Zeitschrift: Helvetica Physica Acta
Band: 42 (1969)
Heft: 3

Artikel: Impossibility of quantum mechanics in a Hilbert space over a finite field

Autor: Eckmann, J.-P. / Zabey, Ph.Ch.

DOI: https://doi.org/10.5169/seals-114075

Impossibility of Quantum Mechanics in a Hilbert Space over a Finite Field

by **J.-P. Eckmann** and **Ph. Ch. Zabey**
Institute of Theoretical Physics, University of Geneva

(12. VIII. 68)

Abstract. In this paper, we show that the lattice of propositions of a quantum mechanical system cannot be represented as subspaces of Hilbert Space with coefficients from a finite field.

The only exceptions are the two dimensional lattices, for which the restriction on the field is only that it may not be of characteristic 2.

1. The Structure of Irreducible Proposition Systems[1])

According to the axiomatic of JAUCH and PIRON, the set of all "yes-no" experiments of a physical system is an irreducible proposition system L, i.e. a partially ordered set with the following properties,

(i) *It is a complete lattice:* every family $\{a_i\}_i$ of elements of L admits a greatest lower bound $\Lambda_i\, a_i$ and a least upper bound $V_i\, a_i$.

(ii) *It is orthocomplemented:* there exists a mapping $a \in L \mapsto a' \in L$ which is involutive ($a'' = a$), decreasing ($a \leqslant b$ implies $b' \leqslant a'$) and such that $a \; V \; a' = I$, where $I = V_{a \in L}\, a$ is the greatest element of L, we define also $0 = I' = \Lambda_{a \in L}\, a$ as the least element of L.

(iii) *It is weakly modular:* if $a \leqslant b$, then $a = (a \; V \; b') \; \Lambda \; b$.

(iv) *It is atomic:* every non zero element admits an atom as lower bound; by atom we mean a non zero element p such that $0 < x \leqslant p$ implies $x = p$.

(v) *It satisfies the covering law:* if p is an atom and a any element such that $a \; \Lambda \; p = 0$, then $(p \; V \; a') \; \Lambda \; a$ is an atom.

(vi) *It is irreducible:* for every pair of atoms (p, q), there exists a third atom r such that $p \; V \; q = p \; V \; r = q \; V \; r$.

The following example ensures the compatibility of these properties; denote by $L(V)$ the set of all biorthogonal manifolds of a euclidian or of a unitary space V. The order in $L(V)$ is given by the inclusion, and the orthocomplementation by taking the orthogonal complement. This set $L(V)$ is a genuine irreducible proposition system.

Conversely, one shows that an irreducible proposition system L can be realized by the set of all biorthogonal manifolds of a vector space V over some field F, the

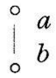
[1]) This section is mostly an extract from: J. M. JAUCH, *Foundations of Quantum Mechanics* (Chapter 8).

orthocomplementation defining on F an involutive antiautomorphism $\alpha \mapsto \bar{\alpha}$[2]) and on V a scalar product, that is a non degenerate sesquilinear hermitian form $S: V \times V \to F$. The field F over which V is defined is, up to an isomorphism, determined by the algebraic structure of L. Usually one takes for F either the field $\mathcal{R}$ of real numbers, or the field $\mathcal{C}$ of complex numbers, or the field $\mathcal{H}$ of quaternionic numbers; each of these is a complete valuated field.

The purpose of this paper is the study of finite dimensional vector spaces over finite fields; such fields are necessarily complete, for they admit only the trivial valuation $|0| = 0$ and $|\alpha| = 1$ for $\alpha \neq 0$.

Remarks:

1. In the following we shall say subspace for biorthogonal manifold.
2. We can use a graphical representation for lattices; a point will figure an element, and a "climbing" line an order relation.



means $a \leq b$.

In the case of a lattice $L(V)$, it will be sufficent to give all possible inclusions between any subspace and a subspace of immediately higher dimension.

2. Finite Fields

Let F be a finite field, and u its unit element. A theorem by Wedderburn[3]) states that F is always abelian. The prime field of F, defined as the subfield generated by u, is isomorphic to the field $\mathcal{Z}_p$, where p is a prime number called the characteristic of the field ($\mathcal{Z}_p$ stands for the field of integers modulo p). Thus F is a finite extension of $\mathcal{Z}_p$ with dimension d over $\mathcal{Z}_p$, and its order is p^d. One knows that to each power p^d of a prime number p ($d \geqslant 1$) there exists, up to an isomorphism, a unique field with p^d elements; one usually writes it as $GF(p, d)$[4]); its multiplicative group is cyclic of order $p^d - 1$.

