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On the Analyticity Properties of the N-Body Scattering Amplitude in Non-Relativistic Quantum Mechanics

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Abstract. We consider the scattering of N non-relativistic, spinless, distinguishable particles interacting via two-body superpositions of Yukawa potentials. The on-energy-shell amplitude is studied as a function of the total center-of-mass kinetic energy E and for physical values of the 'angular' variables $\boldsymbol{x}_i = (1/k)\, \boldsymbol{p}_i$, $\boldsymbol{y}_i = (1/k)\, \boldsymbol{q}_i$, $1 \le i \le N$, $k^2 = E$, where $\boldsymbol{p}_1, \dots, \boldsymbol{p}_N$ and $\boldsymbol{q}_1, \dots, \boldsymbol{q}_N$ are the initial, respectively the final momenta.

It is shown that this amplitude is the boundary value of a function analytic in the energy E in a complex plane cut from $-\infty$ to $-\varrho^2$ for some $\varrho > 0$ and from 0 to $+\infty$ and in all variables $(\boldsymbol{x}_1, \dots, \boldsymbol{y}_N)$ in a neighbourhood of their physical values, up to an algebraic set of codimension 1.

0. Introduction

The exploitation of the analyticity properties of collision amplitudes, Green's functions and vacuum expectation values in relativistic quantum mechanics has been proved to be an important tool for a qualitative understanding of subatomic processes, as well as for the development of certain quantitative approximation procedures.

Although some rigorous results have been already obtained for the simplest collision processes [1], it appears that a further analysis of the more important multiparticle scattering amplitudes encounters considerable difficulties. This motivates the investigation of the non-relativistic N-body problem, since there, rigorous solutions can be shown to exist. It is then hoped that some insight may be gained in the many-particle structure of the S-matrix. It turns out again, that satisfactory analyticity properties can be expected only for an appropriate choice of variables.

Our aim is to show that for certain multiparticle processes, the analytic structure of the exact N-body scattering amplitude in the "physical sheet" is the same as that predicted by the perturbation theory.

In customary notation, the N-particle Hamiltonian with two-body forces

$$H = H_0 + V = H_0 + \sum_{1 \le i < j \le N} V_{ij}$$

will be defined in the total center-of-mass frame, as an operator in the Hilbert space

$$\mathcal{H} = L^2(R^{3(N-1)}) = \left\{ \psi(\boldsymbol{p}_1, \dots, \boldsymbol{p}_N),\ \|\psi\| = \int d^3p_1 \dots d^3p_N\, \delta\Big(\sum_{i=1}^{N} p_i\Big)\, |\psi|^2 < \infty \right\}$$

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with

$$(H_0 \psi)(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N) = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2\, m_i} \psi(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$$

$$(V_{ij} \psi)(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$$
$$= \int d^3q\, v_{ij} \left(\frac{\boldsymbol{p}_i - \boldsymbol{p}_j}{2} - \boldsymbol{q}\right) \psi \left(\boldsymbol{p}_1, \ldots, \frac{\boldsymbol{p}_i + \boldsymbol{p}_j}{2} + \boldsymbol{q}, \ldots, \frac{\boldsymbol{p}_i + \boldsymbol{p}_j}{2} - \boldsymbol{q}, \ldots, \boldsymbol{p}_N\right).$$

We shall always assume that the v_{ij} are real.

Theorem 0.1 (T. Kato [2]). – For any i, j, $1 \leqslant i < j \leqslant N$, let

$$v_{ij}(.) \in L^2(\mathbf{R}^3) + L^\infty(\mathbf{R}^3)\,.$$

Then $\Delta(V_{ij}) \supset \Delta(H_0)$ and for any $a > 0$, there exists a $b < \infty$ such that

$$\| V_{ij} \psi \| \leqslant a \| H_0 \psi \| + b \| \psi \|$$

for all $\psi \in \Delta(H_0)$. The sum $H_0 + V$ is self-adjoint on $\Delta(H_0)$ (and bounded from below).

In the sequel we shall restrict ourselves to square-integrable two-body potentials.

Under this assumption, it may be shown [3] that the Møller operators

$$\Omega^{\substack{\text{out}\\ \text{in}}} = s - \lim_{t \to \pm\infty} e^{iHt}\, e^{-iH_0 t}$$

exist and allow for the definition of an isometric S-matrix

$$S\colon\ \Omega^{\text{out}}\, \mathcal{H} \to \Omega^{\text{in}}\, \mathcal{H}$$

by

$$S\colon\ \Omega^{\text{out}}\, \phi \to \Omega^{\text{in}}\, \phi$$

for all $\phi \in \mathcal{H}$.

We shall further limit ourselves to a class of short range potentials which decrease sufficiently rapidly in configuration space. More specifically, we assume:

(A_0) For any i, j, $1 \leqslant i < j \leqslant N$, $v_{ij}(\boldsymbol{p})$ is holomorphic in

$$\{\boldsymbol{p} \in \mathbf{C}^3\colon\ |Im\, \boldsymbol{p}| < \varkappa\}$$

for some $\varkappa$, $0 < \varkappa < D$.

For any $\varepsilon > 0$, there exist $\theta(\varepsilon) > 3/2$ and $C(\varepsilon) < \infty$ such that

$$|v_{ij}(\boldsymbol{p})| \leqslant C(\varepsilon)\, (1 + |\boldsymbol{p}|)^{-\theta(\varepsilon)}$$

uniformly in

$$\{\boldsymbol{p} \in \mathbf{C}^3\colon\ |Im\, \boldsymbol{p}| \leqslant \varkappa - \varepsilon\}\,.$$

In order to avoid multichannel situations, we assume the potentials to be purely repulsive:

(B) For any i, j, $1 \leqslant i < j \leqslant N$,

$$\boldsymbol{x} \cdot \boldsymbol{\nabla}\, \tilde{v}_{ij}(\boldsymbol{x}) \leqslant 0$$

where $\tilde{v}_{ij}$ is the Fourier transform of v_{ij}:

$$\tilde{v}_{ij}(\boldsymbol{x}) = \int d^3p\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\, v_{ij}(\boldsymbol{p})\,.$$

Assumptions (A_0) and (B) and the virial theorem allow then to prove [4] that the spectrum $\sigma(H)$ of H lies in $[0, \infty)$.

Under assumptions (A_0) and (B) some interesting analyticity properties can be proved for the kernel of the S-matrix (cf. Theorem 3.1). More satisfactory however, is the restriction to superpositions of Yukawa potentials. For simplicity, we consider the class

(A) For any i, j, $1 \leqslant i < j \leqslant N$,

$$v_{ij}(\boldsymbol{p}) = \int_{-\infty}^{+\infty} \frac{d\sigma_{ij}(\lambda)}{\lambda + \boldsymbol{p}^2}$$

where $d\sigma_{ij}$ is a real measure on R^1 satisfying

$$\int_{-\infty}^{+\infty} \lambda \, d\, |\sigma_{ij}(\lambda)| < \infty \tag{0.1a}$$

and with

$$\operatorname{supp} d\sigma_{ij} \subset [\varkappa^2, \infty) , \qquad \varkappa > 0 . \tag{0.1b}$$

Then $v_{ij}(p)$ is square-integrable, due to

$$v_{ij} \, |(\boldsymbol{p})| \leqslant \frac{1}{\varkappa^2 + \boldsymbol{p}^2} \int_{-\infty}^{+\infty} d\, |\sigma_{ij}(\lambda)|$$

and holomorphic in

$$\{\boldsymbol{p} \in \mathsf{C}^3 : \ \boldsymbol{p}^2 \notin [\varkappa^2, \infty)\} .$$

Due to (0.1a), (0.1b), $\boldsymbol{x} \cdot \boldsymbol{\nabla} \, \tilde{v}_{ij}(\boldsymbol{x})$ is well-defined.
We note that the assumption (A) may be generalised to include non-spherically symmetric superpositions of Yukawa potentials as well as potentials which merely possess the analyticity and growth properties of Yukawa potentials.

The requirements (A_0), (B) exclude discrete negative eigenvalues of the Hamiltonian H. It may be shown [5] that the following assumption is sufficient for the control of the discrete spectrum of H in the continuous part $[0, \infty)$ of its spectrum:

(C) For any i, j, $1 \leqslant i < j \leqslant N$, let $\tilde{v}_{ij}(\boldsymbol{x})$ be C^∞ for $\boldsymbol{x} \neq 0$. There exists an α, $0 < \alpha \leqslant 1$, such that for any i, j:

$$\boldsymbol{x} \cdot \boldsymbol{\nabla} \, \tilde{v}_{ij}(\boldsymbol{x}) \leqslant - \alpha \, \tilde{v}_{ij}(\boldsymbol{x}) .$$

Under (A), (B), (C) the S-matrix is unitary [5]:

$$\Omega^{\text{out}} \, \mathcal{H} = \Omega^{\text{in}} \, \mathcal{H} = \mathcal{H} , \quad S \, S^* = S^* \, S = \mathbf{1} .$$

We shall explore certain analytic properties of the N-body scattering amplitude τ defined by

$$\begin{aligned} &\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N \,|\, S \,|\, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N \rangle - \langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N \,|\, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N \rangle \\ &\quad = \delta\Big(\sum_{i=1}^{N} \boldsymbol{p}_i - \sum_{i=1}^{N} \boldsymbol{q}_i\Big) \, \delta\Big(\sum_{i=1}^{N} \mu_i \, \boldsymbol{p}_i^2 - \sum_{i=1}^{N} \mu_i \, \boldsymbol{q}_i^2\Big) \, \tau(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N) . \end{aligned}$$

We have set $\mu_i = 1/2 \, m_i$, $1 \leqslant i \leqslant N$.

In fact, the kernel of the S-matrix is primarily defined in the sense of distributions and then extended by continuity to a functional on $L^2(\mathbf{R}^{3N}) \times L^2(\mathbf{R}^{3N})$, which is antilinear in the first factor and linear in the second.

The general time-independent scattering theory [6] gives $\tau(\boldsymbol{p}_1, \ldots \boldsymbol{q}_N)$ as boundary value in the sense of distributions of the kernel of the N-body T-operator. Let $Im\, z \neq 0$ and

$$R(z) = (z - H)^{-1}$$

be the resolvent of the Hamiltonian H. The T-operator is connected to $R(z)$ via

$$T(z) = V + V\, R(z)\, V \tag{0.2}$$

and to the S-matrix by

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \int d^3p_1 \ldots d^3q_N\, \delta\left(\sum_{i=1}^{N} \boldsymbol{p}_i\right) \delta\left(\sum_{j=1}^{N} \boldsymbol{q}_j\right) \delta\left(\sum_{i=1}^{N} \mu_i\, \boldsymbol{p}_i^2 - \sum_{j=1}^{N} \mu_j\, \boldsymbol{q}_j^2\right) \\
&\qquad \times \varphi^*(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)\, \psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)\ T(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N; \sum_{i=1}^{N} \mu_i\, \boldsymbol{p}_i^2 + i\, \varepsilon) \\
&\qquad = \int d^3p_1 \ldots d^3q_N\, \delta\left(\sum_{i=1}^{N} \boldsymbol{p}_i\right) \delta\left(\sum_{j=1}^{N} \boldsymbol{q}_j\right) \delta\left(\sum_{i=1}^{N} \mu_i\, \boldsymbol{p}_i^2 - \sum_{j=1}^{N} \mu_j\, \boldsymbol{q}_j^2\right) \\
&\qquad \times \varphi^*(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)\, \psi(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)\, \tau(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)
\end{aligned} \tag{0.3}$$

for $\varphi, \psi \in D(\mathbf{R}^{3N})$.

We shall always work in the center-of-mass frame and on the energy shell where

$$\sum_{i=1}^{N} \boldsymbol{p}_i = \sum_{i=1}^{N} \boldsymbol{q}_i = 0\,, \quad \sum_{i=1}^{N} \mu_i\, \boldsymbol{p}_i^2 = \sum_{i=1}^{N} \mu_i\, \boldsymbol{q}_i^2$$

and introduce the variables

$$E = k^2 = \sum_{i=1}^{N} \mu_i\, \boldsymbol{p}_i^2\,, \qquad k \geqslant 0$$

$$\boldsymbol{p}_i = k\, \boldsymbol{x}_i\,, \quad \boldsymbol{q}_i = k\, \boldsymbol{y}_i\,, \quad 1 \leqslant i \leqslant N$$

with

$$\boldsymbol{x}_i, \boldsymbol{y}_j \in \mathbf{R}^3\,, \sum_{i=1}^{N} \boldsymbol{x}_i = \sum_{i=1}^{N} \boldsymbol{y}_i = 0\,, \sum_{i=1}^{N} \mu_i\, \boldsymbol{x}_i^2 = \sum_{i=1}^{N} \mu_i\, \boldsymbol{y}_i^2 = 1\,.$$

Let

$$\Omega_C^N = \left\{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathbf{C}^{3N} : \sum_{i=1}^{N} \boldsymbol{x}_i = 0\,, \sum_{i=1}^{N} \mu_i\, \boldsymbol{x}_i^2 = 1\right\}.$$

Ω_C^N is a complex submanifold on $\mathbf{C}^{3N}$ (with the usual complex structure): we eliminate say $\boldsymbol{x}_N = \sum_{i=1}^{N-1} \boldsymbol{x}_i$ and compute the gradient of the polynomial φ which defines Ω_C^N:

$$\varphi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}) = \sum_{i=1}^{N-1} \mu_i\, \boldsymbol{x}_i^2 + \mu_N \left(\sum_{i=1}^{N-1} \boldsymbol{x}_i\right)^2 - 1\,.$$

Clearly

$$\boldsymbol{\nabla}_i\, \varphi = 2\, \mu_i\, \boldsymbol{x}_i + 2\, \mu_N \sum_{j=1}^{N-1} \boldsymbol{x}_j\,.$$

A straightforward calculation shows that the determinant of the homogeneous system $\nabla_1 \varphi = 0, \ldots, \nabla_{N-1} \varphi = 0$ remains always positive, so that $(\nabla_1 \varphi, \ldots, \nabla_{N-1} \varphi)$ does not vanish on $\{\varphi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}) = 0\}$.

We denote

$$\Omega^N \times \Omega^N = (\Omega_C^N \times \Omega_C^N) \cap \mathsf{R}^{6N} .$$

We shall investigate the analyticity properties of $\tau(k\, \boldsymbol{x}_1, \ldots, k\, \boldsymbol{y}_N)$ on the complex energy shell: $(k, \boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in \mathsf{C}^1 \times \Omega_C^N \times \Omega_C^N$.

Due to (0.3), this amounts to an investigation of the regularity properties of $T(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N; z)$. It may be shown [5] that under the assumptions (A), (B) and (C) and up to an analytic set of codimension 1 in $\Omega_C^N \times \Omega_C^N$, $\tau(k\, \boldsymbol{x}_1, \ldots, k\, \boldsymbol{y}_N)$ is the pointwise limit for $\varepsilon \downarrow 0$ of $T((k + i\, \varepsilon)\, \boldsymbol{x}_1, \ldots, (k + i\, \varepsilon)\, \boldsymbol{y}_N; (k + i\, \varepsilon)^2)$ for real $(\boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in \Omega^N \times \Omega^N$ and for $k \geqslant 0$.

