Local analyticity properties of the n particle scattering amplitude

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Local Analyticity Properties of the n Particle Scattering Amplitude

by **J. Bros** and **V. Glaser**

CERN, Geneva

and **H. Epstein**

I.H.E.S., Bures-sur-Yvette

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Abstract. The connected part $F_c(p)$ of the scattering amplitude $\langle p_1 \ldots p_r | S - 1 | p_{r+1}, \ldots, p_n \rangle$ defined on the mass shell $p_i^2 = m_i^2$ and deduced from a local field theory involving only (stable) particles with strictly positive masses can be represented in a suitable neighbourhood of any physical point p as a finite sum $f_c(p) = \sum_1^N F_i(p)$ of 'partial amplitudes', each $F_i(k)$ analytic in a certain domain $\mathscr{F}_i$ of the complex mass shell $k_i^2 = m_i^2$. The mentioned real neighbourhood lies on the boundary of each $\mathscr{F}_i$. The above decomposition may fail to hold only at points p where any two incoming or any two outgoing four-momenta become parallel (thresholds). The number N as well as the shape of the domains $\mathscr{F}_i$ depend on the number n and on the real neighbourhood considered. For a generic configuration p the intersection of the domains $\mathscr{F}_i$ is empty. When this does not happen, $F_i(p)$ is the boundary value of a single analytic function. This is illustrated on the case of the five-point function, where it is shown that when $D \equiv \det(p_r p_s) > m_1^2 m_2^2 m_3^2$, D being the Gram determinant of the scalar products of the three outgoing momenta p_1, p_2, p_3, the scattering amplitude is the boundary value of a single analytic function. It is also indicated on the same example how these local results may be improved; one finds in the equal mass case $m_r = m$ that the five-point scattering amplitude is the boundary value of a single analytic function whenever $M > 4,8m$, M being the total centre-of-mass energy of the three outgoing momenta.

1. Introduction

In his paper [1], Professor Markus Fierz gave, as early as in 1950, a very lucid analysis of the causal character of the time-ordered amplitudes appearing in the calculation of S matrix elements in field theory. As our contribution to the celebration of his 60th anniversary, we present in this paper an analysis of the analytic structure of the general scattering amplitude involving n particles in a complex neighbourhood of its physical points. We feel that our treatment of the problem is close in spirit—though unfortunately not in style—to the argumentation used in the paper [1]; all the proofs are based solely on the causal factorization and spectral properties of a time-ordered amplitude.

While the analytic properties of scattering amplitudes involving four particles have been extensively studied and are nowadays well understood and well founded on the general principles of local field theory[1]), nothing comparable has been achieved for the case $n \geqslant 5$[2]).

In the present paper, the following will be shown: the connected part $F_c(p)$ of the scattering amplitude:

$$\langle p_1 \ldots p_\nu | S - 1 | p_{\nu+1} \ldots p_n \rangle_c = \delta_4 \left(\sum_1^\nu p_i - \sum_{\nu+1}^n p_j \right) F_c(p)$$

defined on the mass shell $p = (p_1, \ldots, p_n)$, $p_1 + \cdots - p_n = 0$, $p_i^2 = m_i^2$, $p_i \in V_+$, and deduced from a local field theory involving only (stable) particles with strictly positive masses can be represented in a suitable neighbourhood of any point p as a finite sum

$$F_c(p) = \sum_{i=1}^N F_i(p) \tag{D}$$

of 'partial amplitudes', each $F_i(p)$ being the boundary value (in the sense of distributions) of a function $F_i(k)$, $k = p + iq$, analytic in a certain domain $\mathscr{F}_i$ of the complex mass shell $k_i^2 = m_i^2$. The mentioned real neighbourhood lies on the boundary of each $\mathscr{F}_i$. The decomposition (D) fails to hold only at points p where any two incoming or any two outgoing momenta p_i become parallel (thresholds). The number N as well as the shape of the domains $\mathscr{F}_i$ depend on the number n and in general also on the real neighbourhood considered. Only when $n - \nu = 2$ or $\nu = 2$ is the number N independent of the position of the point p on the mass shell, but also in the general case a decomposition (D) can be found—with some loss of information—with an N independent of p and satisfying all the quoted conditions. For a generic configuration p the intersection of the corresponding domains $\mathscr{F}_i$ will be empty:

$$\bigcap_{i=1}^N \mathscr{F}_i = \varnothing .$$

Only when this does not happen will the scattering amplitude be a boundary value of a single analytic function. That these different possibilities do indeed occur is illustrated on examples in section 4. If $\nu = 2$, $n - \nu = 2$ we have $N = 1$, so that the scattering amplitude is everywhere (except possibly at the threshold) the boundary value of a single analytic function. This result for the four-point function (though obtained by a different method) has been known for a long time [6]. In the case of the five-point function ($\nu = 3$, $n - \nu = 2$) we have $N = 3$. In this case the decomposition (D) has been proved some time ago by two of the authors [7]. The method used in [7] is geometrically much more cumbersome, compared to the simple method of this paper, but permits to obtain better local results due to a better exploitation of causality. Therefore the case of the five-point function is discussed in detail in section 4, in order to indicate how the results of this paper could be improved.

The fact that the scattering amplitude is not always the boundary value of a single analytic function but is of the form (D) in the neighbourhood of some Landau singularities was recognized independently in perturbation theory in [8], [9] and [10]. This fact proves the necessity of a decomposition of the type (D), at least in the neighbourhood of some physical points.

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1) For a survey of the methods used and results obtained exclusively on the basis of general principles, compare the review articles [2]–[4].

2) As a most recent summary of the results so far obtained—again on the basis of general principles—cf. [5].

The main mathematical tool used in this paper is the so-called generalized edge-of-the-wedge theorem[3]). It has been proved only recently in full generality in [13] and [14] by the use of a generalization of the ordinary Fourier transform, which was inspired by the paper [9]. On the other hand, the physical problem itself is very much related to the proof of the L.S.Z. reduction formulae achieved some years ago by Hepp [2], [15]. The problem can be formulated as follows: what special continuity properties does the (amputed and truncated) off-mass-shell time-ordered amplitude $\tilde{t}_c(p)$ enjoy thanks to locality and spectrum of the underlying field theory, so that its restriction to the mass shell $p_i^2 \equiv m_i^2$, $i = 1, \ldots, n$, be meaningful. This is a non-trivial problem since the restriction of a general distribution to a lower dimensional manifold is of course meaningless. In the next section we shall first solve this problem again, but in a form which makes the passage to the ambient complex space in section 3 quite transparent and natural. Thus the decomposition (D) can be also viewed as a generalization and a sharper formulation of the well-known results of Hepp.

2. Continuity Properties of $\tilde{t}_c$ Near the Mass Shell

The L.S.Z. formalism gives, as is well known, the following formal prescription for the computation of S matrix elements

$$\langle p_1, \ldots, p_\nu | S - 1 | -p_{\nu+1}, \ldots, -p_n \rangle_c \prod_{i=1}^{\nu} \langle p_i | A(0) | 0 \rangle \prod_{j=\nu+1}^{n} \langle 0 | A(0) | -p_j \rangle$$

$$= \delta_4\left(\sum_1^n p_i\right) \tilde{t}_c(p)|_{p_1, \ldots, p_\nu} \in \overline{V}_+(m), p_{\nu+1}, \ldots, p_n \in \overline{V}_-(m). \quad (1)$$

