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Massless Particles with Definite Helicity are not Weakly Localizable

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Abstract. The concept of localizability has been recently extended and particles such as the photon have been proved to be weakly localizable. This has been possible, however, because both helicities have been simultaneously taken into account. It is proved in this note that massless particles with finite spin $\neq 0$ and irreducible under the (connected) Poincaré group are *not* weakly localizable.

Introduction

It is well-known that the conventional description [1, 2] of localizability of quantum systems by a system of imprimitivities transitive under the Euclidean group of motions excludes relativistic elementary particles with zero mass and finite spin larger than zero from being localizable (exception made of the four-component neutrino). JAUCH and PIRON [3] have recently generalized this frame by essentially abandoning the requirement that position measurements corresponding to overlapping domains be compatible. These authors are thus led to introduce the concept of a *generalized* system of imprimitivities as the adequate mathematical tool to describe localizability in the extended sense. More precisely, let $\mathcal{S}$ denote a single particle quantum system whose states are described by unitary rays in a (separable complex) Hilbert space $\mathcal{H}$, and suppose that the action of the group of motions in the three-dimensional Euclidean space M upon the states of $\mathcal{S}$ is induced by a continuous unitary representation $(a, A) \to U(a, A)$ of its simply connected universal covering group $\mathcal{E} \approx \mathcal{T}_3 \dot{\times} \mathrm{SU}(2)$ [semi-direct product, where SU(2) acts, in the usual way, by automorphisms on the translation normal subgroup $\mathcal{T}_3$]. Let $\mathcal{B}$ denote the family of Borel sets of M, and let $\mathcal{C}$ be the lattice of orthogonal projections in $\mathcal{H}$. We shall say [3] that the system $\mathcal{S}$ admits weak localizability if there exists a map $F: \Delta \in \mathcal{B} \to F(\Delta) \in \mathcal{C}$ satisfying

$$F(\phi) = 0\,, \qquad F(M) = 1 \tag{1}$$

$$\Delta_1 \cap \Delta_2 = \phi \Rightarrow F(\Delta_1)\, F(\Delta_2) = 0 \tag{2}$$

$$F\,(\Delta_1 \cap \Delta_2) = F(\Delta_1) \cap F(\Delta_2) \tag{3}$$

$$U(\boldsymbol{a}, A)\, F(\Delta)\, U^{-1}(\boldsymbol{a}, A) = F\big((\boldsymbol{a}, A)\, \Delta\big) \tag{4}$$

$$F(K) \neq 0 \text{ for some compact } K \tag{5}$$

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where $(\boldsymbol{a}, A)\,\Delta \equiv \{\boldsymbol{r} \in M : \boldsymbol{r} = \boldsymbol{a} + R(A)\,\boldsymbol{r}', \text{ with } \boldsymbol{r}' \in \Delta\}$, $R(A)$ being the rotation associated to A. Conditions (1), (2) and (3) define F as a generalized spectral measure and conditions (4) and (5) express that $\{F(\Delta)\}$ is a non-trivial generalized system of imprimitivities for the representation U of $\mathcal{E}$.

Jauch and Piron have explicitly constructed [3] a generalized system of imprimitivities for the physical photon $[0,1,+] \oplus [0,1,-]$ ($[0,j,\varepsilon]$ denotes a relativistic elementary – under the connected Poincaré group – system of zero mass, spin $j \neq 0$ and helicity ε) which is obviously non-trivial as they claim: take any compact K with non-void interior K^0, and an arbitrary vector-valued C^∞ function $\boldsymbol{f}(\boldsymbol{r})$ with support in K^0. Then $\nabla \times \boldsymbol{f}(\boldsymbol{r})$ is divergenceless, C^∞ and of compact support in K^0. Clearly $\boldsymbol{f}(\boldsymbol{r})$ may be chosen so that $\nabla \times \boldsymbol{f}(\boldsymbol{r}) \not\equiv 0$, wherefrom $\boldsymbol{A}(\boldsymbol{r}) \equiv \nabla \times \boldsymbol{f}(\boldsymbol{r})$ will describe the potential vector of a photon localized in K^0, and *a fortiori*, in K.

The question arises whether elementary particles $[0,j,\varepsilon]$, which were not localizable in the old Newton-Wigner sense, are weakly localizable or not. We intend to prove that the answer is negative, even though we conjecture that all systems $[0,j,+] \oplus [0,j,-]$ are weakly localizable [4].

Section 2 contains some generalities about the one-component realizations of the relativistic elementary systems of zero mass, spin j and helicity ε, which prove useful to deduce our result in Section 3 concerning the non weak-localizability of the systems $[0,j,\varepsilon]$.

One Component Description of Massless Particles with Finite Spin and Definite Helicity

The state of a particle with $m = 0$, finite spin j and fixed helicity ε is determined by the probability amplitude for finding the particle with a definite momentum. Hence it must be possible to describe each such state by a single complex-valued function $\lambda(p)$ with support in the future light cone $C_+ \equiv \{p : p^2 = 0,\ p^0 \geqslant 0\}$ and squared norm

$$\|\lambda\|^2 = \int_{C_+} |\lambda(p)|^2\, d\sigma(p) < \infty, \qquad d\sigma(p) \equiv \frac{d^3\boldsymbol{p}}{2\,|\boldsymbol{p}|}. \tag{6}$$

This fact is well known [5, 6] and the associated irreducible continuous unitary representation of the simply connected universal covering group $\tilde{P} \approx \mathfrak{T}_4 \dot{\times} \mathrm{SL}(2,C)$ of the Poincaré group P may be taken as

$$\big(U(a, A)\,\lambda\big)\,(p) = e^{i a p}\, Q(p, A)\, \lambda\big(\Lambda(A^{-1})\,p\big) \tag{7}$$

with

$$Q(p, A) \equiv \left[\frac{(v^+\,\tilde{p}\,A\,v)}{|(v^+\,\tilde{p}\,A\,v)|}\right]^{-2\varepsilon j}, \quad \tilde{p} \equiv p^0 - \boldsymbol{\sigma}\cdot\boldsymbol{p} \tag{8}$$

$\Lambda(A)$ being the element of the Lorentz group $\mathfrak{L}$ associated to $A \in \mathrm{SL}(2, C)$ and v a non-null arbitrary two-component spinor.