Therefore, we shall study $T(k\, \boldsymbol{x}_1, \ldots, k\, \boldsymbol{y}_N; k^2)$ for $(\boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in \Omega^N \times \Omega^N$ and $k \in \mathsf{C}^1$. It will turn out that this amplitude is holomorphic in

$$\{Im\, k > 0,\ Re\, k \neq 0\} \cup \{0 < Im\, k < \varrho,\ Re\, k = 0\}$$

for some $\varrho > 0$ and for almost all $(\boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in \Omega^N \times \Omega^N$. In other words, we shall obtain analyticity in the complex energy plane with two cuts along the real E-axis, from $-\infty$ to $-\varrho^2$ and from 0 to ∞.

This is a (partial) generalisation of previous results for the two- and three-particle scattering amplitudes [7, 9].

In section I we introduce and discuss briefly a system of coupled linear integral equations of the 2nd kind for the N-body scattering amplitude: the Faddeev-Yakubovskii (F.Y.) equations. We show the complete continuity of the $(N-1)$st iteration of their kernels and define new amplitudes which will be used in the subsequent sections.

In section II we study the iterations of these equations in the framework of perturbation theory.

In section III, the full amplitude as solution of the iterated F.Y. equations, is investigated by means of the Fredholm method.

Some final remarks make up the last section.

1. The Faddeev-Yakubovskii Equations

We first introduce the F.Y. equations [10, 11] for our N-particle system in a slightly different manner as that of Reference [11]: our amplitudes shall have prescribed connectivity properties from the left as well as from the right hand side (two-sided amplitudes). They will turn out to be particularly useful for the derivation of analyticity properties.

For $Im\, z \neq 0$, the resolvents

$$R(z) = (z - H)^{-1} \qquad R_0(z) = (z - H_0)^{-1}$$

of the self-adjoint operators H and H_0 are bounded operators. The off-energy-shell scattering amplitude $T(z)$ (cf. Eq. (0.2)) satisfies the Lippmann-Schwinger equation

$$T(z) = V + V\, R_0(z)\, T(z) \tag{1.1}$$

(and $R(z) = R_0(z) + R_0(z)\, V\, R(z)$) which does uniquely characterise H [10], but has the unconvenient feature that for $N > 2$, no iteration of its kernel $V\, R_0(z)$ is completely continuous.

For the definition of the F.Y. equations we shall use the notation of [11]. Let $\{1, \ldots, N\}$ be the index set for N particles. A partition of $\{1, \ldots, N\}$ into k clusters, $1 \leqslant k \leqslant N$ is denoted by a_k. Inside each cluster the indices are arranged in the natural order. Different partitions into k clusters will be denoted by $a_k, b_k, c_k, \ldots$ or by $a_k^1, a_k^2, \ldots$ There is only one partition a_1 and one a_N. Furthermore, there is a one-to-one correspondence between the partitions a_{N-1} and the pairs of indices (i, j), $1 \leqslant i < j \leqslant N$:

$$a_{N-1} = (\{1\}\,\{2\} \ldots \{i, j\} \ldots \{N\}) \,.$$

A partition a_i is finer than b_j, $a_i \subset b_j$ (or $b_j \supset a_i$) if $i > j$ and if a_i is obtained from b_j by further partitioning some of its clusters.

A chain α_k is a sequence of partitions

$$\alpha_k = (a_k, a_{k+1}, \ldots, a_{N-1}) = (a_k, \alpha_{k+1}) = \ldots = (a_k, a_{k+1}, \ldots, a_n, \alpha_{n+1})$$

where $a_n \supset a_{n+1}$, $k \leqslant n \leqslant N-2$, $(\alpha_{N-1} \equiv a_{N-1})$.

For every partition a_k a channel Hamiltonian H_{a_k} is defined by

$$H_{a_k} = H_0 + V_{a_k} \qquad V_{a_k} = \sum_{a_{N-1} \subset a_k} V_{a_{N-1}}$$

V_{a_k} is the sum of all interactions between particles within the different clusters of a_k. We set $V_{a_N} \equiv 0$.

The corresponding resolvents and off-shell scattering amplitudes are $R_{a_k}(z)$ and $T_{a_k}(z)$. Obviously, $H_{a_1} = H$ and $H_{a_N} = H_0$.

Let a_i be a partition and $k > i$. We shall use matrices $A^k_{a_i}$ of type (k, a_i) whose rows and columns are labeled by chains α_k, β_k with $a_k, b_k \subset a_i$:

$$A^k_{a_i} = \left(A^{\alpha_k, \beta_k}_{a_i}\right).$$

In general, the matrix elements $A^{\alpha_k, \beta_k}_{a_i}$ are linear operators in the Hilbert space $\mathcal{H}$ which are always defined on $\Delta(H_0)$.

Special matrices of type (k, a_i) are those of quasi-diagonal form:

$$A^k_{a_i(0)} = \left(A^{\alpha_k, \beta_k}_{a_k(0)} = A^{\alpha_{k+1}, \beta_{k+1}}_{a_k}\, \delta(a_k, b_k)\right) \tag{1.2a}$$

$$A^k_{a_i(1)} = \left(A^{\alpha_k, \beta_k}_{a_i(1)} = A^{\alpha_{k+2}, \beta_{k+2}}_{a_{k+1}}\, \delta(a_k, b_k)\, \delta(a_{k+1}, b_{k+1})\right) \tag{1.2b}$$

where

$$\delta(a_k, b_k) = \begin{cases} 1 & \text{for } a_k = b_k \\ 0 & \text{otherwise} \end{cases}$$

(1.2a) can be defined for $k \leqslant N-1$ (with the convention that $A^N_{a_{N-1}}$ is a 1×1 matrix) and (1.2b) for $k < N-1$.

We define further the "numerical" matrix $X^k_{a_i}$ by

$$X^{\alpha_k, \beta_k}_{a_i} = \prod_{j=k}^{N-1} [1 - \delta(a_j, b_j)] \prod_{j=k}^{N-2} \delta(b_{j+1} \subset a_j)$$

with

$$\delta(b_{i+1} \subset a_i) = \begin{cases} 1 & \text{for } b_{i+1} \subset a_i \\ 0 & \text{otherwise} \end{cases} .$$

In dealing with matrices of type (k, a_i), it is tacitly assumed that the upper index-chains of $A^{\alpha_k, \beta_k}_{a_i}$ vary only over those α_k, β_k where a_k, $b_k \subset a_i$. Throughout this section $Im\, z \neq 0$ is assumed and z is often dropped as a variable.

The α_k-connected components $T^{\alpha_k}_{a_i}$, $(a_k \subset a_i)$ of any subamplitude T_{a_i} are defined recursively, beginning by

$$T^{\alpha_{N-1}}_{a_i} = V_{a_{N-1}} + V_{a_{N-1}} R_{a_i} V_{a_i} . \tag{1.3}$$

The Faddeev equations [10] for $T^{\alpha_{N-1}}_{a_i}$ take the form

$$T^{\alpha_{N-1}}_{a_i} = T^{\alpha_{N-1}}_{a_{N-1}} + T^{\alpha_{N-1}}_{a_{N-1}} R_0 \sum_{b_{N-1} \neq a_{N-1}} T^{\beta_{N-1}}_{a_i}$$

$$T_{a_i} = \sum_{\alpha_{N-1}} T^{\alpha_{N-1}}_{a_i} \tag{1.4}$$

and are related to the Lippmann-Schwinger equations for T_{a_i} by invertible operations.

In analogy with the matrices $A^k_{a_i}$ of type (k, a_i), we shall consider the different $T^{\alpha_{N-1}}_{a_i}$ as components of a column vector $T^{N-1}_{a_i}$ of type $(N-1, a_i)$.

Then (1.4) can be written in matrix form by defining a 1×1 matrix $M^N_{a_{N-1}}$

$$M^N_{a_{N-1}} = T^{\alpha_{N-1}}_{a_{N-1}} \tag{1.5}$$

and by setting

$$Q^{N-1}_{a_i} = M^{N-1}_{a_{i(0)}} X^{N-1}_{a_i} R_0 . \tag{1.6}$$

We obtain:

$$T^{N-1}_{a_i} = T^{N-1}_{a_{i(0)}} + Q^{N-1}_{a_i} T^{N-1}_{a_i} \tag{1.7}$$

with $T^{\alpha_{N-1}}_{a_{i(0)}} \equiv T^{\alpha_{N-1}}_{a_{N-1}}$.

The F.Y. kernels are defined recursively starting from (1.4), (1.5) and (1.7):

$$M^{\alpha_{k-1}, \beta_{k-1}}_{a_i} = M^{\alpha_{k+1}, \beta_{k+1}}_{a_k} \delta(a_k, b_k)\, \delta(a_{k-1}, b_{k-1}) + \sum_{d_k \subset a_{k-1}} Q^{\alpha_k, \delta_k}_{a_i} M^{\delta_k, \beta_k}_{a_i} . \tag{1.8}$$

We use the convention of summation over all repeatedly occuring indices, the restrictions being indicated under the summation sign: in (1.8), $\sum\limits_{d_k \subset a_{k-1}}$ stands for the summation over all δ_k with $d_k \subset a_{k-1}$ (and of course $a_{k-1} \subset a_i$). Furthermore, we define

$$Q^{\alpha_k, \delta_k}_{a_i} = \sum M^{\alpha_{k+1}, \gamma_{k+1}}_{a_k} \delta(a_k, c_k)\, X^{\gamma_k, \delta_k}_{a_i} R_0$$

or in matrix form

$$Q^k_{a_i} = M^k_{a_i(0)} X^k_{a_i} R_0 .$$

Starting with (1.3), we introduce recursively

$$T^{\alpha_{k-1}}_{a_i} = \sum_{b_k \subset a_{k-1}} Q^{\alpha_k, \beta_k}_{a_i} T^{\beta_k}_{a_i} \tag{1.9}$$

and

$$T^{\alpha_{k-1}}_{a_i(0)} = T^{\alpha_{k-1}}_{a_{k-1}} .$$

It may be shown as in [11] that for $2 \leqslant k \leqslant N - 1$, $k > i$, the $T^{\alpha_k}_{a_i}$ satisfy the following equations on $\Delta(H_0)$:

$$T^k_{a_i} = T^k_{a_i(0)} + Q^k_{a_i} T^k_{a_i} \tag{1.10}$$

and that a solution of (1.10) together with a set of less connected amplitudes $T^{\alpha_j}_{a_j}$, $a_j \subset a_i$, $j > k$, leads to the resolvent R_{a_i} by

$$T_{a_i} = \sum_{\alpha_k} T^{\alpha_k}_{a_i} + \sum_{\alpha_{k+1}} T^{\alpha_{k+1}}_{a_{k+1}} + \ldots + \sum_{\alpha_{N-1}} T^{\alpha_{N-1}}_{a_{N-1}}$$

$$R_{a_i} = R_0 + R_0 T_{a_i} R_0 .$$

Finally, for the full interacting N-particle system:

$$T^{\alpha_2}_{a_1} = T^{\alpha_2}_{a_2} + \sum Q^{\alpha_2, \beta_2}_{a_1} T^{\beta_2}_{a_1} .$$

In order to prepare the proof of the complete continuity of the $(N-1)$st iteration of $Q^2_{a_i}$, we develop a simple graphical interpretation of (1.10).

For square-integrable potentials $V_{a_{N-1}}$ and for $Re\, z < 0$:

$$\| R_0(z) \| \leqslant \frac{1}{|z|} \quad \text{and} \quad \| V_{a_{N-1}} R_0(z) \| \leqslant \frac{\gamma}{|z|}, \ \gamma > 0$$

so that

$$\lim_{Re\, z \to -\infty} \| V_{a_{N-1}} R_0(z) \| = 0 .$$

There exists then a constant $C > -\infty$ such that for $Re\, z < C$:

$$\sum_{a_{N-1}} \| V_{a_{N-1}} R_0(z) \| < \frac{1}{2}$$

the series $\sum_{n=0}^{\infty} [V R_0(z)]^n$ converges uniformly in the Banach algebra of bounded operators on $\mathcal{H}$ and we have

$$T(z) = \sum_{n=0}^{\infty} [V R_0(z)]^n V . \tag{1.11}$$

Every term $V_{i_1 j_1} R_0 V_{i_2 j_2} R_0 \ldots R_0 V_{i_n j_n}$ can be represented by a graph $\mathcal{G}$ in the following way:

N horizontal lines correspond to the particles $1, 2, \ldots, N$. To every $V_{i_\nu j_\nu}$ a vertical line connects the lines for the particles i_ν and j_ν, starting from the left with $V_{i_1 j_1}$ and ending with $V_{i_n j_n}$ [12].

We shall often denote by the same symbol a graph and its analytical contribution. Every graph $\mathcal{G}$ has a unique sequential connectivity $\alpha_i = (a_i, a_{i+1}, \ldots, a_{N-1})$ from the left: $a_{N-1} = (\{1\}, \ldots, \{i_1 j_1\}, \ldots, \{N\})$. If α_j, $j > i$, is the sequential connectivity of $V_{i_1 j_1}, \ldots, V_{i_\tau j_\tau}$ and $V_{i_\nu j_\nu}$ is the next potential which connects two disjoint clusters A_j and B_j in a_j, then $\alpha_{j-1} = (a_{j-1}, \alpha_j)$, where a_{j-1} is obtained from a_j by replacing A_j and B_j by their union $A_j \cup B_j$.

The coarsest partition a_i is called the connectivity of the graph $\mathcal{G}$.

$T(z)$ is the analytic continuation of the sum of all graphs in (1.11) from $\{Re\, z < C\}$ into $\{Im\, z \neq 0\}$. $T_{a_i}^{\alpha_{N-1}}(z)$ is similarly the analytic continuation of the sum of all graphs with sequential connectivity β_k satisfying $b_k \subset a_i$ or $b_k = a_i$ and $\beta_{N-1} = \alpha_{N-1}$ (all these graphs have the same first vertex). The Faddeev equation (1.4) yields thus a cluster decomposition of $T_{a_i}^{\alpha_{N-1}}$ into a trivial part $T_{a_{N-1}}^{\alpha_{N-1}}$ with connectivity α_{N-1} and a remainder with higher connectivity.

Similarly, $T_{a_i}^{\alpha_k}$ is the analytic continuation of the sum of all graphs with sequential connectivity $\beta_n = (b_n, \ldots, b_{k-1}, \alpha_k)$, $b_n \subset a_i$ or $b_n = a_i$ (cf. (1.9), (1.10)).

Since identities between Born series as (1.11) have to hold for a large class of potentials V_{ij} and all z, with $Re\, z < C$, $C = C(V_{ij})$, there is in general no cancellation between different graphs, so that these identities do hold graphically too (private communication from W. Hunziker).