For the sake of simplicity, we shall consider here the usual field theory of a single Bose self-interacting field $A(x)$ describing scalar particles of strictly positive mass m. Here

$$\overline{V}_+(m) = -\overline{V}_-(m) = \{p \in \mathbb{R}_4 : p_0 = +\sqrt{\vec{p}^2 + m^2}\}$$

is the positive mass hyperboloid. $\tilde{t}_c(p)$ is the connected (=truncated) vacuum expectation value of the 'amputated' time-ordered product of n fields:

$$\delta_4\left(\sum_1^n p_i\right) \tilde{t}_c(p) = \int t_c(x) e^{ipx}\, dx, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}_{4n},$$

$$p = (p_1, \ldots, p_n) \in \mathbb{R}_{4n}, \quad px = \sum_1^n p_i x_i, \quad dx = d^4 x_1 \ldots d^4 x_n \quad (2)$$

with

$$t_c(x) = (\Omega, T(x)\Omega)_c \equiv \langle I(x) \rangle_c,$$

$$T(x) = K_{x_1} \ldots K_{x_n} T(A(x_1) \ldots A(x_n)), \quad K_x = \Box_x - m^2. \quad (3)$$

In (2) the energy momentum conservation has been explicitly put in evidence, so that $\tilde{t}_c(p)$ is to be considered as a distribution defined only on the subspace $\sum_1^n p_i = 0$ of $\mathbb{R}_{4n}$.

[3]) See the Theorems 3 and 3′ of section 3. Theorem 3 has been first formulated by A. Martineau in the context of the theory of hyperfunctions by Sato [11]. A special case of this theorem, sufficient for the treatment of the five-point function, was proved in [7]. The authors are very much indebted to A. Martineau, B. Malgrange and J. Lascoux for drawing their attention to this theorem years ago at the Strasbourg meetings. Our special thanks are due to Stora [12], who was the first to insist on the importance of decompositions of the type (D) for field theory.

If we start from the usual Wightman axioms for the field $A(x)$, it is still unknown whether sharp time-ordered products can be constructed. In that case (cf., for example, [2] or [3]) the field operator $A(x)$ should be replaced by its mean value over a finite space-time region, more precisely

$$A(x) \to A_\varphi(x) = \int A(x-y)d^4y, \varphi \in \mathscr{D}(\mathbb{R}_4) \tag{4}$$

with

$$\operatorname{supp} \varphi \subset D = \{x \in \mathbb{R}_4 : |x_0| + |\vec{x}| \leqslant a\}$$

for some finite $a > 0$. The T product of n operators A_φ can then be constructed with the help of the usual step functions

$$T(A_\varphi(x_1) \ldots A_\varphi(x_n)) = \sum_{\pi \in \sigma_n} \theta(x^0_{\pi 1} - x^0_{\pi 2}) \ldots \theta(x^0_{\pi(n-1)} - x^0_{\pi n})$$

$$\cdot A_\varphi(x_{\pi 1}) A_\varphi(x_{\pi 2}) \ldots A_\varphi(x_{\pi n}), \quad \theta(t) = \tfrac{1}{2}(|t| + t) \tag{5}$$

and with such a T product the formal expression (1) is still expected to hold. Care must be only taken to choose φ so that the matrix element $\langle p | A_\varphi(0) | 0 \rangle$ between the vacuum and a one-particle state is $\neq 0$. It follows then from causality and the spectral condition that $\langle p | A_\varphi(0) | 0 \rangle$ is an entire analytic function of p on the complex mass hyperboloid $p^2 = m^2$ (cf., for example, [16]). The same remarks apply also to the case of a Haag-Araki theory: choose any (bounded) operator A in the algebra $\mathscr{A}(D)$ of local observables belonging to the space-time region D defined by (4) for some finite a such that the (entire) function $\langle p | A | 0 \rangle \neq 0$; define the field operator $A(x)$ by $A(x) = e^{iPx} A e^{-iPx}$, where e^{-iPx} is the space-time translation operator. Then the formal recipe (1) is still expected to hold.

