Is a quantum logic a logic?

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In a recent study Jauch and Piron [2] have considered the possibility that a quantum proposition system is an infinite valued logic. They argue that if this is the case then for any two propositions p and q there must exist a conditional proposition $p \to q$. Following Lukasiewicz [3] the truth value $[p \to q]$ of the conditional $p \to q$ is defined as follows: $[p \to q] = \min\{1, 1 - [p] + [q]\}$ where $[p]$ and $[q]$ are the truth values of $[p]$ and $[q]$ respectively. Here $[p] = 1$ is interpreted as 'p is true'. Note that $[p] = 1$ and $[p \to q] = 1$ implies $[q] = 1$ so we have a law of deduction, which is a property that any reasonable logic should possess. Notice further that if $[p \to q] = 1$ and $[q \to r] = 1$ then $[p \to r] = 1$ so that implication is transitive as it should be.

Let $\mathfrak{L}$ be an orthomodular poset (representing some quantum proposition system) and let $\mathcal{S}$ be an order determining (full in [1]) set of states on $\mathfrak{L}$. We further assume that if $m_1, m_2 \in \mathcal{S}$, then $1/2\, m_1 + 1/2\, m_2 \in \mathcal{S}$, that is, $\mathcal{S}$ is closed under the formation of mid-points. We say that $a, b \in \mathfrak{L}$ are *conditional* is there exists $c \in \mathfrak{L}$ such that for all $m \in \mathcal{S}$ $m(c) = \min\{1, m(a') + m(b)\}$. If c exists it is unique. We call c the *conditional* of a and b and write $c = a \to b$. We say that $\mathfrak{L}$ (or, more correctly, the pair $(\mathfrak{L}, \mathcal{S})$) is *conditional* if every pair $a, b \in \mathfrak{L}$ are conditional. Now if $\mathfrak{L}$ is to be a logic with a law of deduction then $\mathfrak{L}$ must be conditional. Jauch and Piron [2] have shown that standard proposition systems (that is, ones that are isomorphic to the lattice of all closed subspaces of a Hilbert space) are not conditional and thus cannot be logics in the usual sense. We generalize their results to the orthomodular posets $\mathfrak{L}$ considered above. In fact we obtain the strong result that $\mathfrak{L}$ is conditional if and only if $\mathfrak{L} = \{0, 1\}$. We then characterize the pairs $a, b \in \mathfrak{L}$ which are conditional.

Undefined terms appear in [1]. If $a \leqslant b'$ we write $a + b$ for $a \vee b$. If $a \leqslant b$ we write $b - a$ for $b \wedge a'$. We first state a useful lemma whose simple proof is left to the reader.

Lemma 1. (i) $m(a \to b) = 1$ if and only if $m(a) \leqslant m(b)$; $m(a \to b) = m(a') + m(b)$ if and only if $m(b) \leqslant m(a) = 1$.

(ii) $m\ (a \to b) = m(b)$ if and only if $m(b) = 1$ or $m(a) = 1$.

This lemma will be frequently used without further comment.

Theorem 2. $\mathfrak{L}$ is conditional if and only if $\mathfrak{L} = \{0, 1\}$.

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Proof: Clearly $\{0, 1\}$ is conditional; in fact $1 = 0 \to 1$ and $0 = 1 \to 0$. Now let $\mathfrak{L}$ be conditional and suppose there exists $a \in \mathfrak{L} - \{0, 1\}$. Then $c = a \to a'$ exists and $m(c) = \min\{1, 2\, m(a')\}$. Since $\mathcal{S}$ is order determining $a' \leqslant c$. Hence there exists $b \in \mathfrak{L}$ such that $a' + b = c$. Now $m(b) = m(c) - m(a') = \min\{m(a), m(a')\}$. Thus $m(b) \leqslant 1/2$ for all $m \in \mathcal{S}$. It follows that $b \leqslant b'$ since $\mathcal{S}$ is order determining. Hence $b = 0$ and $c = a'$. Thus $m(c) = \min\{1, 2\, m(c)\}$ and hence $m(c) = 0$ or 1 for all $m \in \mathcal{S}$. Moreover since $0 < c < 1$ there exist $m_1, m_2 \in \mathcal{S}$ with $m_1(c) = 0$ and $m_2(c) = 1$. Letting $m = 1/2\, m_1 + 1/2\, m_2$ we have $m(c) = 1/2$, a contradiction. Hence $\mathfrak{L} = \{0, 1\}$.

We have seen that, for non-trivial posets $\mathfrak{L}$, not every pair of elements is conditional. We now study the properties of pairs of elements that are conditional.

Lemma 3. If $a \to b$ and $a' \vee b$ exist and are equal then $a\, C\, b$.