The following identity can then easily be proved:

$$T_{a_i}^{\alpha_k} = T_{a_{k+1}}^{\alpha_{k+1}} R_0 \sum_{\substack{c_{N-1} \subset a_k \\ c_{N-1} \not\subset a_{k+1}}} T_{a_i}^{c_{N-1}}$$

$$= T_{a_{N-1}}^{\alpha_{N-1}} R_0 \sum_{b_{N-1} \neq a_{N-1}} T_{a_{N-2}}^{\beta_{N-1}} R_0 \sum_{\substack{c_{N-1} \subset a_{N-2} \\ c_{N-1} \not\subset a_{N-3}}} T_{a_{N-3}}^{\gamma_{N-1}} R_0 \ldots R_0 \sum_{\substack{z_{N-1} \subset a_k \\ z_{N-1} \not\subset a_{k+1}}} T_{a_i}^{\zeta_{N-1}}$$

or, equivalently, upon using (1.1),)1.3):

$$T_{a_i}^{\alpha_k} R_0 = V_{a_{N-1}} R_0 W_{a_{N-1}}^{\alpha_{N-1}} \sum_{b_{N-1} \neq a_{N-1}} V_{b_{N-1}} R_0 W_{a_{N-2}}^{\beta_{N-1}} \cdots \sum_{\substack{z_{N-1} \subset a_k \\ z_{N-1} \not\subset a_{k+1}}} V_{s_{N-1}} R_0 W_{a_k}^{\zeta_{N-1}} \tag{1.12}$$

with bounded operators $W_{aN-1}^{\alpha_{N-1}}, \ldots W_{a_k}^{\zeta_{N-1}}$. The $W_{a_j}^{\lambda_{N-1}}$ are sums of the identity operator and of products of the type $V_{f_{N-1}} R_{h_j}$ and have at most the connectivity a_j, $j > k$.

$W_{a_k}^{\zeta_{N-1}}$ has at most the connectivity a_i.

We investigate now the F.Y. kernel $Q_{a_1}^{\alpha_2, \beta_2} \equiv Q^{\alpha_2, \beta_2}$. Notice first that

$$Q^{\alpha_2, \beta_2} = \sum M_{a_2}^{\alpha_3, \gamma_3} \delta(a_2, c_2) X_{a_2}^{\gamma_3, \delta_3} \delta(b_3 \subset c_2) [1 - \delta(b_2, c_2)] R_0$$

$$\equiv \hat{Q}_{a_2}^{\alpha_3, \beta_3} \delta(b_3 \subset a_2) [1 - \delta(a_2, b_2)] . \tag{1.13}$$

Combining (1.9) and (1.13):

$$T^{\alpha_1}_{a_1} = T^{\alpha_2}_{a_2} R_0 \sum_{C_{N-1} \not\subset a_2} T^{\gamma_{N-1}}_{a_1} = \sum Q^{\alpha_2,\beta_2} T^{\beta_2}_{a_1} = \sum_{\substack{b_3 \subset a_2 \\ b_2 \neq a_2}} \hat{Q}^{\alpha_3,\beta_3}_{a_2} T^{\beta_3}_{b_3} R_0 \sum_{\substack{C_{N-1} \subset b_2 \\ C_{N-1} \not\subset b_3}} T^{\gamma_{N-1}}_{a_1}$$

$$= \sum_{b_3 \subset a_2} \hat{Q}^{\alpha_3,\beta_3}_{a_2} T^{\beta_3}_{b_3} R_0 \sum_{C_{N-1} \not\subset a_2} T^{\gamma_{N-1}}_{a_1} = \sum_{\substack{b_3 \subset a_2 \\ (b_2 \neq a_2)}} \hat{Q}^{\alpha_3,\beta_3}_{a_2} T^{\beta_3}_{b_3} R_0 \sum_{C_{N-1} \not\subset a_2} T^{\gamma_{N-1}}_{a_1} .$$

The comparison of the coefficients of $T^{\gamma_{N-1}}_{a_1}$ gives:

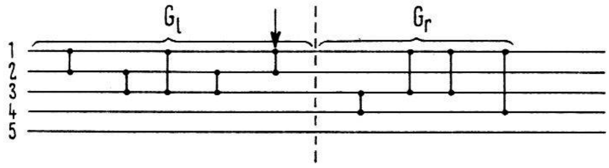
$$T^{\alpha_2}_{a_2} = \sum_{b_3 \subset a_2} \hat{Q}^{\alpha_3,\beta_3}_{a_2} T^{\beta_3}_{b_3} . \tag{1.14}$$

The identity (1.14) provides a simple graphical characterisation of $\hat{Q}^{\alpha_3,\beta_3}_{a_2}$.

$T^{\alpha_2}_{a_2}$ is the sum of all graphs having the sequential connectivity α_2 and similarly for $T^{\beta_3}_{b_3}$, while $\hat{Q}^{\alpha_3,\beta_3}_{a_2}$ is the sum over all graphs which occur in the expansion of $T^{\alpha_2}_{a_2}$ with respect to $T^{\beta_3}_{b_3}$. This expansion is characterised as follows:

To every graph $\mathcal{G}$ in $T^{\alpha_2}_{a_2}$ there exists a unique rightmost vertex $V_{c_{N-1}}$ and a subgraph $\mathcal{G}_r$ to its right, such that $V_{c_{N-1}} R_0 \mathcal{G}_r$ has the connectivity a_2. Then, the subgraph $\mathcal{G}_r$ possesses the sequential connectivity β_3, $b_3 \subset a_2$, $c_{N-1} \not\subset b_3$. On the left of $\mathcal{G}_r$ there remains a subgraph $\mathcal{G}_l$ with a sequential connectivity α_L, $2 \leqslant L \leqslant N-1$.

Example: $N = 5$



$$\begin{aligned} \alpha_2 &= (\{1234\}\{5\},\ \{123\}\{4\}\{5\},\ \{12\}\{3\}\{4\}\{5\}) = (a_2, \alpha_3) \\ c_{N-1} &= \{12\}\{3\}\{4\}\{5\} \\ \beta_3 &= (\{134\}\{2\}\{5\},\ \{1\}\{2\}\{34\}\{5\}),\ \alpha_L = \alpha_3 . \end{aligned}$$

If $L = 2$, then β_3 and c_{N-1} are only restricted by $b_3 \subset a_2$, $c_{N-1} \not\subset b_3$, $c_{N-1} \subset a_2$. If $L > 2$, then one has to require $c_{N-1} \not\subset b_3$, $c_{N-1} \subset a_L$ and

$$\alpha_L \llcorner \beta_3 = \alpha_2 . \tag{1.15}$$

The symbol $\llcorner$ in (1.15) has the following meaning: for any two chains α_i and β_j, the chain $\alpha_i \llcorner \beta_j = \gamma_k$ is defined as $\gamma_k = (c_k, c_{k+1}, \ldots, c_{i-1}, \alpha_i)$, $k \leqslant i$, where the connections $c_{i-1}, \ldots, c_k$ arise from α_i by the successive adjunction of potentials which together produce the sequential connectivity β_j. One can convince oneself that this definition only depends on α_i and β_j and not on the specific way in which β_j is realised.

Conversely, any two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ where $\mathcal{G}_1$ has the sequential connectivity α_L and $V_{c_{N-1}}$ as right-most interaction and $\mathcal{G}_2$ the sequential connectivity β_3, give in the form $\mathcal{G}_1 R_0 \mathcal{G}_2$ rise to a graph in $T_{a_2}^{\alpha_2}$, provided that $c_{N-1} \not\subset b_3, c_{N-1} \subset a_L$ and (1.15) is satisfied.

According to the case whether $c_{N-1} \subset a_{L+1}$ or $c_{N-1} \not\subset a_{L+1}$, we obtain:

$$\hat{Q}_{a_2}^{\alpha_3, \beta_3} = \Sigma^1 \, T_{a_L}^{\alpha_L} R_0 V_{c_{N-1}} R_0 + \Sigma^2 \, T_{a_{L+1}}^{\alpha_{L+1}} R_0 V_{c_{N-1}} R_0 \tag{1.16}$$

In Σ^1 and Σ^2 the summation is over all c_{N-1} and all α_L such that $\alpha_L \llcorner \beta_3 = \alpha_2$, $c_{N-1} \not\subset b_3$ and $c_{N-1} \subset a_L$. In Σ^2 one has in addition the restriction $c_{N-1} \not\subset a_{L+1}$ and the convention $T_{a_N}^{\alpha_N} R_0 \equiv \mathbf{1}$.

Using (1.12) and (1.16) we have thus the

Lemma 1.1. –

$$\hat{Q}_{a_2}^{\alpha_3, \beta_3} = \Sigma^3 \, V_{i_1 j_1} R_0 W_{J1} V_{i_2 j_2} R_0 W_{J2} \dots V_{i_L j_L} R_0 W_{JL} \tag{1.17}$$

where Σ^3 extends over all sequences $J = ((i_1, j_1), \dots, (i_L, j_L))$ satisfying

$$(i_1, j_1) \llcorner (i_2, j_2) \llcorner \dots \llcorner (i_L, j_L) \llcorner \beta_3 = \alpha_2$$

and where $W_{J1}, \dots, W_{JL}$ are bounded operators which are finite sums of the identity operator and of products of the type $V_{d_{N-1}} R_{n_k}$. W_{Jk} has at most the connectivity of $(i_1, j_1) \llcorner \dots \llcorner (i_k, j_k)$.

We are now prepared to prove the

Theorem 1.1. – For $Im\, z \neq 0$ and square-integrable potentials $V_{a_{N-1}}$:

$$[(Q^2)^{N-1}]^{\alpha_2^1, \alpha_2^N} = \sum_{\alpha_2^2, \dots, \alpha_2^{N-1}} Q^{\alpha_2^1, \alpha_2^2} \, Q^{\alpha_2^2, \alpha_2^3} \dots Q^{\alpha_2^{N-1}, \alpha_2^N} \tag{1.18}$$

is a Hilbert-Schmidt operator (HS operator):

$$\int \prod_{j=1}^{N} d^3p_i \, d^3p_i' \, \delta\left(\sum_{j=1}^{N} \boldsymbol{p}_i\right) \delta\left(\sum_{j=1}^{N} \boldsymbol{p}_i'\right) |([(Q^2)]^{N-1})^{\alpha_2^1 \alpha_2^N} (\boldsymbol{p}_1, \dots, \boldsymbol{p}_N, \boldsymbol{p}_1', \dots, \boldsymbol{p}_N'; z)|^2 < C(z)$$

with $C(z) < \infty$.

Proof. – We bring first $(Q^2)^{N-1}$ into a more convenient form: due to (1.13), $(Q^2)^{N-1}$ may be expressed in terms of $\hat{Q}_{a_2^i}^{\alpha_3^i, \alpha_2^{i+1}}$ for which Lemma 1.1 gives an explicit representation. Notice that in (1.18) the sum extends over all $\alpha_2^2, \dots, \alpha_2^{N-1}$ satisfying

$$\alpha_2^1 \llcorner \alpha_2^2 = \alpha_1^1, \dots, \alpha_2^i \llcorner \alpha_2^{i+1} = \alpha_1^i, \dots, \alpha_2^{N-1} \llcorner \alpha_2^N = \alpha_1^{N-1}.$$

Let $\alpha_{L(i)}^i$ be the minimal left-chain in $\hat{Q}_{a_2^i}^{\alpha_3^i, \alpha_3^{i+1}}$, i.e. let $L(i)$ be the largest integer such that $\alpha_{L(i)}^i \llcorner \alpha_3^{i+1} = \alpha_2^i$. Then clearly:

$$\alpha_{L(i)}^i \llcorner \alpha^{i+1} = \alpha_{s(i)}^i$$

where $s(i) \leqslant \min\{L(i), i\}$.

Furthermore

$$\alpha_{L(i)}^i \llcorner \alpha_3^{i+1} = \alpha_2^i$$

and from (1.13),

$$\alpha_{L(i)}^i \llcorner \alpha_2^{i+1} = \alpha_1^i .$$

Therefore

$$\alpha_1^1 = \alpha_{L(1)}^1 \llcorner\!\!\_\, \alpha_2^2 = \alpha_{L(1)}^1 \llcorner\!\!\_\, \alpha_{L(2)}^2 \llcorner\!\!\_\, \alpha_3^3 \,.$$

We make the induction assumption that for $1 \leqslant k < N - 1$:

$$\alpha_1^1 = \alpha_{L(1)}^1 \llcorner\!\!\_\, \alpha_{L(2)}^2 \llcorner\!\!\_\, \ldots \llcorner\!\!\_\, \alpha_{L(k-1)}^{k-1} \llcorner\!\!\_\, \alpha_k^k \,.$$

Since obviously

$$\alpha_k^k \llcorner\!\!\_\, a_{k-1}^k \llcorner\!\!\_\, \ldots \llcorner\!\!\_\, a_{s(k)}^k = \alpha_{s(k)}^k$$

and

$$\alpha_1^1 \llcorner\!\!\_\, a_{k-1}^k \llcorner\!\!\_\, a_{k-2}^k \llcorner\!\!\_\, \ldots \llcorner\!\!\_\, a_{s(k)}^k = \alpha_1^1$$

we obtain

$$\begin{aligned}\alpha_1^1 &= \alpha_{L(1)}^1 \llcorner\!\!\_\, \alpha_{L(2)}^2 \llcorner\!\!\_\, \ldots \llcorner\!\!\_\, \alpha_{L(k-1)}^{k-1} \llcorner\!\!\_\, a_{k-1}^k \llcorner\!\!\_\, \ldots \llcorner\!\!\_\, a_{k(s)}^k \\ &= \alpha_{L(1)}^1 \llcorner\!\!\_\, \alpha_{L(2)}^2 \llcorner\!\!\_\, \ldots \llcorner\!\!\_\, \alpha_{L(k-1)}^{k-1} \llcorner\!\!\_\, \alpha_{L(k)}^k \llcorner\!\!\_\, \alpha_{k+1}^{k+1}\end{aligned}$$

since α_N^N is trivial, we have finally

$$\alpha_1^1 = \alpha_{L(1)}^1 \llcorner\!\!\_\, \alpha_{L(2)}^2 \llcorner\!\!\_\, \ldots \llcorner\!\!\_\, \alpha_{L(N-1)}^{N-1} \,.$$

Together with Lemma 1.1 this gives the representation

$$[(Q^2)^{N-1}]^{\alpha_2^1, \alpha_2^N} = \sum_J V_{i_1 j_1} R_0 W_{J1} V_{i_2 j_2} R_0 W_{J2} \ldots V_{i_{N-1} j_{N-1}} R_0 W_{J\,N-1} \qquad (1.19)$$

with summation over all sequences $J = ((i_1, j_1), \ldots, (i_{N-1}, j_{N-1}))$ such that

$$(i_1, j_1) \llcorner\!\!\_\, (i_2, j_2) \llcorner\!\!\_\, \ldots \llcorner\!\!\_\, (i_{N-1}, j_{N-1}) = \alpha_1^1$$

and bounded operators W_{Jk} which are again linear combinations of the identity operator and products $V_{d_{N-1}} R_{k_j}$ and have at most the connectivity of

$$(i_1, j_1) \llcorner\!\!\_\, (i_2, j_2) \llcorner\!\!\_\, \ldots \llcorner\!\!\_\, (i_k, j_k) \,.$$

Therefore the arguments of [13] are directly applicable to prove the complete continuity of (1.19).

For later use we now introduce amplitudes with specific sequential connectivities from the left as well as from the right.