The problem we want to discuss in this section is the following: $\tilde{t}_c(p)$ is a tempered distribution $\in \mathscr{S}'(\mathbb{R}_{4(n-1)})$ (also in the Haag-Araki case although $t_c(x)$ may then be chosen to be a bounded continuous function in x space) and the restriction of a distribution to a manifold, such as the mass shell as required by equation (1), has in general no sense. How do the properties of causality and spectrum of the function $t_c(x)$ following from the general principles make this restriction nevertheless possible? The proof of the restrictability to the mass-shell was given by Hepp [2], [15], who showed that $\tilde{t}_c(p)$ had continuity properties such that (1) can be defined as a distribution *on* the mass shell in the set of three-vectors $\vec{p}_1, \ldots, \vec{p}_n$, $p^0_i = \pm\sqrt{\vec{p}_i^2 + m^2}$, $i = 1, \ldots, n$, provided it is applied to test function $\varphi(\vec{p}_1, \ldots, \vec{p}_n) \in \mathscr{D}(\mathbb{R}_{3n})$ which vanish in a neighbourhood of any two parallel momenta.

We shall now rederive this result in a form more adapted for the passage to the complex reserved for the next section.

Let us first state the properties of the time-ordered amplitude t_c on which all the considerations of this paper will be based.

1. *Causality*

$$\langle T(X) \rangle_c = \langle T(I_1) T(I_2) \cdots T(I_p) \rangle_c \quad \text{if } [I_r] \gtrsim [I_s]$$
$$\text{for all } r < s,\ r, s = 1, 2, \ldots, p \tag{C}$$

where $X = \bigcup_{r=1}^{p} I_r$ and $I_r \cap I_s = \varnothing$ for $r \neq s$. Here some explanation of the notation is needed: $X = \{1, 2, \ldots, n\}$ is the set of indices numbering the different space-time points of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_{4n}$, I_s is any subset of X and by an abuse of notation, we write $T(x_{i_1},\ldots,x_{i_\mu}) = T(I)$ for the amputated T product of μ operators, where $I=\{i_1,i_2,\ldots,i_\mu\} \subset X$. Consequently, we write indifferently: $T(x)=T(X)$.

$$[I]=\bigcup_{i\in I}\{x_i\}\subset \mathbb{R}_4 \tag{6}$$

is the collection of the single points $\{x_i\}$, $i\in I$, considered as a subset of the Minkowski space $\mathbb{R}_4$. Finally, for two sets A, $B\subset \mathbb{R}_4$

$$A \gtrsim B \text{ means } A\cap\{B+\overline{V}_-\}=\varnothing, \tag{7}$$

i.e., the set A does not intersect the 'past causal shadow' of the set B. In the case of a 'sharp' T product condition (C) is simply the well-known factorization property of a T product in case the argument X can be decomposed into several clusters in a mutually acausal position. In the Wightman or Haag-Araki case, when the T product is given by equation (5), the condition (C) still holds, provided we define for any $I\subset X$:

$$[I]=\bigcup_{i\in I} D(x_i) \tag{8}$$

with

$$D(x_i)=\{x\in\mathbb{R}_4 : |x^0-x_i^0|+|\vec{x}-\vec{x}_i|\leqslant a\}$$

[cf., equation (4)]. We shall denote the expression (8) by $[I]_a$ and consider (6) as a limiting case for $a=0$: $[I]=[I]_0$.

The condition (C) is the *full* causality condition for the n point Green's function. In the following we shall exploit only the special case of the decomposition into two clusters

$$\langle T(X)\rangle_c=\langle T(X\backslash I)\,T(I)\rangle_c \tag{C$'$}$$

when $[X\backslash I]_a \gtrsim [I]_a$[4]).