A simple computation shows that the standard raising and lowering operators $J_\pm$ and the third component J^3 of the angular momentum operator $\boldsymbol{J}$ have the following expressions in spherical polar co-ordinates for $\boldsymbol{p}$:

$$J_\pm = e^{\pm i\varphi}\left\{\pm\,\partial_\theta + i\cot\theta\,\partial_\varphi + \varepsilon j \tan\frac{\theta}{2}\right\} \qquad J^3 = -i\,\partial_\varphi + \varepsilon j. \tag{9}$$

Consequently any eigenfunction $\lambda_{JJ}(p)$ of $\boldsymbol{J}^2, J^3$ with eigenvalues $J(J+1), J$ and support in C_+ can be written as

$$\lambda_{JJ}(p) = a(|\boldsymbol{p}|)\left(\sin\frac{\theta}{2}\right)^{J-\varepsilon j}\left(\cos\frac{\theta}{2}\right)^{J+\varepsilon j} e^{i(J-\varepsilon j)\varphi}. \tag{10}$$

Note that $J \equiv j \pmod 1$ and that the requirement that λ_{JJ} be of finite norm implies $J \geqslant j$, i.e. the least angular momentum the system $[0, j, \varepsilon]$ can have equals J.

Systems [0, j, ε] are not Weakly Localizable

We intend now to prove that particles $[0, j, \varepsilon]$ with $j > 0$ are not weakly localizable. Suppose the contrary, and let $\{F(\Delta)\}$ be the corresponding non-trivial generalized system of imprimitivities. If K is a compact set satisfying (5), and S is a closed sphere $S \equiv \{\boldsymbol{r} \in M : |\boldsymbol{r}| \leqslant \varrho\}$ such that $K \subseteq S$, it is plain that $F(S) \neq 0$. But $(\boldsymbol{0}, A) S = S$ for every $A \in \mathrm{SU}(2)$. Therefore (4) implies

$$[U(\boldsymbol{0}, A), F(S)] = 0 \quad \text{for} \quad A \in \mathrm{SU}(2). \tag{11}$$

Hence the non-null subspace $F(S)\,\mathcal{H}$ reduces the representation U of SU(2) and thus it contains some non-null vector λ_{JJ} of the form (10).

On the other hand, $S \cap (\boldsymbol{a}, 1) S = \phi$ for $|\boldsymbol{a}| > 2\varrho$ and therefore $U(\boldsymbol{a}, 1)\lambda_{JJ}$ is orthogonal to λ_{JJ} for every such $\boldsymbol{a}$. This means that

$$\int e^{i\boldsymbol{a}\boldsymbol{p}}\, |\lambda_{JJ}(|\boldsymbol{p}|, \boldsymbol{p})|^2 \frac{d^3\boldsymbol{p}}{2|\boldsymbol{p}|} = 0 \quad \text{for} \quad |\boldsymbol{a}| > 2\varrho. \tag{12}$$

But (12) requires [7] that $f(\boldsymbol{p}) \equiv (1/2|\boldsymbol{p}|)\,|\lambda_{JJ}(|\boldsymbol{p}|, \boldsymbol{p})|^2$ should be extendable to an entire function on $M + iM$ of exponential type $\leqq 2\varrho + \delta$ (δ arbitrary > 0). However, we may always write

$$f(\boldsymbol{p}) = b(|\boldsymbol{p}|)\,|\boldsymbol{p}|^{-2[J]}(p_1^2 + p_2^2)^J \left(\frac{|\boldsymbol{p}| + p^3}{|\boldsymbol{p}| - p^3}\right)^{\varepsilon j} \tag{13}$$

where $[J]$ stands for the largest integer $\leqslant J$, and hence the condition that $f(\boldsymbol{p})$ be entire analytic as a function of p^1 when $p^2 = p^3 = 0$ implies that $b(|\boldsymbol{p}|) = C(\boldsymbol{p}^2)$, $C(\boldsymbol{p}^2)$ being an entire function of $\boldsymbol{p}^2$. Take now $p^1 = p^2 = p^3 = \alpha/\sqrt{3}$; then

$$g(\alpha) \equiv f\left(\frac{\alpha}{\sqrt{3}}, \frac{\alpha}{\sqrt{3}}, \frac{\alpha}{\sqrt{3}}\right) \Big/ C(\alpha^2)\,\alpha^{2(J-[J])}$$

should either have a pole or be regular at $\alpha = 0$. This is obviously not the case, for

$$\lim_{\alpha \to \pm 0} g(\alpha) = \left(\frac{2}{3}\right)^J \left(2 \mp \sqrt{3}\right)^{\varepsilon j}. \tag{14}$$

The proof is now complete, Q.E.D.

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