The amplitude

$$N_{a_i}^{\alpha_{N-1}, \beta_{N-1}} = V_{a_{N-1}} \delta(a_{N-1}, b_{N-1}) + V_{a_{N-1}} R_{a_i} V_{b_{N-1}} = M_{a_i}^{\alpha_{N-1}, \beta_{N-1}}$$

satisfies the Faddeev equations

$$\begin{aligned}N_{a_i}^{\alpha_{N-1}, \beta_{N-1}} &= N_{a_i(0)}^{\alpha_{N-1}, \beta_{N-1}} + \sum Q_{a_i}^{\alpha_{N-1}, \gamma_{N-1}} N_{a_i}^{\gamma_{N-1}, \beta_{N-1}} \\ &= N_{a_i(0)}^{\alpha_{N-1}, \beta_{N-1}} + \sum N_{a_i}^{\alpha_{N-1}, \beta_{N-1}} \overline{Q}_{a_i}^{\delta_{N-1}, \beta_{N-1}}\end{aligned}$$

where

$$N_{a_i(0)}^{\alpha_{N-1}, \beta_{N-1}} = T_{a_{N-1}} \delta(a_{N-1}, b_{N-1})$$

and

$$\bar{Q}_{a_i}^{N-1} = R_0 X_{a_i}^{N-1} M_{a_i}^{N-1}$$

in analogy with (1.6).

Let

$$\overline{X}_{a_i}^{\alpha_k, \beta_k} = X_{a_i}^{\beta_k, \alpha_k}$$

$$\overline{M}_{a_i}^{\alpha_{N-1}, \beta_{N-1}} = M_{a_i}^{\alpha_{N-1}, \beta_{N-1}} \qquad \overline{M}_{a_{N-1}}^{\alpha_N, \beta_N} = T_{a_{N-1}}$$

and define recursively for $3 \leqslant k \leqslant N-1$:

$$\overline{M}_{a_i}^{\alpha_{k-1}, \beta_{k-1}} = \overline{M}_{a_k}^{\alpha_{k+1}, \beta_{k+1}} \delta(a_k, b_k)\, \delta(a_{k-1}, b_{k-1}) + \sum_{d_k \subset a_{k-1}} \overline{M}_{a_i}^{\alpha_k, \delta_k} \overline{Q}_{a_i}^{\delta_k, \beta_k}$$

with

$$\overline{Q}_{a_i}^{k-1} = R_0 \overline{X}_{a_i}^{k-1} \overline{M}_{a_i(0)}^{k-1}.$$

We can then define recursively $N_{a_i}^{\alpha_{k-1}, \beta_{k-1}}$ by raising the left and the right connectivity:

$$N_{a_i}^{\alpha_{k-1}, \beta_{k-1}} = \sum_{\substack{c_k \subset a_{k-1} \\ d_k \subset b_{k-1}}} Q_{a_i}^{\alpha_k, \gamma_k} N_{a_i}^{\gamma_k, \delta_k} \overline{Q}_{a_i}^{\delta_k, \beta_k}.$$

The amplitudes $N_{a_i}^{\alpha_k, \beta_k}$ satisfy the F.Y. equations

$$N_{a_i}^{\alpha_k, \beta_k} = N_{a_i(0)}^{\alpha_k, \beta_k} + \sum Q_{a_i}^{\alpha_k, \gamma_k} N_{a_i}^{\gamma_k, \beta_k} = \overline{N}_{a_i(0)}^{\alpha_k, \beta_k} + \sum N_{a_i}^{\alpha_k, \delta_k} \overline{Q}_{a_i}^{\delta_k, \beta_k}.$$

By iterating from both sides L times, $L = 1, 2, \ldots$ we obtain with $Q_{a_1}^2 = Q$, and $N_{a_1}^2 = N$:

$$N^{\alpha_2, \beta_2} = N_L^{\alpha_2, \beta_2} + \sum (Q^L)^{\alpha_2, \gamma_2} N^{\gamma_2, \delta_2} (\overline{Q}_L)^{\delta_2, \beta_2}. \tag{1.20}$$

The inhomogeneity $N_L^{\alpha_2, \beta_2}$ in (1.20) can be entirely expressed in terms of amplitudes for subsystems of the N-particle system.

Thus, N^{α_2, β_2} is represented, up to the inhomogeneity, as a sum over two-sided amplitudes N^{γ_2, δ_2} which are "sandwiched" between (for large L) highly connected kernels $(Q_L)^{\alpha_2, \gamma_2}$ and $(\overline{Q}^L)^{\delta_2, \beta_2}$.

In conclusion, we remark that the F.Y. equations for subsystems a_i, $1 < i \leqslant N-1$, enjoy, in the relative momentum Hilbert spaces $\mathcal{H}_{a_i}$, similar properties as announced in Lemma 1.1 and Theorem 1.1 for the kernel $Q_{a_1}^2$:

Theorem 1.2. –

$$Q_{a_i}^{\alpha_{i+1}, \beta_{i+1}} = \hat{Q}_{a_{i+1}}^{\alpha_{i+2}, \beta_{i+2}} \delta(b_{i+2} \subset a_{i+1}) \left[1 - \delta(a_{i+1}, b_{i+1})\right]$$

satisfies on $\mathcal{H}_{a_i}$ a representation of the type (1.17). For $Im\, z \neq 0$ and $V_{a_{N-1}} \in L^2(R^3)$ for all $a_{N-1} \subset a_i$,

$$[(Q_{a_i}^{i+1})^{N-i}]^{\alpha_{i+1}, \beta_{i+1}}$$

is a HS operator in $\mathcal{H}_{a_i}$.

The proof of this theorem is entirely similar to the proof of Theorem 1.1.

2. Analyticity Properties in Perturbation Theory

The investigation of the analyticity properties of the perturbation series (1.10) is an important first step before studying the exact amplitude. We shall see in the next section that for a class of two-body potentials, the exact amplitudes have in the "physical sheet" the same analyticity properties as those which we shall derive here for an arbitrary term of the Born series.

A connected graph is called c-connected (c positive integer), if it can be decomposed into c connected subgraphs by $c - 1$ vertical cuts (without cutting a potential line).

The graphs in the Born series may be classified into

(i) disconnected graphs

(ii) weakly connected graphs (less than c_0-connected)

(iii) strongly connected graphs (more than c_0-connected)

The value of c_0 will be precised in the sequel ($c_0 = 2L$ in Theorem 2.1).

Due to the momentum-conservation δ-functions, disconnected diagrams are only defined on certain subvarieties in the variables $\boldsymbol{x}_1, \ldots, \boldsymbol{y}_N$. The exact scattering amplitudes for the corresponding process can be obtained by the convolution prescription from the amplitudes of the subprocesses [12, 14].

Weakly connected amplitudes have rescattering singularities, whose positions depend in a complicated manner on the configuration of both the initial and final momenta (cf. [9] for $N = 3$).

Amplitudes of sufficiently high connectivity have only threshold singularities at loci which do not couple incoming and outgoing momenta. For their characterisation, let $a_i = (a(1), a(2), \ldots, a(i))$ be a partition of $\{1, \ldots, N\}$ into i clusters $a(\nu)$, $1 \leqslant \nu \leqslant i$. Let

$$\boldsymbol{x}_{a(\nu)} = \sum_{k \in a(\nu)} \boldsymbol{x}_k \qquad \mu_{a(\nu)} = \Big(2 \sum_{k \in a(\nu)} m_k\Big)^{-1}.$$

Then

$$\Omega^N_{a_i} = \left\{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \Omega^N\colon \sum_{\nu=1}^{i} \mu_{a(\nu)}\, \boldsymbol{x}^2_{a(\nu)} = 1\right\}$$

is the subvariety of Ω^N where all the relative momenta within the subsystems $a(1), \ldots, a(i)$ vanish. We set

$$\Omega^N_0 = \Omega^N - \bigcup_{a_i, i>1} \Omega^N_{a_i}$$

and

$$H_\varrho = \{k \in \mathbf{C}^1\colon Im\, k \geqslant 0\} - \{k = i\lambda\,,\ \lambda \geqslant \varrho > 0\} - \{k = 0\}\,.$$

The mapping $k \to E = k^2$ maps the interior of H_ϱ biholomorphically onto the complex E-plane with the cuts $(-\infty, -\varrho^2]$ and $[0, +\infty)$.

Using the two-sided amplitudes N^{α_2,β_2} and (1.20) the strongly connected amplitudes are generated by representing $T(z)$ in the form

$$T = T_L + \sum_{\alpha_2, \ldots, \delta_2} (Q^L)^{\alpha_2,\delta_2}\, N^{\delta_2,\gamma_2}\, (\bar{Q}^L)^{\gamma_2,\beta_2} \tag{2.1}$$

with a sufficiently large L.

The main result of this section is contained in

Theorem 2.1. – For two-body potentials of the class (A), (B) and for L sufficiently large, there exists a $\varrho > 0$ such that every term in the perturbation series for $T - T_L$ is

(i) Hölder continuous for $k \in H_\varrho$ and $(\boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in \Omega^N \times \Omega^N$

(ii) holomorphic for $k \in H_\varrho$ and $(\boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in \Omega_0^N \times \Omega_0^N$.

Remark. – A function $f: M \subset C^1 \times \Omega_C^N \times \Omega_C^N \to C^1$ is called holomorphic in the set M, if for every point $P \in M$, there exists a neighbourhood $U(P) \subset C^1 \times \Omega_C^N \times \Omega_C^N$ and a convergent power series f_P in the local coordinates of $U(P)$ such that $f = f_P$ on $M \cap U(P)$.

The proof of Theorem 2.1 will be given in a series of lemmas. An arbitrary graph $\mathcal{G}$ contributing to $T - T_L$ can be splitted in the form

$$\mathcal{G} = \mathcal{G}_1 \, R_0 \, \mathcal{G}_2 \, R_0 \, \mathcal{G}_3 \tag{2.2}$$

where $\mathcal{G}_1 \, R_0$ and $R_0 \, \mathcal{G}_3$ contribute to $(Q^L)^{\alpha_2, \delta_2}$ resp. $(\overline{Q}^L)^{\gamma_2, \beta_2}$, while $\mathcal{G}_2$ contributes to N^{δ_2, γ_2}.

According to the "Feynman rules" [12], $\mathcal{G}$ is holomorphic in $k, \boldsymbol{x}_1, \ldots, \boldsymbol{y}_N$, if in its Feynman integral the real contour of loop momenta avoids all singularities of the potentials and free propagators and if the integral converges absolutely with all its derivatives with respect to $k, \boldsymbol{x}_1, \ldots, \boldsymbol{y}_N$.

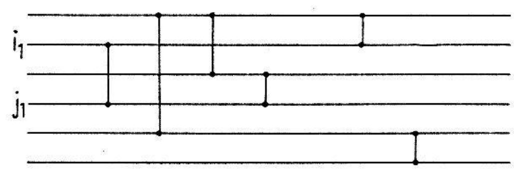
We first consider $\mathcal{G}_1$ in (2.2) and make a provisory choice of the loop momenta according to the following algorithm, which is closely related to a construction by P. Federbush [15]:

Let $V_{i_1, j_1}, V_{i_2, j_2}, \ldots, V_{i_t j_t}$ be the sequence of potentials in $\mathcal{G}_1$ from left to right. Let $\boldsymbol{k}_1^0, \ldots, \boldsymbol{k}_N^0$, $\boldsymbol{k}_i^0 = k \, \boldsymbol{x}_i$, $1 \leqslant i \leqslant N$, be the external momenta on the left of the graph. Define

$$\boldsymbol{k}_l^1 = \begin{cases} \boldsymbol{k}_l^0 & l \neq i_1, j_1 \\ \mu_{j_1} (\mu_{i_1} + \mu_{j_1})^{-1} (\boldsymbol{k}_{i_1}^0 + \boldsymbol{k}_{j_1}^0) & l = i_1 \\ \mu_{i_1} (\mu_{i_1} + \mu_{j_1})^{-1} (\boldsymbol{k}_{i_1}^0 + \boldsymbol{k}_{j_1}^0) & l = j_1 \, . \end{cases} \tag{2.3}$$

Suppose the (i_1)st particle line be not connected with any other line before the (j_1)st line ("longest segment").

Example:



Put the first loop momentum $\boldsymbol{s}^1$ on this "longest segment" and define

$$\boldsymbol{s}_l^1 = \begin{cases} 0 & l \neq i_1, j_1 \\ \boldsymbol{s}^1 & l = i_1 \quad (\text{'longest segment'}) \, . \\ -\boldsymbol{s}^1 & l = j_1 \end{cases}$$

The first intermediate state carries the momenta $\boldsymbol{k}_l^1 + \boldsymbol{s}_l^1$, $1 \leqslant l \leqslant N$, with

$$\sum_{l=1}^{N} \boldsymbol{k}_l^1 = \sum_{l=1}^{N} \boldsymbol{s}_l^1 = 0 .$$

Let $\boldsymbol{k}_l^\varrho$ and $\boldsymbol{s}_l^\varrho$, $1 \leqslant l \leqslant N$, be chosen for all intermediate states following V_{i_ϱ, j_ϱ}, $1 \leqslant \varrho \leqslant r < t$. After $V_{i_{r+1} j_{r+1}}$ we introduce

$$\boldsymbol{k}_l^{r+1} = \begin{cases} \boldsymbol{k}_l^r & l \neq i_{r+1}, j_{r+1} \\ \mu_{j_{r+1}} (\mu_{i_{r+1}} + \mu_{j_{r+1}})^{-1} (\boldsymbol{k}_{i_{r+1}}^r + \boldsymbol{k}_{j_{r+1}}^r) & l = i_{r+1} \\ \mu_{i_{r+1}} (\mu_{i_{r+1}} + \mu_{j_{r+1}})^{-1} (\boldsymbol{k}_{i_{r+1}}^r + \boldsymbol{k}_{j_{r+1}}^r) & l = j_{r+1} \end{cases} \tag{2.4}$$

and

$$\boldsymbol{s}_l^{r+1} = \begin{cases} \boldsymbol{s}_l^r & l \neq i_{r+1}, j_{r+1} \\ \boldsymbol{s}^{r+1} & l = i_{r+1} \text{ ('longest segment')} \\ \boldsymbol{s}_{j_r}^r + \boldsymbol{s}_{i_r}^r - \boldsymbol{s}^{r+1} & l = j_{r+1} \end{cases} \tag{2.5}$$

again with

$$\sum_{l=1}^{N} \boldsymbol{k}_l^{r+1} = \sum_{l=1}^{N} \boldsymbol{s}_l^{r+1} = 0 .$$

Lemma 2.1. – The rth energy denominator

$$D_r = k^2 - \sum_{l=1}^{N} \mu_l \, (\boldsymbol{k}_l^r + \boldsymbol{s}_l^r)^2$$

never vanishes for $\boldsymbol{s}^1, \ldots, \boldsymbol{s}^r \in \boldsymbol{R}^3$, real $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \Omega_0^N$ and $Im\, k \neq 0$.

Proof. – According to (2.3, (2.5), the $(k)^{-1} \boldsymbol{k}_l^r$, $1 \leqslant l \leqslant N$, are linear combinations of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$. We set

$$\boldsymbol{k}_l^r = k\, \boldsymbol{x}_l^r \,, \quad 1 \leqslant l \leqslant N \,, \; k = \varkappa\, e^{i\varphi} \,, \; \varkappa > 0 \,, \quad \varphi \neq n\pi \,, \; n \text{ integer.}$$

Then a simple calculation gives

$$D_r = -\varkappa^2 \left[1 - \sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^r)^2\right] - \sum_{l=1}^{N} \mu_l (\boldsymbol{s}_l^r)^2 + Im\, D_r \cdot \operatorname{cotg}\varphi + i\, Im\, D_r \,. \tag{2.6}$$

Now, either $Im\, D_r \neq 0$, in which case the lemma holds trivially, or $Im\, D_r = 0$ and then $D_r < 0$ for all real $\boldsymbol{s}^1, \ldots, \boldsymbol{s}^r$, provided that

$$\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^r)^2 < 1 .$$

By construction, one has for all $r \geqslant 0$ and $\varrho > r$:

$$\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^\varrho)^2 \leqslant \sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^r)^2 \leqslant 1 .$$

Therefore either $\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^r)^2 < 1$ for some $r \geqslant 0$ and then the same inequality holds for all $\varrho > r$, or $\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^r)^2 = 1$ and then also for all $0 \leqslant \varrho < r$.

