Zeitschrift:	Helvetica Physica Acta
Band:	42 (1969)
Heft:	5
Artikel:	Causality in S-matrix theory. II
Autor:	Chandler, Colston
DOI:	https://doi.org/10.5169/seals-114092

## Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. <u>Mehr erfahren</u>

## **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. <u>En savoir plus</u>

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. <u>Find out more</u>

# Download PDF: 15.07.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

# Causality in S-Matrix Theory, II

### by Colston Chandler

Seminar für Theoretische Physik der ETH Zürich

## (3. II. 69)

Abstract. Two scattering processes are discussed for which there exist points in the physical region of the mass shell at which the analytic S-matrix cannot be represented as the boundary value of a single analytic function. At such points the S-matrix must instead be represented as a sum of at least two such boundary value terms.

#### I. Introduction

A basic assumption of analytic S-matrix theory is that, apart from energymomentum conservation delta functions, the connected parts of the momentum space S-matrix are boundary values of functions holomorphic in the complex mass shell [1]. The question studied here is whether a *single* boundary value term suffices for each scattering function  $T_c$  at each point of its physical region.

The answer to this question is no, at least within the framework of an earlier paper [2] in which the holomorphy assumption was justified on the basis of a causality requirement. In that paper the question was formulated in the following way. Consider a scattering process involving a total of n initial and final particles, and let  $\overline{K} = (\overline{k_1}, \ldots, \overline{k_n})$  be the set of mathematical energy-momentum vectors of the particles. The vectors  $\overline{k_i}$  are related to the physical energy-momentum vectors  $\overline{p_j}$  of the particles by  $\overline{k_i} = \sigma_j \overline{p_j}$ , where

$$\sigma_j = +1$$
 for final particles,  
= -1 for initial particles. (1.1)

The point  $\overline{K}$  belongs to the *n*-particle mass shell

$$\mathcal{M}_{n} = \{ K \mid K = (k_{1}, \dots, k_{n}), \ k_{j}^{2} = (k_{j0})^{2} - k^{2} = m_{j}^{2}, \ \sigma_{j} k_{j0} > 0, \ \sum k_{j} = 0 \} , \quad (1.2)$$

where the  $m_j$  are the particle<sup>1</sup>). Suppose, in addition, that  $\overline{K}$  does not belong to  $\mathcal{M}_{n,0}$ , the set of points of  $\mathcal{M}_n$  at which at least two initial particle energy-momentum vectors are parallel or at least two final particle energy-momentum vectors are parallel<sup>2</sup>). Then only a finite number of different leading positive- $\alpha$  Landau surfaces  $L_g^+$ , where g belongs to a finite index set I, can pass through  $\overline{K}$  and each has a local representation

$$L_g^+ = \{ K \mid K \in \mathcal{M}_n, \ \Lambda_g(K) = 0 \} .$$

$$(1.3)$$

<sup>1)</sup> Spin and other quantum numbers are not important here and are suppressed.

<sup>&</sup>lt;sup>2</sup>) Points of  $\mathcal{M}_{n,0}$  are excluded because of the well known difficulty that at such points the particles with parallel energy-momentum vectors interact over an infinite time span.

Colston Chandler

The functions  $\Lambda_g$  are real analytic functions in some 4 *n*-dimensional neighborhood of  $\overline{K}$ , and the gradients

where

$$\nabla A_g(K) = (\gamma_1, \ldots, \gamma_n),$$
(1.4)

$$\gamma_j^{\mu} = \left(\frac{\partial \Lambda_g}{\partial k_{j\mu}}\right) \, (\overline{K}) \, , \tag{1.5}$$

are nonzero. In addition, the gradients  $\nabla \Lambda_{g}$  are not of the form

$$U_0(\overline{K}) = (a + t_1 \,\overline{k}_1, \dots, a + t_n \,\overline{k}_n) \tag{1.6}$$

where *a* is any four-vector and the  $t_i$  are real. The problem is then to find a point *K* through which pass surfaces  $L_g^+$  that can be labeled by two distinct index sets  $I_1$  and  $I_2$ . The Landau diagrams  $D_g$  corresponding to the surfaces  $L_g^+$ ,  $g \in I_1$ , are all contractions of a single diagram  $\overline{D}_1$ , and those corresponding to  $L_g^+$ ,  $g \in I_2$ , are all contractions of  $\overline{D}_2$ . If at such a point a relation<sup>3</sup>)

$$\sum_{I_1} \lambda_g \, \nabla \Lambda_g(\overline{K}) = -\sum_{I_2} \lambda_g \, \nabla \Lambda_g(\overline{K}) + U_0(K) \tag{1.7}$$

is possible with non-negative  $\lambda_g$  (but some nonzero  $\lambda_g$  on each side), then at least two boundary-value terms are needed to represent  $T_c$  at  $\overline{K}$ .