2. *Invariance under space-time translations*

$t_c(x)$ and all the distributions appearing in (C) are supposed to be invariant under $(x_1,\ldots,x_n)\to(x_1+a,\ldots,x_n+a)$ for all $a\in\mathbb{R}_4$. All these distributions depend therefore on $n-1$ four-vectors, e.g., $\xi_r=x_r-x_n$, $r=1,\ldots,n-1$. In order not to break the symmetry under permutations, we shall not hesitate to express this fact by saying simply $t_c\in\mathscr{S}'(\mathbb{R}_{4(n-1)})$ without specifying a co-ordinate system in $\mathbb{R}_{4(n-1)}$. The Fourier transform of such a distribution, say $\tilde{t}_c$, will be also an element of $\mathscr{S}'(\mathbb{R}_{4(n-1)})$. Again, for the sake of symmetry, it will be considered as a function of n four vectors $p_1,\ldots,p_n$ linked by the relation $p_1+\cdots+p_n=0$ without specification of a co-ordinate system, always in the sense of formula (2), in which invariance under translations has been taken into account. The subspace $p_1+\cdots+p_n=0$ of $\mathbb{R}_{4n}$ will be simply called $\mathbb{R}_{4(n-1)}$. Note that the causality condition is translationally invariant.

3. *The spectral condition*

The Fourier transform of $t_{cI}=\langle T(X\backslash I)\,T(I)\rangle_c$ has the following support property:

$$\operatorname{supp}\tilde{t}_{cI}\subset\{p\in\mathbb{R}_{4(n-1)}: p_{X\backslash I}\in\overline{V}_+(M_{X\backslash I})\} \tag{Sp}$$

[4]) Usually (C) is stated only for the time components of the points x_i in a special Lorentz frame. That the local commutativity of the fields really imples (C) for the T product defined by (5) requires an (easy) proof (e.g., by induction on the number of arguments).

if $I \neq \varnothing$ and X. Here

$$p_J = \sum_{i \in J} p_i \tag{9}$$

and $\overline{V}_+(M_J)$ denotes the following closed sets in $\mathbb{R}_4$:

$$\overline{V}_+(M_J) = \overline{V}_+(2m) = \{p \in \mathbb{R}_4 : p^0 \geqslant \sqrt{\vec{p}^2 + 4m^2}\} \quad \text{if } |I| = 1 \text{ or } n-1$$

$$\overline{V}_+(M_J) = \overline{V}_+(m, 2m) = \overline{V}_+(m) \cup \overline{V}_+(2m) \quad \text{if } 1 < |J| < n-1 \tag{10}$$

where $|J|$ denotes the number of elements in the set J.

The condition (Sp) is obtained by 'inserting intermediary states' between the operators $T(X\backslash I)$ and $T(I)$ and taking into account the assumed spectral structure of the energy momentum operator of the theory. Because of the truncation, the vacuum state does not contribute and because of the amputation, the one-particle state does not contribute when $T(I)$ or $T(X\backslash I)$ consist of single field operators. Here again the complete spectral condition would involve the Fourier transform of the general 'cluster' $\langle T(I_1) \ldots T(I_p)\rangle_c$, but for our purposes (Sp) will suffice.

The list of our assumptions being complete, take any point p in momentum space $\mathbb{R}_{4(n-1)}$. For any such point and any proper subset I of X, we will have either $p_I \in \overline{V}_+(M_I)$ or $p_I \in \overline{V}_-(M_I) \equiv -\overline{V}_+(M_I)$ or $p_I \in \complement(\overline{V}_+(M_I) \cup \overline{V}_-(M_I))$. More precisely, let $\mathscr{P}*(X)$ denote the set of proper subsets of $X = \{1, \ldots, n\}$ and define, given $p = (p_1, \ldots, p_n) \in \mathbb{R}_{4(n-1)}$, the following three subsets of $\mathscr{P}*(X)$:

$$\mathscr{K} = \{I \in \mathscr{P}*(X) : p_I \in \complement(\overline{V}_+(M_I) \cup \overline{V}_-(M_I))\}$$

$$\mathscr{S}_\pm = \{I \in P*(X) : p_I \in \overline{V}_\pm(M_I)\}. \tag{11}$$

In the definition of $\mathscr{S}_\pm$ either the upper or the lower sign holds throughout. The collections of sets $\mathscr{K}$, $\mathscr{S}_+$ and $\mathscr{S}_-$ have the following properties

$$\mathscr{K} \cup \mathscr{S}_+ \cup \mathscr{S}_- = \mathscr{P}*(X), \quad \mathscr{K} \cap \mathscr{S}_+ = \mathscr{S}_+ \cap \mathscr{S}_- = \varnothing \tag{12a}$$

$$I \in \mathscr{K} \Leftrightarrow X\backslash I \in \mathscr{K} \tag{12b}$$

$$I \in \mathscr{S}_+ \Leftrightarrow X\backslash I \in \mathscr{S}_- \tag{12c}$$