Under an automorphism of $F = GF(p, d)$, u and therefore the elements of the prime field $\mathcal{Z}_p$ are invariant. The group Aut(F) of automorphisms is cyclic of order d; each automorphism of F can be written as:

$$\alpha \in F \mapsto \alpha^{(p^\delta)} \in F \quad (\delta = 0, 1, 2, \ldots, d-1) .$$

The group Aut(F) has as generating element the automorphism $\alpha \mapsto \alpha^p$.

F has a non trivial involution (that is an automorphism of order 2) if and if only d is even. Evidently, in that case the automorphism

$$\alpha \in F \mapsto \alpha^{(p^c)} \in F \quad (c = d/2)$$

is a non trivial involution, and there is no other one possible. We write:

$$\bar{\alpha} = \alpha^{(p^c)}.$$

We say that $GF(p, 2c)$ is of *complex type* whereas $GF(p, 2c+1)$ is called of *real type*, and we shall use the terms and notations commonly adopted, except for $|\alpha|^2 = \alpha\bar{\alpha}$, because a finite field admits only the trivial valuation.

---

[2]) $\bar{\bar{\alpha}} = \alpha$; $\overline{\alpha+\beta} = \bar{\alpha}+\bar{\beta}$; $\overline{\alpha\beta} = \bar{\beta}\,\bar{\alpha}$.

[3]) See E. Artin, *Geometric Algebra* (Chapter 1, section 8).

[4]) GF = Galois Field.

3. Lattices $L(p, d, n)$

Let V be a vector space of dimension n over the field $GF(p, d)$. Let $L(p, d, n)$ denote the lattice of all subspaces of V. We intend to calculate the number $N_k(p, d, n)$ of subspaces of dimension k ($0 \leqslant k \leqslant n$) and the number $L_{k, k+1}(p, d, n)$ of subspaces of dimension $k + 1$ containing one of the subspaces of dimension k ($0 \leqslant k < n$).

First note that each subspace of dimension k has p^{dk} elements, or in other words $p^{dk} - 1$ non zero vectors.

We calculate the number $T_k(p, d, n)$ of ordered k-frames[5]) of V; evidently, $T_n(p, d, n)$ denotes the number of ordered bases of V. We proceed by the construction of an ordered k-frame. There are $p^{dn} - 1$ possibilities to choose the first vector of the frame; afterwards there remain $p^{dn} - p^d$ vectors in V which are linearly independent from the first chosen, that is there are $p^{dn} - p^d$ possibilities to choose the second vector of the frame; there remain $p^{dn} - p^{2d}$ vectors linearly independent from the first two chosen, and out of these we choose the third one; and so on. It follows that

$$T_k(p, d, n) = \pi_{i=0}^{k-1} (p^{dn} - p^{di}) \qquad (0 < k \leqslant n) . \tag{3.1}$$

We define $T_0(p, d, n) = 1$. As there are $T_k(p, d, k)$ ordered bases for one k-dimensional subspace, $N_k(p, d, n)$ is given by

$$N_k(p, d, n) = \frac{T_k(p, d, n)}{T_k(p, d, k)} . \tag{3.2}$$

$L_{k, k+1}(p, d, n)$ is calculated as follows. A subspace of dimension $k + 1$ contains $N_k(p, d, k + 1)$ subspaces of dimension k. As there are $N_{k+1}(p, d, n)$ subspaces of dimension $k + 1$, the total number of inclusions is $N_k(p, d, k + 1)\, N_{k+1}(p, d, n)$. Now there are $N_k(p, d, n)$ subspaces of dimension k, and as each of them is included in the same number $L_{k, k+1}(p, d, n)$ of subspaces of dimension $k + 1$, we have

$$L_{k, k+1}(p, d, n) = \frac{N_k(p, d, k+1) N_{k+1}(p, d, n)}{N_k(p, d, n)} = T_1 (p, d, n - k) . \tag{3.3}$$

The formulas (3.2) and (3.3) characterize completely the structure of the lattice $L(p, d, n)$.

The next problem discussed in this paper is the following: describe all possible orthocomplementations of $L(p, d, n)$. This description is possible for all such lattices and is given by the following theorem.

Theorem: Let $L(p, d, n)$ be defined as before. An orthocomplementation is possible only if $p \neq 2$ and $n = 2$, for each value of d. In this case, there are

$$\frac{(2\,q)!}{q!\,2^q} \qquad (2\,q = p^d + 1)$$

different ways to realize the orthocomplementation.

The proof of this theorem is divided into three parts:

1st part: (section 4) $n = 2$.

2nd part: (section 5) $n > 2$ and d odd (real fields).

3rd part: (section 6) $n > 2$ and d even (complex fields).

[5]) A *k-frame* is a set of k linearly independent vectors; a *basis* is a total frame.

4. Two-dimensional Vector Spaces

In a two-dimensional vector space, an orthocomplementation is an involutive permutation of the 1-dimensional subspaces which leaves no element invariant. Evidently, the number of 1-dimensional subspaces must be even for such an involution to exist. But we know that:

$$N_1(p, d, 2) = \frac{p^{2d}-1}{p^d-1} = p^d + 1 .$$

If $p = 2$, $p^d + 1$ is odd and there is no orthocomplementation.