In Section II an example of such a point is given. This point is an example of the type I points of Ref. [2]. Then in Section III an example is given of a point with the additional property that the two index sets  $I_1$  and  $I_2$  are not disjoint. This is an example of a type II point.

No new fundamental problems appear to arise from the existence of these points. They pose no obstacle to holomorphic continuation of scattering functions, for example. This is because the union of all such points is a subset  $\mathcal{L}_1^+$  of relative measure zero of  $\mathcal{L}^+$ , the union of all positive- $\alpha$  Landau surfaces [2]. Paths of continuation can simply go around  $\mathcal{L}_1^+$ . Even in general arguments involving arbitrary paths of continuation only a small amount of additional algebra is needed to allow for points in  $\mathcal{L}_1^+$ [3]. This paper merely confirms that this extra effort is more than just a precaution and is actually necessary.

## II. Type I Point

The first example involves the elastic scattering of four particles, two of mass M and two of mass m, M > m. The mathematical energy-momentum vectors of the particles are given by

$$\overline{k_j} = (\overline{k_j}_0, \overline{k_j}) = \sigma_j \, m_j (\cosh \theta_j, \sinh \theta_j \, \hat{e}) , \qquad (2.1)$$

where the  $\sigma_i$  are defined by (1.1) and  $\hat{e}$  is an arbitrary (3-dimensional) unit vector. The initial particles are specified by  $1 \leq j \leq 4$  and the final particles by  $5 \leq j \leq 8$ . The indices are chosen so that

$$M = m_1 = m_2 = m_5 = m_6$$
,  $m = m_3 = m_4 = m_7 = m_8$ . (2.2)

760

<sup>&</sup>lt;sup>3</sup>) The symbol  $U_0(\overline{K})$  always means a quantity of the form (1.6).

Finally the angles  $\theta_i$  are chosen so that

$$\theta_i = -\theta_{i+1}$$
,  $j$  odd, (2.3)

$$heta_1 > heta_5$$
 , (2.4)

$$\theta_{3} = -\frac{1}{2} (\theta_{1} - \theta_{5}) + \sinh^{-1} \left[ (M/m) \sinh \frac{1}{2} (\theta_{1} + \theta_{5}) \right], \qquad (2.5)$$

$$\theta_{7} = \frac{1}{2} \left( \theta_{1} - \theta_{5} \right) + \sinh^{-1} \left[ (M/m) \sinh \frac{1}{2} \left( \theta_{1} + \theta_{5} \right) \right].$$
(2.6)

It is easily verified that the point  $\overline{K}$  so defined belongs to  $\mathfrak{M}_8$ . The conditions  $\overline{k_j^2} = m_j^2$  and  $\sigma_j \,\overline{k_j}_0 > 0$  are trivial consequences of (2.1). Momentum conservation is a trivial consequence of (2.3), and energy conservation,

$$0 = \Sigma \sigma_j m_j \cosh \theta_j = 2 M \left( \cosh \theta_5 - \cosh \theta_1 \right) + 2 m \left( \cosh \theta_7 - \cosh \theta_3 \right)$$
(2.7)

follows immediately from (2.5) and (2.6).

It is also easy to show that  $\theta_1$  and  $\theta_3$  can be chosen so that  $\overline{K}$  does not belong to  $\mathcal{M}_{8,0}$ . The condition that any two of the vectors (2.1), say  $\overline{k_i}$  and  $\overline{k_j}$ ,  $i \neq j$ , are collinear is that  $\theta_i = \theta_j$ . Enumeration of the various possibilities shows that  $\theta_1 = \theta_3$  is the only possible equality under the conditions (2.4) and M > m. Even this equality is impossible in a small neighborhood of  $\theta_1 = \theta_5$ , as can be seen from (2.5). Thus, for  $\theta_1 - \theta_5$  sufficiently small, the point  $\overline{K}$  not only does not lie on  $\mathcal{M}_{8,0}$  but in fact has the property that *no* two of the vectors  $\overline{k_j}$ , initial *or* final, are collinear<sup>4</sup>).