Let $a_i = (a(1), \ldots, a(i))$, $i \geqslant 1$, be the partition corresponding to

$$(i_1, j_1) \llcorner (i_2, j_2) \llcorner \ldots \llcorner (i_r, j_r) .$$

We claim that $\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^r)^2 = 1$, if and only if

$$\sum_{\nu=1}^{i} \mu_{a(\nu)} \boldsymbol{x}_{a(\nu)}^2 = 1 . \tag{2.7}$$

The condition (2.7) is necessary and sufficient for $r = 1$, since $\sum_{l=1}^{N} \mu_l \boldsymbol{x}_l^2 = 1$ and

$$\sum_{\substack{l \neq i_1, j_1 \\ l=1}}^{N} \mu_l \boldsymbol{x}_l^2 + \left[\mu_{i_1}\left(\frac{\mu_{j_1}}{\mu_{i_1}+\mu_{j_1}}\right)^2 + \mu_{j_1}\left(\frac{\mu_{i_1}}{\mu_{i_1}+\mu_{j_1}}\right)^2\right] (\boldsymbol{x}_{i_1} + \boldsymbol{x}_{j_1})^2 = 1 \tag{2.8}$$

are compatible, if and only if the relative momenta of the (i_1)st and (j_1)st particles vanish, that is

$$\mu_{i_1} \boldsymbol{x}_{i_1} = \mu_{j_1} \boldsymbol{x}_{j_1}$$

which is equivalent to (2.7).

Assume now that for some $1 \leqslant p < r$, the identity $\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^p)^2 = 1$ is equivalent with the vanishing of all relative momenta within the clusters of the partition $b_h = (b(1), \ldots, b(h))$ corresponding to $(i_1, j_1) \llcorner \ldots \llcorner (i_p, j_p)$. If $V_{i_{p+1}, j_{p+1}}$ connects two particles within the same cluster $b(\nu)$, $1 \leqslant \nu \leqslant h$, then evidently $\boldsymbol{x}_l^p = \boldsymbol{x}_l^{p+1}$, $1 \leqslant l \leqslant N$. If, on the other hand $V_{i_{p+1} j_{p+1}}$ connects two particles within different clusters $b(\nu) \neq b(\nu')$, then a calculation similar to (2.8) shows that

$$\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^p)^2 = \sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^{p+1})^2$$

is equivalent with the vanishing of the relative momentum of $b(\nu)$ and $b(\nu')$.

From (2.6) one deduces the

Lemma 2.2. – For $s^1, \ldots, s^r \in R^3$, real $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \Omega^N$ and $Im\, k \neq 0$, D_r can only vanish for

$$\boldsymbol{s}_1^r = \boldsymbol{s}_2^r = \ldots = \boldsymbol{s}_N^r = 0 .$$

The Hölder continuity as announced in Theorem 2.1, will be an immediate consequence of this lemma, by a power counting argument as in [5].

A sequence of potentials $V_{i_\varrho j_\varrho}$, $1 \leqslant \varrho \leqslant r$, is called connecting, if $(i_1, j_1) \llcorner \ldots \llcorner (i_r, j_r)$ has the connectivity a_1.

Lemma 2.3. – There exists a $\delta = \delta(\mu_1, \ldots, \mu_N)$, $0 < \delta < 1$, such that for every connecting sequence of potentials $V_{i_\varrho j_\varrho}$, $1 \leqslant \varrho \leqslant r$:

$$\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^r)^2 \leqslant \delta < 1 .$$

Proof. – Let $\boldsymbol{x}_l^0 = \boldsymbol{x}_l$, $1 \leqslant l \leqslant N$ and $\boldsymbol{k}_l^r = k\,\boldsymbol{x}_l^r$ be defined by (2.3) and (2.4). Then one has

$$\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^r)^2 - \sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^{r+1})^2 = (\mu_{i_{r+1}} + \mu_{j_{r+1}})^{-1} (\mu_{i_{r+1}}\, \boldsymbol{x}_{i_{r+1}}^r - \mu_{j_{r+1}}\, \boldsymbol{x}_{j_{r+1}}^r)^2 .$$

If the sequence $J = ((i_1, j_1), \ldots, (i_r, j_r))$ is connecting, then the quadratic form

$$Q_J(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = \sum_{\varrho=0}^{r-1} (\mu_{i_{\varrho+1}} + \mu_{j_{\varrho+1}})^{-1} (\mu_{i_{\varrho+1}}\, \boldsymbol{x}_{i_{\varrho+1}}^{\varrho} - \mu_{j_{\varrho+1}}\, \boldsymbol{x}_{j_{\varrho+1}}^{\varrho})^2$$

is positive definite on Ω^N. Therefore:

$$1 \geqslant 1 - \sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^r)^2 = Q_J(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = 1 - \delta_J(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$$

for $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \Omega^N$.
It remains to show that

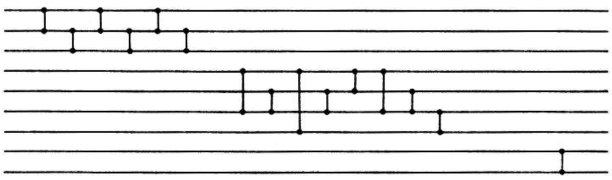
$$\delta = \max_{J, (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \Omega^N} \delta_J(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) < 1 . \tag{2.9}$$

This follows by complete induction:

We assume that the lemma holds for all subsystems of less than N particles. The set of all possible sequences J in (2.9) may be then reduced to a finite subset.

First we identify sequences where one and the same potential occurs consecutively several times, with a sequence with only one of the iterated potential occuring. Next, we consider those sequences where a subsequence generates a strongly connected subgraph $a_i = (a(1), \ldots, a(i))$, without any of its particle lines being connected in-between to other lines.

Example:



Due to the induction hypothesis and up to an infinitesimal error (provided that all connectivities are sufficiently high), these sequences lead to the same quadratic forms Q_J as a sequence where the relative momenta within $a(1), a(2), \ldots, a(i)$ are reduced to zero.

Noting that Ω^N is a compact $3\,(N-1) - 1$ dimensional surface in $\mathbf{R}^{3N}$, lemma 2.3 is proved by a finite, repeated application of this reduction procedure, since for fixed given J and as a consequence of the Heine-Borel theorem, $\max\limits_{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)} \delta_J(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) < 1$.

If $V_{i_1 j_1}, \ldots, V_{i_r j_r}$ is n times connecting, then

$$\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^r)^2 \leqslant \delta^n .$$

Thus, for highly connected diagrams the translations $\boldsymbol{x}_l^r$ tend to zero. Therefore one can in sufficiently connected graphs introduce loop momenta $\boldsymbol{s}^r \in R^3$ independent of the $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$, with D_r remaining different from zero for $Im\, k \neq 0$.

Let $\boldsymbol{k}_l^\sigma$, $\boldsymbol{s}_l^\sigma$ be chosen as before for $1 \leqslant \sigma \leqslant r$ and let $(i_1, j_1), \ldots, (i_r, j_r)$ be n-fold connecting with $n \geqslant c_0 (\mu_1, \ldots, \mu_N)$. – Assume that the sequence $(i_{r+1}, j_{r+1}), \ldots, (i_t, j_t)$ is connecting. Then we modify the prescription (2.4) in such a way that the $\boldsymbol{x}_l^t$, $1 \leqslant l \leqslant N$, vanish identically:

By assumption, there is a connecting subsequence of $N-1$ potentials, say $(i_{\varrho_1}, j_{\varrho_1}), \ldots, (i_{\varrho_{N-1}}, j_{\varrho_{N-1}})$ within $(i_{r+1}, j_{r+1}), \ldots, (i_t, j_t)$. Consider first the corresponding subgraph. It can be easily seen that one may select the intermediate $\boldsymbol{x}_l^\lambda$, $1 \leqslant l \leqslant N$, $\varrho_1 \leqslant \lambda \leqslant \varrho_{N-1}$ to be appropriate partial sums of the $\boldsymbol{x}_l^r$, such that the $\boldsymbol{x}_l^{\varrho_{N-1}}$, $1 \leqslant l \leqslant N$, reduce to zero after the complete connection.

Then the remaining potentials of the full original sequence do not give rise to new closed loops and may be inserted into the subgraph by requiring that all $\boldsymbol{x}_l^\lambda$, $\varrho_1 \leqslant \lambda \leqslant \varrho_{N-1}$, $1 \leqslant l \leqslant N$, remain unchanged at these new vertices. Momentum conservation is trivially satisfied at each new vertex.

Throughout this procedure, we choose the $\boldsymbol{s}_l^\delta$ according to (2.5). Clearly, for $s > r$:

$$\sum_{l=1}^{N} \mu_l (\boldsymbol{x}_l^s)^2 < 1$$

and Lemma 2.1 remains valid for this new choice of loop momenta.

Our final choice of the loop momenta will be a combination of these two algorithms. One begins with (2.4), (2.5). If a subsequence of the potentials, $V_{i_\varrho j_\varrho}$, $V_{i_\lambda j_\lambda}$, $\ldots$, $V_{i_\tau j_\tau}$, generates a subsystem $\{l_1, \ldots, l_g\} \subset \{1, \ldots, N\}$ of high connectivity without any of the particles within $\{l_1, \ldots, l_g\}$ being connected with the remaining ones between the ϱ^{th} and the τ^{th} step, then we procede after $V_{i_\tau j_\tau}$ as follows:

$$\boldsymbol{k}_l^\tau = \begin{cases} \boldsymbol{k}_l^{\tau-1} & l \neq i_\tau, j_\tau \\ \dfrac{1}{g} \displaystyle\sum_{\nu=1}^{g} \boldsymbol{k}_{l_\nu}^{\tau-1} & l = i_\tau \text{ ('longest segment')} \\ -\displaystyle\sum_{\substack{l \neq j_\tau \\ l=1}}^{N} \boldsymbol{k}_l^\tau & l = j_\tau \end{cases} \tag{2.10}$$

and $\boldsymbol{s}_l^\tau$ as in (2.5).

If the connectivity of the subgraph is sufficiently high, then the derivations of the $\boldsymbol{k}_l^\tau$ from the mean value $1/g \sum_{\nu=1}^{g} \boldsymbol{k}_{l_\nu}^{\tau-1}$ are small, so that the energy denominator D_τ remains different from zero as in Lemma 2.1.

Furthermore, if the sequence $V_{i_\varrho j_\varrho}, \ldots, V_{i_\tau j_\tau}, V_{i_{\tau+1} j_{\tau+1}}, \ldots$ continues to connect particles from $\{l_1, \ldots, l_g\}$ entirely among themselves and if one proceeds according to our second prescription, then after a complete connection the deviations from the mean cluster momentum vanish and each particle line within this subsystem will carry the momentum $1/g \sum_{\nu=1}^{g} \boldsymbol{k}_{l_\nu}^\tau$ as the only k-dependence.

The analogous prescription will be applied to the graph $\mathcal{G}_3$ from the right. If the iteration order L in (2.1) is sufficiently large, then in $\mathcal{G}_2$ the dependence of the loop momenta on the external momenta has "died out" and they can be fitted together through $\mathcal{G}_2$.

We note that for each subsystem of less than N particles and for any sequence of potentials connecting just the particles within this subsystem, only the relative momentum of the subsystem with respect to the remaining cluster(s) survives. Thus, we have the

Lemma 2.4. – Using the two prescriptions described above for the choice of loop momenta from the left in $\mathcal{G}_1$ and similarly from the right in $\mathcal{G}_3$, there exists only a finite number of graphs in $T - T_L$ which carry different repartitions of the external momenta.

We now turn to the analyticity properties of the potentials. The argument of a potential $V_{i_r j_r}$, after which the momenta have been chosen according to (2.4), (2.5) is

$$\frac{1}{\mu_{i_r}+\mu_{j_r}}\left(\mu_{i_r}\boldsymbol{k}_{i_r}^{r-1}-\mu_{j_r}\boldsymbol{k}_{j_r}^{r-1}\right)+\boldsymbol{s}_{i_r}^{r-1}-\boldsymbol{s}^r\,. \tag{2.11}$$

For $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \Omega^N$, there exists obviously a $\xi = \xi(\mu_1, \ldots, \mu_N) < \infty$ such that

$$\frac{1}{\mu_{i_r}+\mu_{j_r}}\left|\mu_{i_r}\boldsymbol{x}_{i_r}^{r-1}-\mu_{j_r}\boldsymbol{x}_{j_r}^{r-1}\right| \leqslant \xi$$

for all $r - 1$, i_r, j_r. For real $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \Omega^N$ and all $\boldsymbol{s}^1, \ldots, \boldsymbol{s}^r \in R^3$ the argument (2.11) lies in the regularity domain of $V_{i_r j_r}$ provided that

$$|Im\,k| \leqslant \frac{\varkappa}{\xi}, \qquad k \neq 0\,. \tag{2.12}$$

This follows immediately, if one notices that for all real $\boldsymbol{s}$ and $\boldsymbol{x} \in R^3$ with $|\boldsymbol{x}| \leqslant \xi$ and for all real λ with $\varkappa^2 \leqslant \lambda$, the condition

$$\lambda + (k\,\boldsymbol{x} + \boldsymbol{s})^2 \neq 0$$

gives the maximal analyticity strip (2.12).

If our second prescription is used, then the deviations of the relevant $\boldsymbol{x}_l^\varrho$ from their mean cluster value are already sufficiently small (for sufficiently high connectivity), so that ξ may be retained as upper bound for the coefficient of the k-dependent part of the argument of the potential.

Therefore any graph $\mathcal{G}$ in the perturbation series for $T - T_L$ is holomorphic for $k \in \{0 < Im\,k < \varkappa/\xi\}$ and for real $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_N) \in \Omega_0^N \times \Omega_0^N$.

We define

$$\varrho = \frac{\varkappa}{\xi}\,.$$

If we set

$$k = e^{i\varphi}\,\check{k}\,, \quad 0 < Im\,\check{k} < \varrho\cos\varphi\,, \quad |\varphi| < \frac{\pi}{2}$$

and "rotate" the loop momenta $\boldsymbol{s}^t$ in $\mathcal{G}$ by setting

$$\boldsymbol{s}^t = e^{i\varphi}\,\check{\boldsymbol{s}}^t\,,\ \check{\boldsymbol{s}}^t \in R^3 \tag{2.13}$$

then all energy denominators D_t stay away from zero for real $(\boldsymbol{x}_1, \ldots, \boldsymbol{y}_N)$ in $\Omega_0^N \times \Omega_0^N$. (2.13) reads

$$\lambda + (e^{i\varphi} \check{k}\, \boldsymbol{x} + e^{i\varphi} \check{s})^2 \neq 0$$

and remains satisfied for all real $\check{\boldsymbol{s}}$ and $\lambda \geqslant \varkappa^2$, provided that

$$|Im\, \check{k}| < \varrho \cos\varphi\,, \quad |\varphi| < \frac{\pi}{2}\,.$$

Thus, the Feynman integral over the rotated contour $\boldsymbol{s}^t = e^{i\varphi}\, \check{\boldsymbol{s}}^t$, $\boldsymbol{s}^t \in R^3$ defines a function of k and $(\boldsymbol{x}_1, \ldots, \boldsymbol{y}_N)$ holomorphic for real $(\boldsymbol{x}_1, \ldots, \boldsymbol{y}_N)$ in $\Omega_0^N \times \Omega_0^N$ and $0 < Im\, (e^{-i\varphi}\, k) < \varrho \cos\varphi$.