These properties are an immediate consequence of $p_I + p_{X\backslash I} = p_X = 0$, of $\overline{V}_+(M_I) = -\overline{V}_-(M_I)$ and of $\overline{V}_+(M_I) + \overline{V}_+(M_J) \subset \overline{V}_+(M_K)$ for any $I, J, K \subset X$. The last relation entails also the following two properties

$$\{I_{1,2} \in \mathscr{S}_\pm, I_1 \cap I_2 = \varnothing, I_1 \cup I_2 \neq X\} \Rightarrow \{I_1 \cup I_2 \in \mathscr{S}_\pm\} \tag{12d}$$

$$\{I_{1,2} \in \mathscr{S}_\pm, I_1 \cup I_2 = X, I_1 \cap I_2 \neq \varnothing\} \Rightarrow \{I_1 \cap I_2 \in \mathscr{S}_\pm\} \tag{12e}$$

the sign + or the sign − holding throughout, which we mention for the sake of completeness. Let us call the collection $(\mathscr{K}, \mathscr{S}_+, \mathscr{S}_-)$ determined by a point p, in the manner just described, a *hypercell*[5]). As a matter of fact, a hypercell is already completely determined by $\mathscr{S}_+$ (or $\mathscr{S}_-$) alone in view of the properties (12a)–(12c).

To a hypercell, we attach an open set $\Omega_{\mathscr{K}, \mathscr{S}_+}$ in momentum space as follows:

$$\Omega_{\mathscr{K}, \mathscr{S}_+} = \{p \in \mathbb{R}_{4(n-1)} : p_I \in \complement(\overline{V}_+(M_I) \cup V_-(M_I)) \text{ for all } I \in \mathscr{K}, p_I \in \complement\overline{V}_-(M_I) \text{ for all } I \in \mathscr{S}_+\} \tag{13}$$

[5]) The name 'cell' is reserved for a very similar collection of subsets of the set X attached to the definition of a generalized retarded function as it will be discussed in section 4.

where $\complement A$ denotes the complement of A in $\mathbb{R}_4$. Every point $p \in \mathbb{R}_{4(n-1)}$ is in an open set $\Omega_{\mathscr{K},\mathscr{S}_+}$ determined uniquely by that point, and there is obviously a finite number of sets $\Omega_{\mathscr{K},\mathscr{S}_+}$ covering the whole of $\mathbb{R}_{4(n-1)}$.

Now, given a hypercell $(\mathscr{K},\mathscr{S}_+)$, the set $\Omega_{\mathscr{K},\mathscr{S}_+}$ was constructed in such a manner that the Fourier transform of $\langle T(X\backslash I)T(I)\rangle_c$ vanishes for all $I \in \mathscr{K} \cup \mathscr{S}_+$ when $p \in \Omega_{\mathscr{K},\mathscr{S}_+}$ as a consequence of (Sp′):

$$\tilde{t}_{cI}(p) = 0 \tag{14}$$

for all $p \in \Omega_{\mathscr{K},\mathscr{S}_+}$ and all $I \in \mathscr{K} \cup \mathscr{S}_+$. If we define then for any $I \in \mathscr{P}^*(X)$ a 'retarded amplitude' r_I by the formula:

$$r_I(x) = \langle T(X)\rangle_c - \langle T(X\backslash I)T(I)\rangle_c \tag{15}$$

the following relation will hold for its Fourier transform:

$$\tilde{t}_c(p) = \tilde{r}_I(p) \tag{16}$$

for all

$$p \in \Omega_{\mathscr{K},\mathscr{S}_+} \text{ and } I \in \mathscr{K} \cup \mathscr{S}_+.$$

Because of (C) r_I has the support property

$$r_I(x) = 0 \text{ in } U_{a,I} = \{x \in \mathbb{R}_{4(n-1)} : [X\backslash I]_a \gtrsim [I]_a\} \tag{17}$$

Note that $U_{a,I}$ is an open set.