Proof: There exists $d \in \mathfrak{L}$ such that $b + d = a' \vee b$. We show $d \leqslant a'$. Otherwise there exists $m \in \mathcal{S}$ such that $m(d) > m(a')$. Then $m(a' \vee b) = m(b) + m(d) > 1 - m(a) + m(b)$ so $m(a) > m(b)$. Hence $m(a' \vee b) = m(a \to b) = 1 - m(a) + m(b)$, a contradiction. Now there exists $e \in \mathfrak{L}$ with $d + e = a'$. We show $e \leqslant b$. Otherwise there exists $m \in \mathcal{S}$ with $m(e) > m(b)$. Then $m(a') = m(d) + m(e) > m(d) + m(b) = m(a' \vee b) \geqslant m(a')$, a contradiction. Hence there exists $f \in \mathfrak{L}$ with $b = f + e$, $a' = d + e$ and $f \leqslant b \leqslant d'$ so that $a'\, C\, b$. Thus $a\, C\, b$.

Lemma 4. If $c = a \to b$ exists then $a' \leqslant c$ and $b \leqslant c$.

Proof: If $a' \nleqslant c$ then there exists $m \in \mathcal{S}$ such that $m(c) < m(a')$. Hence $m(c) < 1$ and $1 - m(a) + m(b) = m(c) < 1 - m(a)$. Thus $m(b) < 0$, a contradiction. That $b \leqslant c$ is immediate.

We say that $\mathcal{S}$ is *sufficient* if $0 \neq a \in \mathfrak{L}$ implies there exists $m \in \mathcal{S}$ with $m(a) = 1$.

Theorem 5. Let $\mathcal{S}$ be sufficient and assume that $a' \vee b$ exists. Then $a \to b$ exists if and only if $a \leqslant b$ or $b \leqslant a$.

Proof: Clearly, if $a \leqslant b$ then $a \to b = 1$ and if $b \leqslant a$, then $a \to b = a' + b$. Conversely, assume $c = a \to b$ exists. By Lemma 4 $c \geqslant a' \vee b$. Hence there exists $d \in \mathfrak{L}$ such that $(a' \vee b) + d = c$. Suppose $d \neq 0$. Then there exists $m \in \mathcal{S}$ such that $m(d) = 1$. Hence $m(a') = m(b) = 0$ and $m(c) = 1 - m(a) + m(b) = 0$, a contradiction. Therefore $d = 0$ and $c = a' \vee b$. It now follows from Lemma 3 that $a\, C\, b$. Suppose a and b are not comparable. Then $a \wedge b < a$ and $a \wedge b < b$. Hence there exists $m_1, m_2 \in \mathcal{S}$ such that $m_1\, (a - (a \wedge b)) = 1$ and $m_2\, (b - (a \wedge b)) = 1$. It follows that $m_1(a) = m_2(b) = 1$ and $m_1(b) = m_1\, (a \wedge b) = m_2(a) = m_2\, (a \wedge b) = 0$. Let $m = 1/2\, (1/2\, m_1 + 1/2\, m_2) + 1/2\, m_1 = 3/4\, m_1 + 1/4\, m_2$. Then $m\, (a \wedge b) = 0$ and $m(b) = 1/4 < 3/4 = m(a)$. Hence $m(a') + m(b) = m(c) = m\, (a' \vee b) = m\, (a' + (a \wedge b)) = m(a') + m\, (a \wedge b)$. Thus $m(b) = m\, (a \wedge b)$, a contradiction.

Corollary 6. Let $\mathcal{S}$ be sufficient and $a' \vee b$ exist. If $a \to b$ exists, then $a \to b = a' \vee b$, $b \to a$ exists, $b' \vee a$ exists, and $b \to a = b' \vee a$.

The proofs of the previous theorems depend heavily on the fact that $\mathcal{S}$ is order determining, sufficient or both. If we strengthen $\mathcal{S}$ still further we obtain a stronger result. We say that $\mathcal{S}$ is *strongly order determining* if $\{m \in \mathcal{S} : m(a) = 1\} \subseteq \{m \in \mathcal{S} : m(b) = 1\}$

implies that $a \leqslant b$. It can be shown that strongly order determining implies both order determining and sufficiency. (The converse fails; see [1].) Notice that the set of states on the lattice of all closed subspaces of a Hilbert space is strongly order determining.

Theorem 7. If S is strongly order determining, then $a \rightarrow b$ exists if and only if $a \leqslant b$ or $b \leqslant a$.

Proof: As in Theorem 5, if a and b are comparable, then $a \rightarrow b$ exists. Now assume $c = a \rightarrow b$ exists. Suppose $a \nleqslant b$ and $b \nleqslant a$. Then there exists $m_0, m_1 \in S$ such that $m_0(a) = 1$, $m_0(b) < 1$, $m_1(a) < 1$ and $m_1(b) = 1$. Note that $m_0(c) = m_0(b)$ and $m_1(c) = 1$. Let $m = 1/2\, m_0 + 1/2\, m_1$. Then $m(a) = 1/2 + 1/2\, m_1(a) < 1$, $m(b) = 1/2\, m_0(b) + 1/2 < 1$ and $m(c) = m(b)$. This last sentence contradicts Lemma 1 (ii). Hence a and b are comparable.

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