In the non-empty intersection

$$\{0 < Im\, k < \varrho\} \cap \{0 < Im\, (k\, e^{-i\varphi}) < \varrho \cos\varphi\}$$

both analytic functions coincide, since the integrand are uniformly decreasing at infinity so that the Cauchy formula can be applied.

In this way, we can continue each Feynman integral into the upper k-plane except for $k = i\,\lambda$, $\lambda \geqslant \varrho > 0$ and also across the real axis with the origin excluded. Since

$$H_\varrho \subset \bigcup_{|\varphi| < \pi/2} \{k = e^{i\varphi}\, \check{k}\,,\ 0 < Im\, \check{k} < \varrho \cos\varphi\}$$

this completes the proof of part (ii) of Theorem 2.1. The Hölder continuity in $(k, \boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in H_\varrho \times \Omega^N \times \Omega^N$ follows from the remark after Lemma 2.2.

3. Fredholm Theory

In this section we shall derive for the exact N-particle amplitude $T - T_L$ analyticity properties similar to those announced in Theorem 2.1 for an arbitrary term of its Born series. The proof proceeds in two steps:
(i) For all purely repulsive holomorphic two-body potentials of short range (assumptions (A_0) and (B)) a function

$$\langle k\, \boldsymbol{x}_1, \ldots, k\, \boldsymbol{x}_N\, |\, (T - T_L)\ (k^2)\, |\, k\, \boldsymbol{y}_1, \ldots, k\, \boldsymbol{y}_N \rangle \tag{3.1}$$

can be defined which is holomorphic in $(k, \boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in \{0 < Im\, k < \varrho\} \times \Omega_0^N \times \Omega_0^N$. Here one exploits only the analyticity properties of the potentials in order to define a function holomorphic in the external momenta $k\, \boldsymbol{x}_1, \ldots, k\, \boldsymbol{y}_N$. Also the operator-analyticity of the resolvents $R_{a_j}(z)$ outside the spectrum $[0, \infty)$ of H_{a_j} will be used.

However, the F.Y. equations will be only required for establishing a cluster decomposition and could be replaced by the Weinberg equations [12].
(ii) The analytic continuation into $H_\varrho \times \Omega_0^N \times \Omega_0^N$ will rely heavily on the special analyticity of superpositions of Yukawa potentials, since we shall define resolvents $R_{a_j}(z)$ on rotated contours of the type (2.13).

It will turn out that the uncontrollable singularities in the "unphysical sheet" do not interfere with the analytic continuation. The absence of spurious solutions [16] to the F.Y. equations is important in order to control the singularities of $R_{a_j}(z)$ for $0 < \varphi < \pi/2$ without any further technical assumption beyond (B).

An infinitesimal analytic continuation of (3.1) into the "unphysical sheet" across the right-hand cut in the k^2-plane will be possible using the results of K. HEPP [5], if there are no bound states for positive energies. This can be controlled by assumption (C).

For the definition of (3.1), we use again the representation (2.1). In the same way we decompose the amplitudes T_{a_j} for subsystems a_i, $i > 1$, which occur in Q^L and in $\overline{Q}^L$ into (lower) iterations of the F.Y. series for T_{a_j} and amplitudes $N_{a_i}^{\varrho_{i+1}, \tau_{i+1}}$ which are sandwiched between (within a_i) highly connected sequences of amplitudes of finer partitions $b_j \subset a_i$.

After a finite number of steps, all non-perturbative contributions to $T - T_L$ will be sandwiched between sufficiently connected sequences of potentials and amplitudes of subsystems $b_j \subset a_i$.

Then, the dependence on the external momenta $k\,\boldsymbol{x}_1, \ldots, k\,\boldsymbol{y}_N$, routed through the diagram as in section II, is only within the arguments of the potentials and free resolvents and through the argument z of the $N_{a_i}^{\varrho_{i+1}, \tau_{i+1}}(z)$. For $a_i = (a(1), \ldots, a(i))$, $a(\nu) = \{(\nu 1), (\nu 2), \ldots, (\nu k_\nu)\} \subset \{1, \ldots, N\}$, let

$$\hat{N}_{a_i}^{\varrho_{i+1}, \tau_{i+1}}(\hat{\boldsymbol{p}}_{11}, \ldots, \hat{\boldsymbol{p}}_{1k_1}, \ldots, \hat{\boldsymbol{p}}_{i1}, \ldots, \hat{\boldsymbol{p}}_{ik_i}, \hat{\boldsymbol{q}}_{11}, \ldots, \hat{\boldsymbol{q}}_{ik_i}; z)$$

be the kernel of $N_{a_i}^{\varrho_{i+1}, \tau_{i+1}}(z)$ in the Hilbert space $\mathcal{H}_{a_i}$ with the center-of-mass movement of the clusters $a(1), \ldots, a(i)$ factored out:

$$\sum_{k \in a(\nu)} \hat{\boldsymbol{p}}_k = \sum_{k \in a(\nu)} \hat{\boldsymbol{q}}_k = 0\,, \quad 1 \leqslant \nu \leqslant i\,. \tag{3.2}$$

Then the kernel of $N_{a_i}^{\varrho_{i+1}, \tau_{i+1}}(k^2)$ in the N-particle Hilbert space $\mathcal{H}$ reads

$$\begin{aligned}\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N \,|\, N_{a_i}^{\varrho_{i+1}, \tau_{i+1}}(k^2) \,|\, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N \rangle &= \prod_{\nu=1}^{i} \delta\,(\boldsymbol{p}_{a(\nu)} - \boldsymbol{q}_{a(\nu)}) \\ &\times \hat{N}_{a_i}^{\varrho_{i+1}, \tau_{i+1}}\Bigg(\boldsymbol{p}_{11} - \frac{\boldsymbol{p}_{a(1)}}{k_1}, \ldots, \boldsymbol{p}_{ik_i} - \frac{\boldsymbol{p}_{a(i)}}{k_i}, \boldsymbol{q}_{11} - \frac{\boldsymbol{q}_{a(1)}}{k_1}, \ldots, \boldsymbol{q}_{ik_i} - \frac{\boldsymbol{q}_{a(i)}}{k_i}; \\ &\times k^2 - \sum_{\nu=1}^{i} \mu_{a(\nu)}\, \boldsymbol{p}_{a(\nu)}^2\Bigg)\end{aligned} \tag{3.3}$$

with $\mu_{a(\nu)}$, $\boldsymbol{p}_{a(\nu)}$, $\boldsymbol{q}_{a(\nu)}$ defined as in (3.2).

The integrations over the loop momenta eliminate the δ-functions in (3.3). The results of section II guarantee further that in highly connected sandwiches, the relative momenta

$$\boldsymbol{p}_{11} - \frac{\boldsymbol{p}_{a(1)}}{k_1}, \ldots, \boldsymbol{q}_{k_i} - \frac{\boldsymbol{q}_{a(i)}}{k_i}$$

are independent of k, $\boldsymbol{x}_1, \ldots, \boldsymbol{y}_N$.

Finally, the center-of-mass momenta $\boldsymbol{p}_{a(\nu)}$, $\boldsymbol{q}_{a(\nu)}$ involve loop momenta $\boldsymbol{s}^1, \ldots, \boldsymbol{s}^t$ and $k\,\boldsymbol{x}_1, \ldots, k\,\boldsymbol{y}_N$ typically in the form

$$\boldsymbol{p}_{a(\nu)} = k\,\boldsymbol{x}_{a(\nu)} + \boldsymbol{s}_{a(\nu)}$$

where the $\boldsymbol{s}_{a(\nu)}$ are linear combinations of $\boldsymbol{s}^1, \ldots, \boldsymbol{s}^t$. Therefore, on Ω_0^N $k^2 - \sum_{\nu=1}^{i} \mu_{a(\nu)} \times \boldsymbol{p}_{a(\nu)}^2$ never intersects the spectrum $[0, \infty)$ of H_{a_i} (the free center-of-mass movement of the clusters $a(\nu)$ has been already factored out):

If $Im \left(k^2 - \sum_{\nu=1}^{i} \mu_{a(\nu)} \boldsymbol{p}_{a(\nu)}^2\right) = 0$, then equation (2.6) shows that

$$k^2 - \sum_{\nu=1}^{i} \mu_{a(\nu)} \boldsymbol{p}_{a(\nu)}^2 < 0 .$$

The analyticity property of the two-body potentials of class (A_0) is sufficient to prove, as in section II, that the $(k, \boldsymbol{x}_1, \ldots, \boldsymbol{y}_N)$-dependence in the arguments of the v_{ij} always varies within their domain of analyticity. The estimates of [13] guarantee again the absolute convergence of all integrals and we obtain the

Theorem 3.1. – Unter the assumptions (A_0) and (B), the highly connected N-body amplitude $\langle k\, \boldsymbol{x}_1, \ldots, k\, \boldsymbol{x}_N \mid (T - T_L)\, (k^2) \mid k\, \boldsymbol{y}_1, \ldots, k\, \boldsymbol{y}_N\rangle$ is holomorphic in

$$S^0 = \{0 < Im\, k < \varrho\} \times \Omega_0^N \times \Omega_0^N$$

and Hölder continuous in $\{0 < Im\, k < \varrho\} \times \Omega^N \times \Omega^N$.

We now turn to the construction of an analytic continuation

$$\langle k\, \boldsymbol{x}_1, \ldots, k\, \boldsymbol{x}_N \mid (T^\varphi - T_L^\varphi)\, (k^2) \mid k\, \boldsymbol{y}_1, \ldots, k\, \boldsymbol{y}_N\rangle$$

of (3.1) into

$$S^\varphi = \{0 < Im\, (k\, e^{-i\varphi}) < \varrho \cos\varphi\, ,\ \mathrm{sgn}\, \varphi\, Re\, k > 0\} \times \Omega_0^N \times \Omega_0^N$$

for $0 < \varphi < \pi/2$. We notice that $\bigcup_{|\varphi| < \pi/2} S^\varphi$ coincides with $H_\varrho \times \Omega_0^N \times \Omega_0^N$.

A function with the required analyticity properties can be constructed, once we have kernels $\hat{N}_{a_i}^\varphi\, (\hat{\boldsymbol{p}}_{11}, \ldots, \hat{\boldsymbol{q}}_{i k_i}; z)$, $1 \leqslant i \leqslant N-1$, $\hat{N}_{a_i}^\varphi \equiv \hat{N}_{a_i}^{\varrho_{i+1}, \tau_{i+1}, \varphi}$ defined on contours

$$e^{-i\varphi}\, \hat{\boldsymbol{p}}_{11} \in R^3\, , \ldots, e^{-i\varphi}\, \hat{\boldsymbol{q}}_{i k_i} \in R^3 \tag{3.4}$$

in the Hilbert space $\mathcal{H}_{a_i}^\varphi$ with the Lebesgue measure over (3.4) and the center-of-mass movements factored out:

$$\sum_{l \in a(\nu)} \hat{\boldsymbol{p}}_l = \sum_{l \in a(\nu)} \hat{\boldsymbol{q}}_l = 0\, ,\ 1 \leqslant \nu \leqslant i\, . \tag{3.5}$$

Let us assume that these kernels exist and are holomorphic for all z within

$$Z^\varphi = \begin{cases} \{z \in C^1 - \{0\}\, ,\ 2\varphi < \arg z < 2\pi\} & \text{if } \varphi \geqslant 0 \\ \{z \in C^1 - \{0\}\, ,\ 0 < \arg z < 2\pi - 2\varphi\} & \text{if } \varphi < 0 \end{cases}$$

and that

$$\int_{R^{6N}} \prod_{l=1}^{N} d^3p_l\, d^3q_l \prod_{\nu=1}^{i} \delta \left(\sum_{k \in a(\nu)} \boldsymbol{p}_k - \sum_{k \in a(\nu)} \boldsymbol{q}_k\right)$$
$$\times \left| \hat{N}_{a_i}^\varphi\, (\hat{\boldsymbol{p}}_{11}^\varphi, \ldots, \hat{\boldsymbol{q}}_{i k_i}^\varphi; z) \left(z - \sum_{l=1}^{N} \mu_l\, \boldsymbol{q}_l^2\, e^{2i\varphi}\right)^{-1} \right|^2 < \infty$$

with their HS-norm tending to zero for $Re\, z \to -\infty$. We have set $\boldsymbol{p}^\varphi = e^{i\varphi}\, \boldsymbol{p}$.

If $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N$ is a 2 N-tuple of 3-vectors satisfying

$$\sum_{i=1}^{N} \boldsymbol{p}_i = \sum_{i=1}^{N} \boldsymbol{q}_i = 0 \tag{3.6a}$$

$$e^{-i\varphi}\left(\boldsymbol{p}_{\nu l} - \frac{\boldsymbol{p}_{a(\nu)}}{k_\nu}\right) \in R^3 \,, \; e^{-i\varphi}\left(\boldsymbol{q}_{\nu l} - \frac{\boldsymbol{q}_{a(\nu)}}{k_\nu}\right) \in R^3, \; 1 \leqslant \nu \leqslant i \,, \; 1 \leqslant l \leqslant k_\nu \tag{3.6b}$$

$$\boldsymbol{p}_{a(\nu)} = k \, \boldsymbol{x}_{a(\nu)} + \boldsymbol{s}_{a(\nu)} \,, \;\; \boldsymbol{x}_{a(\nu)} \in R^3, \;\; \check{\boldsymbol{s}}_{a(\nu)} = \boldsymbol{s}_{a(\nu)} \, e^{-i\varphi} \in R^3 \tag{3.6c}$$

$$\sum_{\nu=1}^{i} \mu_{a(\nu)} \, \boldsymbol{x}^2_{a(\nu)} < 1 \,, \; 0 < Im \, (e^{-i\varphi} \, k) < \varrho \cos\varphi \,, \;\; \operatorname{sgn} \varphi \, Re \, k > 0 \tag{3.6d}$$

then we claim that

$$\hat{N}^{\varphi}_{a_i}\left(\boldsymbol{p}_{11} - \frac{\boldsymbol{p}_{a(1)}}{k_1}, \ldots, \boldsymbol{q}_{ik_i} - \frac{\boldsymbol{q}_{a(i)}}{k_i} \; ; \; k^2 - \sum_{\nu=1}^{i} \mu_{a(\nu)} \, \boldsymbol{p}^2_{a(\nu)}\right)$$

is well-defined.