The important property of the point  $\overline{K}$  is that it lies on the intersection

$$J = \bigcap_{g=1}^{4} L_g^+$$
 (2.8)

of the leading positive- $\alpha$  Landau surfaces  $L_{g}^{+}$ ,  $1 \leq g \leq 4$ , of the triangle diagrams of Figures 1 and 2. To show, for example, that  $\overline{K}$  belongs to  $L_{1}^{+}$ , one must first compute

$$\begin{aligned} x_1 &= (2\ M^2)^{-1} \left[ (k_1 + k_2 + k_4 + k_7)^2 - 2\ M^2 \right] = \cosh\left(\theta_1 - \theta_5\right), \\ x_2 &= (2\ M^2)^{-1} \left[ (\overline{k_3} + \overline{k_8})^2 - 2\ M^2 \right] = -\cosh\left(\theta_1 + \theta_5\right), \\ x_3 &= (2\ M^2)^{-1} \left[ (\overline{k_5} + \overline{k_6})^2 - 2\ M^2 \right] = \cosh 2\theta_5. \end{aligned}$$
(2.9)

Direct substitution of (2.9) into the Landau equation

$$\Lambda_1 = 1 - x_1^2 - x_2^2 - x_3^2 - 2 x_1 x_2 x_3 = 0$$
(2.10)

shows that  $\overline{K}$  belongs to the (general- $\alpha$ ) Landau surfaces  $L_1$  [4]. Equation (2.4) insures that

$$1 < x_1, x_3 < -x_2$$
, (2.11)

so that  $\overline{K}$  is in fact on  $L_1^+$  (see Fig. 3). The proof that  $\overline{K}$  lies on each of the other surfaces  $L_g^+$  is similar.

<sup>&</sup>lt;sup>4</sup>) Thus, the point  $\overline{K}$  does not lie on the boundary of the physical region in the space of Lorentz invariant variables.



Triangle diagrams  $D_1$  and  $D_2$ with heavy intermediate particles.









Vol. 42, 1969

#### Causality in S-Matrix Theory, II

Consider now the quantity

$$U = \nabla \Lambda_1(\overline{K}) + \nabla \Lambda_2(\overline{K}) = (u_1, \dots, u_8).$$
(2.12)

The values of the vectors  $u_j$  are computed by substitution of (2.1) into the definitions (1.4) and (1.5):

$$u_{1} = u_{2} = u ,$$

$$u_{3} = u_{8} = u - \alpha \sinh \theta_{5} \overline{k_{2}} ,$$

$$u_{4} = u_{7} = u - \alpha \sinh \theta_{5} \overline{k_{1}} ,$$

$$u_{5} = u_{6} = u - \alpha \sinh \theta_{5} k_{1} + \alpha \sinh \theta_{1} \overline{k}_{6} ,$$

$$\alpha = 8 M^{-2} \cosh^{2} \theta_{5} [\sinh \theta_{1} + \sinh \theta_{5}] .$$
(2.13)

The value of u is unimportant.

Consider next the quantity

$$V = \nabla \Lambda_3(\overline{K}) + \nabla \Lambda_4(\overline{K}) = (v_1, \dots, v_8) .$$
(2.14)

The values of the vectors  $v_j$  are also computed by substitution of (2.1) into (1.4) and (1.5):

$$\begin{aligned} v_1 &= v_6 = v - \beta \sinh \theta_7 \, k_4 ,\\ v_2 &= v_5 = v - \beta \sinh \theta_7 \, \overline{k_3} ,\\ v_3 &= v_4 = v ,\\ v_7 &= v_8 = v - \beta \sinh \theta_7 \, \overline{k_3} + \beta \sinh \theta_3 \, \overline{k_8} ,\\ \beta &= 8 \, m^{-2} \cosh^2 \theta_7 \, [\sinh \theta_3 + \sinh \theta_7] . \end{aligned}$$

$$(2.15)$$

The value of v is of no consequence.

