - z_0) G(z) = (z - z_0) D(z) + I(z) (z - z_0) G(z).$$

Since  $D(z)$  and  $I(z)$  are holomorphic in a neighborhood of  $z_0$ , we obtain, for  $z \rightarrow z_0$ :

$$P = I(z_0) P, \quad (23)$$

hence  $P$  is a compact projection and therefore of finite rank. This concludes the proof of Theorem 2.

## 7. FADDEEV'S EQUATIONS

(23) shows that, for  $z \notin \sigma_c$ ,

$$Hf = zf \text{ implies } f = I(z)f.$$

For  $N > 2$ , however, the reverse is not proved and probably false. Thus we cannot conclude that  $(1 - I(z))^{-1}$  exists (for  $z \notin \sigma_c$ ) whenever  $z$  is not an eigenvalue of  $H$ . From all we know, the poles of  $(1 - I(z))^{-1}$  need not even be real.

In the case  $N = 3$ , this annoying feature is avoided by L. D. FADDEEV [10] in the following way: he splits the GREEN's function into components

$$G(z) = G_0(z) + \sum_{\alpha} R_{\alpha}(z)$$

where  $R_{\alpha} = G_0 V_{\alpha} G$  or, in terms of graphs,  $R_{\alpha} = \Sigma$  (of all graphs with  $V_{\alpha}$  as the left-most interaction). Instead of (22), L. D. FADDEEV treats a linked set of equations for the components  $R_{\alpha}$ :

$$R_{\alpha} = (G_{\alpha} - G_0) + \sum_{\beta \neq \alpha} G_{\alpha} V_{\alpha} R_{\beta}$$

where  $G_{\alpha} = (z - H_0 - V_{\alpha})^{-1}$  is the disconnected part of  $R_{\alpha}$ . This splitting into components has the effect that, for  $z \notin \sigma_c$ , the homogeneous equations

$$Hf = zf \text{ and } f_{\alpha} = \sum_{\beta \neq \alpha} G_{\alpha} V_{\alpha} f_{\beta}$$

are equivalent, if connected by

$$f = \sum_{\alpha} f_{\alpha}, \quad f_{\alpha} = G_0 V_{\alpha} f.$$

The author has tried, unsuccessfully so far, to find a generalization of FADDEEV's equations to  $N > 3$  which still has this property. The analysis of graphs suggests several ways of splitting  $G(z)$  – or  $C(z)$  – into components (an example is (18)), and linked sets of equations for these components can be derived along the lines followed in Section 5. The unsolved problem is to show whether or not nontrivial solutions of the homogeneous equations correspond to bound states.

### Appendix

In this appendix we formulate and prove a theorem which seems to be known but which we couldn't find in the literature.

#### Theorem

- (a) Let  $C$  be the closure, in the sense of the operator norm, of the set of operators of finite rank on a Banachspace  $B$ .
- (b) Let  $A(z)$  be a  $C$ -valued function of the complex variable  $z$ , which is holomorphic in an open, connected region  $G$ .
- (c) Suppose that  $(1 - A(z_0))^{-1}$  exists for some  $z_0 \in G$ . Then  $(1 - A(z))^{-1}$  is meromorphic in  $z$  for  $z \in G$ , i.e. its only possible singularities in  $G$  are poles.

#### Remark:

If  $B$  is a Hilbertspace, then  $C$  is the set of all compact operators on  $B$ . To our knowledge, it is still open if the same is true for any Banachspace  $B$ .

#### Proof:

(a) The theorem holds if there exists a subspace  $R \subset B$ ,  $\dim R < \infty$ , such that the range of  $A(z)$  is contained in  $R$  for all  $z \in G$ . To show this, let  $a(z)$  be the restriction of  $A(z)$  to  $R$ . Then

$$(1 - A(z))^{-1} = 1 + (1 - a(z))^{-1} A(z), \quad (24)$$

whenever the right-hand side exists. In any basis of  $R$ , the determinant of  $1 - a(z)$  is holomorphic in  $z \in G$  and not identically zero, for  $a(z_0)f = f$  implies  $f = 0$ , by (c). Hence  $(1 - a(z))^{-1}$  and, by (24),  $(1 - A(z))^{-1}$  are meromorphic in  $G$ .

(b) Now let  $A(z)$  satisfy the hypothesis of the theorem. For any  $z \in G$  there exists an operator  $F(z)$  of finite rank such that  $\|A(z) - F(z)\| < 1/2$ , and, since  $A(z)$  is continuous in  $z$ , a neighborhood  $U(z) \subset G$  of  $z$  such that

$$\|A(z') - F(z)\| < 1 \text{ if } z' \in U(z). \quad (25)$$

Let  $z_1$  be an arbitrary point in  $G$ . We have to show that  $(1 - A(z))^{-1}$  is meromorphic in a neighborhood of  $z_1$ . By the Heine-Borel theorem, there exists a sequence of points  $s_1 \dots s_N$  in  $G$  such that

$$z_0 \in U(s_1), \quad z_1 \in U(s_N), \quad U(s_i) \cap U(s_{i+1}) \text{ non-empty for } i = 1 \dots N-1.$$

Let  $e(z) = A(z) - F(s_1)$ .  $e(z)$  is holomorphic in  $G$  and  $\|e(z)\| < 1$  for  $z \in U(s_1)$ , by (25). Thus  $(1 - e(z))^{-1}$  is holomorphic in  $U(s_1)$ , and

$$1 - A(z) = [1 - F(s_1) (1 - e(z))^{-1}] (1 - e(z)).$$

For  $z \in U(s_1)$ ,  $F(s_1) (1 - e(z))^{-1}$  is of finite rank, with range contained in the range of  $F(s_1)$ , and holomorphic in  $z$ . Also, by (c),  $[1 - F(s_1) (1 - e(z_0))^{-1}]^{-1}$  exists. Thus, by part (a) of the proof,  $(1 - A(z))^{-1}$  is meromorphic in  $U(s_1)$ . Since  $U(s_1) \cap U(s_2)$  is open and non-empty, there exists a new  $z_0 \in U(s_2)$  for which  $(1 - A(z_0))^{-1}$  exists. By the same analysis, we find that  $(1 - A(z))^{-1}$  is meromorphic in  $U(s_2) \dots U(s_N)$ .

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