In fact, the relative momenta lie on the rotated contour (3.4). Secondly,

$$k^2 - \sum_{\nu=1}^{i} \mu_{a(\nu)} \, \boldsymbol{p}^2_{a(\nu)} \in Z^{\varphi} \,.$$

For, assume, for instance, $\varphi > 0$ and let $b \notin Z^{\varphi}$

$$\check{b} = \beta \, e^{i\psi} \,, \;\; \beta \geqslant 0 \,, \;\; -2\,\varphi \leqslant \psi \leqslant 0$$

$$\check{k} = \varkappa \, e^{i\theta} \,, \;\; \varkappa > 0 \,, \;\; 0 < \theta < \frac{\pi}{2} - \varphi \qquad \left(0 < \varphi < \frac{\pi}{2}\right).$$

Then a simple calculation shows that

$$\begin{aligned} D = \check{k}^2 - \sum_{\nu=1}^{i} \mu_{a(\nu)} \, [\check{k} \, \boldsymbol{x}_{a(\nu)} + \check{\boldsymbol{s}}_{a(\nu)}]^2 - \check{b} &= -\,\varkappa^2 \left[1 - \sum_{\nu=1}^{i} \mu_{a(\nu)} \, \boldsymbol{x}^2_{a(\nu)}\right] \\ &- \sum_{\nu=1}^{i} \mu_{a(\nu)} \, \check{\boldsymbol{s}}^2_{a(\nu)} - \beta \sin(\theta - \psi) \, \frac{1}{\sin\theta} + Im \, D \operatorname{cotg}\theta + i \, Im \, D \end{aligned} \tag{3.7}$$

never vanishes for (3.6 a)–(3.6 d) and a similar result holds for $0 > \varphi > -\pi/2$. Therefore, the singularities in the "unphysical sheet" $\complement Z^{\varphi}$ do not interfere with the analyticity of the kernels $N^{\varphi}_{a_i}$ in their argument z.

If we define $\langle k \, \boldsymbol{x}_1, \ldots, k \, \boldsymbol{x}_N \mid (T^{\varphi} - T^{\varphi}_L) \mid k \, \boldsymbol{y}_1, \ldots, k \, \boldsymbol{y}_N \rangle$ for $(k, \boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in S^{\varphi}$ by the same representation as the one obtained by reducing (2.1), with $N^{\varrho_{i+1}, \tau_{i+1}}_{a_i}$ replaced by $N^{\varrho_{i+1}, \tau_{i+1}, \varphi}_{a_i}$ and all loop momenta on rotated contours, $\boldsymbol{s} \, e^{-i\varphi} \in R^3$, then by Theorem 2.1, this function will have the required analyticity properties.

For $z \in \bigcap_{0 \leqslant \varphi' \leqslant \varphi} Z^{\varphi'}$, assume in addition that the kernels $\hat{N}^{\varphi}_{a_i}(\hat{\boldsymbol{p}}_{11}, \ldots, \hat{\boldsymbol{q}}_{ik_i}; z)$ are holomorphic in $\hat{\boldsymbol{p}}_{11}, \ldots, \hat{\boldsymbol{q}}_{ik_i}$ (satisfying (3.5)) in a neighbourhood of the set

$$\bigcup_{0 \leqslant \varphi' \leqslant \varphi} \{\hat{\boldsymbol{p}}_{11} \, e^{-i\varphi'} \in R^3, \ldots, \hat{\boldsymbol{q}}_{ik_i} \, e^{-i\varphi'} \in R^3\}$$

and satisfy for $\hat{\boldsymbol{p}}_{11}, \ldots, \hat{\boldsymbol{q}}_{ik_i} \in R^3$ and $0 \leqslant \varphi' \leqslant \varphi$ resp. $-\pi/2 < \varphi \leqslant \varphi' \leqslant 0$:

$$\hat{N}^{\varphi}_{a_i}(e^{i\varphi'}\,\hat{\boldsymbol{p}}_{11}, \ldots, e^{i\varphi'}\,\hat{\boldsymbol{q}}_{ik_i}; z) = \hat{N}^{\varphi'}_{a_i}(e^{i\varphi'}\,\hat{\boldsymbol{p}}_{11}, \ldots, e^{i\varphi'}\,\hat{\boldsymbol{q}}_{ik_i}; z)\,.$$

Thereby, $\hat{N}^0_{a_i} = \hat{N}_{a_i}$ is defined by (3.3).

If furthermore the $\hat{N}^{\varphi}_{a_i}$ decrease sufficiently rapidly at infinity, uniformly in $e^{i\varphi'}\,\hat{\boldsymbol{p}}_{11}, \ldots, e^{i\varphi'}\,\hat{\boldsymbol{q}}_{ik_i}$, then we may again distort the contour of integration without crossing any singularity of the integrand and obtain

$$\begin{aligned}&\langle k\,\boldsymbol{x}_1, \ldots, k\,\boldsymbol{x}_N \,|\, (T - T_L)\,(k^2) \,|\, k\,\boldsymbol{y}_1, \ldots, k\,\boldsymbol{y}_N\rangle \\ &\quad = \langle k\,\boldsymbol{x}_1, \ldots, k\,\boldsymbol{x}_N \,|\, (T^{\varphi} - T^{\varphi}_L)\,(k^2) \,|\, k\,\boldsymbol{y}_1, \ldots, k\,\boldsymbol{y}_N\rangle\end{aligned}$$

for $(k, \boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in S^{\varphi} \cap S^0$.

Since $S^{\varphi} \cap S^0 \neq \phi$ for all $|\varphi| < \pi/2$, both functions have a common analytic continuation into $S^{\varphi} \cup S^0$.

Thus, we may already formulate the

Theorem 3.2. – Under the assumptions (A) and (B), $\langle k\boldsymbol{x}_1, \ldots, k\boldsymbol{x}_N \,|\, (T - T_L)\,(k^2) \,|\, k\boldsymbol{y}_1, \ldots, k\boldsymbol{y}_N\rangle$ as defined by Theorem 3.1, has an analytic continuation into $H_{\varrho} \times \Omega^N_0 \times \Omega^N_0$ and is Hölder continuous in $H_{\varrho} \times \Omega^N \times \Omega^N$.

Proof. – As outlined above, we only have to prove the existence of kernels $\hat{N}^{i+1,\varphi}_{a_i}(\hat{\boldsymbol{p}}_{11}, \ldots, \hat{\boldsymbol{q}}_{ik_i}; z)$ on rotated contours, for $z \in Z^{\varphi}$. The construction proceeds inductively according to an increasing connectivity of the partitions a_i, $i > 1$, of $\{1, \ldots, N\}$.

We start with some partition a_{N-1} and review well-known results [7, 9] for a two-particle system.

For definiteness, assume $\varphi > 0$ and set $V = V_{a_{N-1}}$, $\mu = \mu_{a_{N-1}}$, $\mathcal{H}^{\varphi} = \mathcal{H}^{\varphi}_{a_{N-1}}$ $H = H_{a_{N-1}}$.

For $z \in \boldsymbol{C}^1 - \{0\}$, $\arg z \neq 2\,\varphi$, $\boldsymbol{p}^{\varphi} = e^{i\varphi}\,\boldsymbol{p}$, $\boldsymbol{q}^{\varphi} = e^{i\varphi}\,\boldsymbol{q}$, $\boldsymbol{p}, \boldsymbol{q} \in R^3$ the kernel.

$$v\,(\boldsymbol{p}^{\varphi} - \boldsymbol{q}^{\varphi})\,(z - \mu\,\boldsymbol{q}^2\,e^{2i\varphi})^{-1}\,e^{3i\varphi} \tag{3.8}$$

is a HS operator in $\mathcal{H}^{\varphi}$ with its HS-norm tending to zero for $Re\,z \to -\infty$. Therefore, the equation

$$\begin{aligned}t^{\varphi}(\boldsymbol{p}^{\varphi}, \boldsymbol{q}^{\varphi}; z)\,(z - \mu\,\boldsymbol{q}^2\,e^{2i\varphi})^{-1} &= v\,(\boldsymbol{p}^{\varphi} - \boldsymbol{q}^{\varphi})\,(z - \mu\,\boldsymbol{q}^2\,e^{2i\varphi})^{-1} \\ &+ \int_{R^3} d^3r\,e^{3i\varphi}\,v\,(\boldsymbol{p}^{\varphi} - \boldsymbol{r}^{\varphi})\,\frac{1}{z - \mu\,\boldsymbol{r}^2\,e^{2i\varphi}}\,t^{\varphi}(\boldsymbol{r}^{\varphi}, \boldsymbol{q}^{\varphi}; z)\,\frac{1}{z - \mu\,\boldsymbol{q}^2\,e^{2i\varphi}}\end{aligned} \tag{3.9}$$

has as solution a HS operator

$$t^{\varphi}(\boldsymbol{p}^{\varphi}, \boldsymbol{q}^{\varphi}; z)\,(z - \mu\,\boldsymbol{q}^2\,e^{2i\varphi})^{-1} \tag{3.10}$$

holomorphic in $z \in \{\arg z \neq 2\,\varphi,\ z \neq 0\}$, provided (Fredholm alternative) that there are no L^2-solutions f^{φ} to the homogeneous equation

$$f^{\varphi}(\boldsymbol{p}^{\varphi}) = \int_{R^3} d^3r\,e^{3i\varphi}\,v\,(\boldsymbol{p}^{\varphi} - \boldsymbol{r}^{\varphi})\,\frac{1}{z - \mu\,\boldsymbol{r}^2\,e^{2i\varphi}}\,f^{\varphi}(\boldsymbol{r}^{\varphi})\,. \tag{3.11}$$

Necessary and sufficient for the existence of a square-integrable solution of (3.11) is the vanishing of the Fredholm determinant $D^\varphi(z)$ [18]:

$$D^\varphi(z) = 1 + \sum_{n=1}^{\infty} D_n^\varphi(z)$$

$$D_n^\varphi(z) = \frac{(-1)^n}{n!} \begin{vmatrix} 0 & n-1 & 0 & \ldots 0 & 0 & 0 \\ \sigma_2^\varphi & 0 & n-2 & \ldots 0 & 0 & 0 \\ \sigma_3^\varphi & \sigma_2^\varphi & 0 & \ldots 0 & 0 & 0 \\ \vdots & & & & & \\ \sigma_{n-1}^\varphi & \sigma_{n-2}^\varphi & \sigma_{n-3}^\varphi & \ldots \sigma_2^\varphi & 0 & 1 \\ \sigma_n^\varphi & \sigma_{n-1}^\varphi & \sigma_{n-2}^\varphi & \ldots \sigma_3^\varphi & \sigma_2^\varphi & 0 \end{vmatrix} \tag{3.12}$$

where σ_i^φ is the trace of the ith iteration of the kernel (3.8). We observe that for $2\varphi < \arg z < 2\pi$ and $0 \leqslant \varphi' \leqslant \varphi$, the iterated traces

$$\sigma_n^{\varphi'}(z) = \int_{R^{3n}} \prod_{j=1}^{n} \frac{e^{3i\varphi}\, d^3p_j}{z - \mu\, \boldsymbol{p}_j^2\, e^{2i\varphi}}\, v\,(\boldsymbol{p}_1^\varphi - \boldsymbol{p}_2^\varphi)\, v\,(\boldsymbol{p}_2^\varphi - \boldsymbol{p}_3^\varphi) \ldots v\,(\boldsymbol{p}_n^\varphi - \boldsymbol{p}_1^\varphi)$$

are independent of φ':

$$\sigma_n^{\varphi'}(z) = \sigma_n^0(z)\,.$$

The deformation of the contour does not lead to a contribution at infinity due to the uniform decrease of the potentials and free propagators at infinity. Moreover, for $2\varphi < \arg z < 2\pi$

$$D^\varphi(z) = D^0(z) \neq 0\,.$$

For, a zero of the Fredholm determinant $D^0(z)$ in Z^φ would lead to a non-trivial L^2-solution f^0 of (3.11) and thus to an eigenvector

$$g^0(\boldsymbol{p}) = \frac{f^0(\boldsymbol{p})}{z - \mu\, \boldsymbol{p}^2} \tag{3.13}$$

of the Hamiltonian H^0 with eigenvalue $z \in Z^\varphi$. This is however in contradiction to the spectrum $\sigma(H^0) = [0, \infty)$, which follows from (A), (B).

We remark that (3.10) is even meromorphic in $\{\arg z \neq 2\varphi\}$ [17]. But the singularities in the "unphysical sheet", which are poles in this case, become more and more complicated for multiparticle systems and it is a lucky circumstance that our process of analytic continuation does not lead into the "unphysical sheet" at all (cf. Eq. (3.7)).

As a consequence of (3.9), we may write

$$\begin{aligned} t^\varphi(\boldsymbol{p}^\varphi, \boldsymbol{q}^\varphi; z)\,(z - \mu\, \boldsymbol{q}^2\, e^{2i\varphi})^{-1} &= v\,(\boldsymbol{p}^\varphi - \boldsymbol{q}^\varphi)\,(z - \mu\, \boldsymbol{q}^2\, e^{2i\varphi})^{-1} \\ &+ \int_{R^3} d^3r\, e^{3i\varphi}\, v\,(\boldsymbol{p}^\varphi - \boldsymbol{r}^\varphi)\, \frac{1}{z - \mu\, \boldsymbol{r}^2\, e^{2i\varphi}}\, v\,(\boldsymbol{r}^\varphi - \boldsymbol{q}^\varphi)\, \frac{1}{z - \mu\, \boldsymbol{q}^2\, e^{2i\varphi}} \\ &+ \int_{R^6} d^3r\, d^3s\, e^{6i\varphi}\, v\,(\boldsymbol{p}^\varphi - \boldsymbol{r}^\varphi)\, \frac{1}{z - \mu\, \boldsymbol{r}^2\, e^{2i\varphi}}\, t^\varphi(\boldsymbol{r}^\varphi, \boldsymbol{s}^\varphi; z)\, \frac{1}{z - \mu\, \boldsymbol{s}^2\, e^{2i\varphi}} \\ &\times v(\boldsymbol{s}^\varphi - \boldsymbol{q}^\varphi)\, \frac{1}{z - \mu\, \boldsymbol{q}^2\, e^{2i\varphi}}\,. \end{aligned} \tag{3.14}$$

From this representation it follows that $t^\varphi(\boldsymbol{p}^\varphi, \boldsymbol{q}^\varphi; z)$ is analytic in $\boldsymbol{p}^\varphi$ and $\boldsymbol{q}^\varphi$, since

$$\lambda + (\boldsymbol{p}^\varphi - \boldsymbol{r}^\varphi)^2 \neq 0\,, \quad \lambda + (\boldsymbol{s}^\varphi - \boldsymbol{q}^\varphi)^2 \neq 0$$

for all $\boldsymbol{r}, \boldsymbol{s} \in R^3$ and all $\lambda \geqslant \varkappa^2$, if only $| Im\, \boldsymbol{p} | < \varkappa \cos\varphi$ resp. $| Im\, \boldsymbol{q} | < \varkappa \cos\varphi$.

It remains to prove that for $0 \leqslant \varphi' \leqslant \varphi$ and $\boldsymbol{p}, \boldsymbol{q} \in R^3$

$$t^\varphi(\boldsymbol{p}^{\varphi'}, \boldsymbol{q}^{\varphi'}; z) = t^{\varphi'}(\boldsymbol{p}^{\varphi'}, \boldsymbol{q}^{\varphi'}; z)\,. \tag{3.15}$$

Firstly, for $| Im\, (e^{i(\varphi-\varphi')}\, \boldsymbol{p}) | < \varkappa \cos\varphi$, $| Im\, (e^{i(\varphi-\varphi')}\, \boldsymbol{q}) | < \varkappa \cos\varphi$, this follows by using the representation of $t^\varphi(\boldsymbol{p}^\varphi, \boldsymbol{q}^\varphi; z)\; (z - \mu\, \boldsymbol{q}^2\, e^{2i\varphi})^{-1}$ by a Fredholm series [18]. Inserting this into (3.14) as in [9], one can apply the Cauchy formula to prove Equation (3.15).