The remarkable thing about these two quantities is that they satisfy

$$U = -\lambda V + U_0(\overline{K}) , \qquad (2.16)$$

where

$$\lambda = [\alpha M \sinh \theta_1 \sinh \theta_5] [\beta m \sinh \theta_3 \sinh \theta_7]^{-1}.$$
(2.17)

The quantity  $U_0(\overline{K})$  is of the form (1.6) with

$$a = u + \lambda v - t_1 \overline{k_1} - t_3 \overline{k_3},$$
  

$$t_1 = t_2 = \alpha \sinh \theta_5,$$
  

$$t_3 = t_4 = \lambda \beta \sinh \theta_7,$$
  

$$t_5 = t_6 = \alpha \sinh \theta_1,$$
  

$$t_7 = t_8 = \lambda \beta \sinh \theta_3.$$
(2.18)

Equation (2.16) is, of course, just (1.7) rewritten. It follows that the scattering function  $T_c$  for this process must be the sum of at least two boundary value terms. A similar conclusion must hold for a continuum of nearby points K. This follows from the fact that the properties of  $\overline{K}$  do not depend on the exact values of  $\theta_1$ ,  $\theta_5$  and  $\hat{e}$ ,

Colston Chandler

but only on the provision that  $\theta_1 - \theta_5$  be small enough. Hence the continuum has dimension of at least four.

Finally, the point  $\overline{K}$  defined by (2.1) is a type I point. This means that one of the boundary value terms in the representation of  $T_c$  at  $\overline{K}$  is singular only on  $L_1^+ \cup L_2^+$ , and the other is singular only on  $L_3^+ \cup L_4^+$ . Such a canonical choice of boundary value terms is unknown for type II points, and it is of interest to know if such points exist outside of the set  $\mathfrak{M}_{n,0}^{-5}$ ).

## III. Type II Point

There is a type II point for the elastic scattering of seven particles. Let eight of the fourteen initial and final particles have energy-momentum vectors given by (2.1), and let the remaining energy-momentum vectors  $(\overline{k}_9, \ldots, \overline{k}_{14})$  be any point on the positive- $\alpha$  surface of a triangle diagram. For simplicity assume that the additional particles all have mass M. The initial particles are labeled  $9 \leq j \leq 11$ , and the final particles  $12 \leq j \leq 14$ . The additional energy-momentum vectors are, of course, chosen so that no two of the vectors  $\overline{k}_i$ ,  $1 \leq j \leq 14$ , are collinear.



The point K so defined lies on the intersection

$$J' = \bigcap_{g=5}^{9} L_g^+$$
 (3.1)

<sup>5</sup>) It was shown in Reference 2 that all points of  $\mathcal{M}_{n,0}$  are of type II.

Vol. 42, 1969

of the positive- $\alpha$  Landau surfaces  $L_g^+$  of the diagrams of Figures 4-6. Inspection of the diagrams shows that the two index sets  $I_1$  and  $I_2$  can be chosen to be

$$I_1 = \{5, 6, 9\}, \qquad I_2 = \{7, 8, 9\}.$$
(3.2)

The conclusion that

$$U = \nabla \Lambda_{5}(\overline{K}) + \nabla \Lambda_{6}(\overline{K}) , \qquad V = \nabla \Lambda_{7}(\overline{K}) + \nabla \Lambda_{8}(\overline{K})$$
(3.3)

satisfy (2.16) is valid just as before. The additional quirk is that  $I_1$  and  $I_2$  overlap. This means that the point is of type II and that, unlike the situation of Section II, there is no canonical way to choose the boundary value terms for the representation of the scattering amplitude  $T_c$ .



Triangle diagram  $D_9$  with heavy intermediate particles.

## Acknowledgements

It is a pleasure to thank Dr. H. P. STAPP, who contributed so heavily to Ref. [2] upon which this note is based, for his continued interest. I am also indebted to the Seminar für Theoretische Physik for their kind hospitality and support.

#### References

- [1] See, for example, G. F. CHEW, The Analytic S-Matrix (W. A. Benjamin, Inc., New York 1966).
- [2] C. CHANDLER and H. P. STAPP, J. Math. Phys. 10, 826 (1969). This paper is summarized in C. CHANDLER, Phys. Rev. 174, 1749 (1968). For later developments see D. IAGOLNITZER, Spacetime Properties and Physical Region Analyticity in S-Matrix Theory, Berkeley preprint of 1968 Boulder lectures.
- [3] H. P. STAPP, J. Math. Phys. 9, 1548 (1968).
- [4] R. J. EDEN, P. V. LANDSHOFF, D. I. OLIVE and J. C. POLKINGHORNE, *The Analytic S-Matrix* (Cambridge University Press, New York 1966), p. 62.