If $p \neq 2$, p is odd and thus $p^d + 1$ is even, and there are orthocomplementations. Their number is equal to the number of pairings of $2q = p^d + 1$ elements, this number is:

$$(2q - 1)(2q - 3) \ldots 3 \cdot 1 = \frac{(2q)!}{2^q q!} .$$

We summarize:

Lemma: A lattice $L(2, d, 2)$ admits no orthocomplementation. A lattice $L(p, d, 2)$, with $p \neq 2$, admits

$$\frac{(2q)!}{2^q q!} \qquad (2q = p^d + 1)$$

orthocomplementations.

5. Vector Spaces over Real Fields, with Dimension $n > 2$

The fields $GF(p, 2c + 1)$ have no non trivial involution. It follows that an orthocomplementation must be induced by some *bilinear form* $B : V \times V \to F = GF(p, 2c + 1)$. Before giving the main lemma of this second part, let us recall a few general notions.

A *quadratic space* is a pair (V, B) consisting of a vector space V and a bilinear symmetric form $B : V \times V \to F$. Let x be any vector of a quadratic space. It is said to be *isotropic* when $B(x, x) = 0$; note that the zero vector is always isotropic. The space V itself is said to be *isotropic* when it contains a non zero isotropic vector.

Remark: In a euclidian or in a unitary space, the scalar product, being non degenerate, does not admit any non zero isotropic vector; thus such a space is never isotropic.

An orthocomplementation of $L(p, 2c + 1, n)$ is induced by a bilinear symmetric form bringing V into a non isotropic space. We recall now the following important result: "A quadratic space over a finite field is isotropic if its dimension is greater than or equal to 3"[6]). As a corollary we have now our main lemma.

Lemma: A lattice $L(p, 2c + 1, n)$, with $n \geqslant 3$, admits no orthocomplementation.

6. Vector Spaces over Complex Fields, with Dimension $n > 2$

These fields have a non trivial involution $\alpha \mapsto \bar{\alpha}$. So we have two possibilities to choose the form defining the orthocomplementation.

[6]) See O. T. O'Meara, *Introduction to Quadratic Forms* (Section 62).

A. *The involution is the identity*

Then, again, the orthocomplementation is induced by a bilinear symmetric form, and the arguments of the previous section apply; there is no such form.

B. *The involution is not trivial*

In this case the orthocomplementation is induced by a sesquilinear hermitian form $S : V \times V \to F$, which, in a suitable basis $(e_1, \ldots, e_n)$, is diagonal:

$$S(x, y) = \Sigma_{i=1}^{n} \alpha_i \bar{x}_i y_i$$

where $x = \Sigma_i x_i e_i$ and $y = \Sigma_i y_i e_i$. The hermiticity of S implies $\bar{\alpha}_i = \alpha_i$ for all i's. Let $\tilde{V}$ denote a vector space of same dimension as V but defined over the subfield of F of the elements invariant under the involution $\alpha \mapsto \bar{\alpha}$; this subfield, denoted by $\tilde{F}$, corresponds to the real axis of F. $\tilde{V}$ may be identified to a subset of V. The restriction of S to $\tilde{V}$ defines a bilinear symmetric form $B : \tilde{V} \times \tilde{V} \to \tilde{F}$:

$$x, y \in \tilde{V} \Rightarrow B(x, y) = S(x, y) .$$

As $\dim \tilde{V} = \dim V \geqslant 3$, $\tilde{V}$ is isotropic for B, after the arguments of the previous section, and hence V is also isotropic for S.

We have therefore proved our last lemma.

Lemma: A lattice $L(p, 2\,c, n)$, with $n \geqslant 3$, admits no orthocomplementation.

The three lemmas of the sections 4, 5 and 6 are sufficient to prove the theorem we stated in section 3.

7. Concluding Remarks and Acknowledgements

Only a lattice $L(p, d, 2)$, with $p \neq 2$, admits at least one orthocomplementation. Such a lattice can always be imbedded in the proposition system of the polarization states of a photon, i.e. the lattice of subspaces of the unitary space $\mathbb{C}^2$.

We would like to thank Prof. J. M. Jauch for having suggested this problem and for his advice during its solution.

This research was partially supported by the National Science Foundation.

Bibliography

E. Artin, *Geometric Algebra* (Interscience Publ., New York, 1957).

R. Baer, *Linear Algebra and Projective Geometry* (Academic Press, 1952).

J. M. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley, 1968).

O. T. O'Meara, *Introduction to Quadratic Forms* (Springer-Verlag, 1963).

B. L. van der Waerden, *Algebra* (Springer-Verlag, 1966).