Here the square-integrability of $v(\boldsymbol{s}^\varphi - \boldsymbol{q}^\varphi)$ in $\boldsymbol{s}$ for fixed $\boldsymbol{q}$, uniformly in φ and of $v\,(\boldsymbol{p}^\varphi - \boldsymbol{r}^\varphi)\; (z - \mu\, \boldsymbol{r}^2\, e^{2i\varphi})^{-1}$ in $\boldsymbol{r}$ are important for the existence of $t^\varphi(\boldsymbol{r}^\varphi, \boldsymbol{s}^\varphi; z)$. Then (3.15) is proved by analytic continuation.

Finally we remark that since the HS-norm of the kernel (3.8) tends to zero for $Re\, z \to -\infty$, the Born series for (3.10) converges for sufficiently small $Re\, z$ and the HS-norm of (3.10) also tends to zero for $Re\, z \to -\infty$.

We shall now prove inductively similar properties for the kernels $\hat{N}^\varphi_{a_i}\, R^\varphi_0$ which we define as solutions of the F.Y. equations for subsystems:

$$\hat{N}^\varphi_{a_i}\, R^\varphi_0 = \hat{N}^\varphi_{a_i(0)}\, R^\varphi_0 + \hat{Q}^\varphi_{a_i}\, \hat{N}^\varphi_{a_i}\, R^\varphi_0 = \hat{\hat{N}}^\varphi_{a_i(0)}\, R^\varphi_0 + \hat{N}^\varphi_{a_i}\, R^\varphi_0 (R^\varphi_0)^{-1}\, \hat{\hat{Q}}^\varphi_{a_i}\, R^\varphi_0\,. \tag{3.16}$$

We have set:

$$\hat{Q}^\varphi_{a_i} = \hat{Q}^{i+1,\varphi}_{a_i}\,.$$

For all subsubsystems $b_j \subset a_i$, $j > i$, we assume the existence of kernels $\hat{N}^\varphi_{b_j}$ on rotated contours with the following properties:

(a) $\hat{N}^\varphi_{b_j}(z)\; R^\varphi_0(z)$ is a HS operator in the relative momentum Hilbert space $\mathcal{H}^\varphi_{b_j}$, holomorphic for $z \in Z^\varphi$, with its HS-norm going to zero for $Re\, z \to -\infty$.

(b) The kernel of $\hat{N}^\varphi_{b_j}(z)$ is holomorphic in a complex neighbourhood of the relative coordinates on the rotated contour (cf. (3.4), (3.5)) and for $z \in Z^\varphi$.

For $\varphi' \geqslant 0$ sufficiently small

$$\hat{N}^\varphi_{b_j}(\hat{\boldsymbol{p}}_{11}\, e^{-i\varphi'}, \ldots, \hat{\boldsymbol{q}}_{jl_j}\, e^{-i\varphi'}; z) = \hat{N}^{\varphi-\varphi'}_{b_j}(\hat{\boldsymbol{p}}_{11}\, e^{-i\varphi'}, \ldots, \hat{\boldsymbol{q}}_{jl_j}\, e^{-i\varphi'}; z) \tag{3.17}$$

and then by analytic continuation, also for all φ', $0 \leqslant \varphi' \leqslant \varphi$. Here, $\hat{N}^0_{b_j} = \hat{N}_{b_j}$, with $\hat{N}_{b_j}$ as defined in (3.3).

(c) The cluster decomposition properties of the $\hat{N}^\varphi_{b_j}$ are the same as those of $\hat{N}_{b_j}$ on the unrotated contour. The $\hat{N}^\varphi_{b_j}\, R^\varphi_0$ can be represented by a Fredholm series in terms of amplitudes of subsubsystems $c_k \subset b_j$.

Notice that the 2nd property in (c) entails the first one. Under these assumptions we shall study the iterated Equations (3.16).

For $L \geqslant 1$, assumption (c) allows for a decomposition and reduction of

$$[\hat{Q}^{\varphi}_{a_i}(z)]^L \hat{N}^{\varphi}_{a_i(0)}(z) \; R^{\varphi}_0(z) \tag{3.18}$$

into a sum of products of terms with the same structure as those in Theorem 1.2 and equation (1.17). Assumption (a) and the method of [13] allow then to estimate (3.18) and prove that it is a HS operator for $z \in Z^{\varphi}$ in the relative momentum Hilbert space $\mathcal{H}^{\varphi}_{a_i}$.

Furthermore, the operators

$$[\hat{Q}^{\varphi}_{a_i}(z)]^L, \quad (R^{\varphi}_0(z))^{-1} [\hat{\bar{Q}}^{\varphi}_{a_i}(z)]^L R^{\varphi}_0(z)$$

are of HS-class in $\mathcal{H}^{\varphi}_{a_i}$ for $z \in Z^{\varphi}$ and $L \geqslant N - i$.

Therefore the iterated equations

$$\begin{aligned}[\hat{N}^{\varphi}_{a_i}(z) - \hat{N}^{\varphi}_{a_i(0)}(z)] \; R^{\varphi}_0(z) &= \sum_{L=1}^{K-1} [\hat{Q}^{\varphi}_{a_i}(z)]^L \hat{N}^{\varphi}_{a_i(0)}(z) \; R^{\varphi}_0(z) \\ &+ [\hat{Q}^{\varphi}_{a_i}(z)^K] \; [\hat{N}^{\varphi}_{a_i}(z) - \hat{N}^{\varphi}_{a_i(0)}(z)] \; R^{\varphi}_0(z)\end{aligned}$$

have as unique solution a HS operator $[\hat{N}^{\varphi}_{a_i}(z) - \hat{N}^{\varphi}_{a_i(0)}(z)] \; R^{\varphi}_0(z)$ analytic in z, for $K \geqslant N - i$, provided that the Fredholm determinant $D^{\varphi}_K(z)$ does not vanish.

Again, $D^{\varphi}_K(z)$ is made up by traces of iterations of the kernel $\hat{Q}^{\varphi}_{a_i}(z)$, similar to (3.12). Every term in the corresponding series involves only expressions containing amplitudes for finer partitions $b_j \subset a_i$. Due to the induction hypotheses, these amplitudes have a representation by Fredholm series, where the Fredholm determinants and the iterated traces in the numerators are always identical to those for $\varphi = 0$.

By further reduction one arrives at sums over perturbation-theoretic expressions involving potentials and free propagators on rotated contours and Fredholm determinants for subsystems b_n, $n > j$, which are independent of φ.

Furthermore, whenever intermediate momenta of "spectator particles" occur in the argument z of a Fredholm determinant, it will be of the general form

$$z - \sum_{\nu=1}^{n} \mu_{b(\nu)} \, \boldsymbol{p}^2_{b(\nu)}$$

and will have a real part tending to $-\infty$ for large $\boldsymbol{p}^2_{b(\nu)}$ and $z \in Z^{\varphi}$.

Due to our assumptions (A), (B), the Fredholm determinants $\left[D^{\varphi}_{b_n}\left(z - \sum_{\nu=1}^{n} \mu_{b(\nu)} \boldsymbol{p}^2_{b(\nu)}\right)\right]^{-1}$ are uniformly bounded in this region.

Therefore the asymptotic behaviour of the multiple Feynman integrals are those of the Born series, in which, due to the known uniform decrease of potentials and free propagators at infinity, a rotation of contours and the use of the Cauchy formula are always possible (cf. section II).

For $z \in Z^{\varphi}$ we obtain thus

$$D_K^{\varphi}(z) = D_K^{\varphi'}(z) \, , \; 0 \leqslant \varphi' \leqslant \varphi \tag{3.19}$$

where $D_K^0(z)$ is the Fredholm determinant of the iterated F.Y. equations on the unrotated contour.

Due to the spectrum property $\sigma(H_{a_i}) = [0, \infty)$, it may be shown that for every $z \in Z^{\varphi}$, an integer $K \geqslant N - i$ can be found such that $D_K^0(z) \neq 0$. For, $D_K^0(z) = 0$ entails [19] that there exists an integer $0 \leqslant n < K$ and an $f_K \in \mathcal{H}_{a_i}$ such that

$$f_K = e^{i(2\pi/K)n} \, Q_{a_i}(z) \, f_K \, .$$

The spectrum property $\sigma(H_{a_i}) = [0, \infty)$ excludes the value $n = 0$, as in (3.13). If $D_K^0(z) = 0$ for all $K \geqslant N - i$, then one could construct infinitely many different eigenvalues of $Q_{a_i}(z)$ on the unit circle. This hovewer, is in contradiction with the compactness of $[Q_{a_i}(z)]^{N-i}$ in $\mathcal{H}_{a_i}$.

Thus, for every compact set in Z^{φ} there exists a finite open covering $\{U_{\alpha}^K\}$ such that $D_K^0(z) \neq 0$ for $z \in U_{\alpha}^K$. For $Re\, z$ negative and sufficiently small, the Neumann series converges. Therefore the HS-norm of $\hat{N}_{a_i}^{\varphi}(z) \, R_0^{\varphi}(z)$ tends to zero for $Re\, z \to - \infty$.

This finishes the induction step for the assumption (a) for a_i itself. Assumption (c) follows either by analytic continuation in z from the convergent Born series in $Re\, z < - C$, $C > 0$, sufficiently large or from the Fredholm series for $\hat{N}_{a_i}^{\varphi}(z) \, R_0^{\varphi}(z)$.

The analyticity of $\hat{N}_{a_i}^{\varphi} \, (\hat{\boldsymbol{p}}_{11}, \ldots, \hat{\boldsymbol{q}}_{ik_i}, z)$ in the external momenta can be exhibited by a representation as in equation (1.20). If $L \geqslant N - i$ and if one selects loop momenta according to the "longest segment" rule of section II, then the dependence on the external momenta will be only contained in the arguments z of the amplitudes for subprocesses and in the free propagators. This proves then the first part of (b).

The equality (3.17) follows most easily by using the representation of $N_{a_i}^{\varphi}$ by the Fredholm series, sandwiched between $[\hat{Q}_{a_i}^{\varphi}(z)^{N-i}$ and $[\hat{\bar{Q}}_{a_i}^{\varphi}(z)]^{N-i}$.

Theorem 1.2 and the representations similar to (1.19) exhibit clearly by induction that

$$\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N \,|\, V_{i_1 j_1} \, R_0 \, W_{J1} \ldots V_{i_{N-1} j_{N-1}} \, R_0 \, W_{J_{N-1}} \,|\, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N \rangle$$

is for fixed $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N$ square-integrable in $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N$:

Namely $\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N \,|\, V_{i_1 j_1} \, R_0 \,|\, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N \rangle$ is square-integrable in the relative $\boldsymbol{q}$-momentum $(\mu_{i_1} + \mu_{j_1})^{-1} \, (\mu_{i_1} \, \boldsymbol{q}_{i_1} - \mu_{j_1} \, \boldsymbol{q}_{j_1})$ for fixed relative $\boldsymbol{p}$-momentum. This property remains valid after application of the bounded operator W_{J1} (which operates in the Hilbert space $\mathcal{H}_{(i_1, j_1)}$). Assume that

$$\langle \boldsymbol{p}_1, \ldots, \boldsymbol{p}_N \,|\, V_{i_1 j_1} \, R_0 \, W_{J1} \ldots V_{i_k j_k} \, R_0 \, W_{J k} \,|\, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N \rangle$$

is square-integrable in the relative $\boldsymbol{q}$-momenta in the Hilbert space $\mathcal{H}_{a_l}$, $a_l = (i_1, j_1) \llcorner\!\!\!\_\_ \ldots \llcorner\!\!\!\_\_ (i_k, j_k)$ for fixed relative $\boldsymbol{p}$-momenta.

Multiplying with $\langle \boldsymbol{q}_1, \ldots, \boldsymbol{q}_N \,|\, V_{i_{k+1} j_{k+1}} \, R_0 \,|\, \boldsymbol{q}_1', \ldots, \boldsymbol{q}_N' \rangle$ and integrating over $\boldsymbol{q}_1', \ldots, \boldsymbol{q}_N'$ (after elimination of the momentum-conservation δ-functions this amounts to an ordinary product), one obtains a square-integrable function in the relative q'-momenta in the Hilbert space $\mathcal{H}_{a_m}$ with $a_m = (i_1, j_1) \llcorner\!\!\_\_ \ldots \llcorner\!\!\_\_ (i_{k+1}, j_{k+1})$, for fixed relative p-momenta.

This again remains valid after the application of the bounded operator $W_{J\,k+1}$ which operates in $\mathcal{H}_{a_m}$.

Similarly, the sandwiching kernels are square-integrable in their respective "internal" momenta for fixed external momenta, with L^2-norms which depend continuously on φ.

Reduction as in the proof of (3.19) and using the relative, uniform, absolute convergence of the Fredholm series [18] for $\hat{N}^{\varphi}_{a_i}(z)\ R^{\varphi}_0(z)$ lead to (3.17) by a simple inspection of the individual terms.

This completes the proof of Theorem 3.2.

4. Conclusions

The initial objective of this investigation was to lay a foundation to a systematic and mathematically rigorous derivation of analyticity properties for the N-body scattering amplitudes in non-relativistic quantum mechanics. It has turned out that our results give rise to more questions than to definite answers.

We have concentrated on the analyticity properties of the "true" N-body scattering amplitude $T - T_L$. The sum T_L of the first few iterations of the F.Y. equations is known from the solutions of the lower-body problems. For $(k, \boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in H_\varrho \times \Omega^N \times \Omega^N$, the singularities of these terms are expected to be confined to a finite number of Landau varieties, since the highly connected remainders of lower-body amplitudes T_{a_i} in T_L have again only threshold singularities and since therefore effectively, only a finite number of diagrams are relevant.

Let $(\Omega^N \times \Omega^N)_{NL}$ be the complement in $\Omega^N \times \Omega^N$ of finitely many Landau varieties. Then the complete N-body amplitude is expected to be holomorphic for $(k, \boldsymbol{x}_1, \ldots, \boldsymbol{y}_N) \in H_\varrho \times (\Omega^N \times \Omega^N)_{NL}$.

However, a more detailed elucidation of the nature of these physical region singularities should be possible and is of interest, e.g. for cluster properties of the S-matric.

Analyticity of $T - T_L$ in the cut E-plane is only the first step in the proof of dispersion relations for the N-body scattering amplitude. Up to now, a complete and rigorous proof of dispersion relations has been given only for the two-body scattering amplitude [7]. The behaviour of the amplitude at infinity in complex directions and on the right-hand cut appears fairly well controllable (although tedious estimates and angular integrations are involved [9]). Yet, no results, neither on the existence of boundary values nor on the asymptotic behaviour along the lefthand cut have been obtained.

Another necessary extension of the present results will lie in the inclusion of multiparticle bound states. The results of [9] make already clear that future "dynamical" calculations for the non-relativistic N-body problem will probably not be made on the basis of dispersion relations, but by attacking directly the F.Y. equations.

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