The relations (16) and (17) are fundamental for the rest. In order to derive continuity properties of $\tilde{t}_c(p)$, we multiply both sides of (16) by any infinitely differentiable function $\tilde{\alpha}(p)$ with compact support contained in $\Omega_{\mathscr{K},\mathscr{S}_+}$. Equation (16) becomes

$$t_c(p)\tilde{\alpha}(p) = \tilde{r}_I(p)\tilde{\alpha}(p) \tag{16′}$$

valid in the whole space $\mathbb{R}_{4(n-1)}$ for all $I \in \mathscr{K} \cup \mathscr{S}_+$ and any $\alpha \in \mathscr{D}(\Omega_{\mathscr{K},\mathscr{S}_+})$.

We can choose $\tilde{\alpha}$ so that $\tilde{\alpha}(p) = 1$ in any fixed compact set contained in $\Omega_{\mathscr{K},\mathscr{S}_+}$. By Fourier transformation, (16′) becomes:

$$(t_c * \alpha)(x) = (r_I * \alpha)(x) \tag{16″}$$

in $\mathbb{R}_{4n}$ for all $I \in \mathscr{K} \cup \mathscr{S}_+$.

We now apply to $r_I * \alpha$ the following lemma, first used by Hepp [15].

Lemma 1. *If a tempered distribution $F \in \mathscr{S}'(\mathbb{R}_N)$ vanishes in an open cone $C \subset \mathbb{R}_N$, then for any fixed test function $\alpha \in \mathscr{S}(\mathbb{R}_N)$ the infinitely differentiable function $F * \alpha \in \mathcal{O}_\mu$ is of fast decrease in any closed cone Γ such that $\Gamma\backslash\{x_0\} \subset C$, x_0 being the common apex of Γ and C. We denote this property shortly by: $F * \alpha \in \mathscr{S}(C)$.*

Since $F * \alpha$ is by general theorems always C_∞ and of at most polynomial increase together with all its derivatives at infinity, the lemma is a statement about the asymptotic behaviour at infinity; along any direction contained in C, $(F * \alpha)(x)$ and all its derivatives vanish at infinity faster than any inverse power of the distance from any fixed point in $\mathbb{R}_N$. The proof is an immediate consequence of the very definition $(F * \alpha)(x) = \langle F(y), \alpha(x - y)\rangle$ of the convolution and of the definition of the support of a distribution: when x runs away to infinity within any closed cone Γ of the lemma, the distance of the point x to the support of F tends uniformly to zero.

We choose $F = r_I$ in the above lemma and want to show that

$$r * \alpha \in \mathcal{S}(U_I) \tag{18}$$

with $U_I = U_{0,I}$ for any $\tilde{\alpha} \in \mathcal{S}(\mathbb{R}_{4(n-1)})$. In the case of a 'sharp' T product we have just to put $C = U_I$ in Lemma 1, since then $a = 0$ in (17) and U_I is an open cone in $\mathbb{R}_{4(n-1)}$ with its apex at the origin, as it immediately follows from the definitions (6) and (7). If $a > 0$, it is enough to establish the relation

$$d(x, \complement U_{a,I}) \geqslant d(x, \complement U_I) - ca \tag{19}$$

where $d(x, \ldots)$ is the Euclidean distance of an arbitrary point $x \in \mathbb{R}_{4(n-1)}$ to the complement of $U_{a,I}$, respectively U_I, and c is a constant independent of x. Equation (19) says namely that the distance of a point x to the support of r_I tends to infinity whenever $d(x, \complement U_I)$ tends to infinity, which is precisely what is needed to establish (18). Equation (19) can be inferred from the following very useful explicit representation of the complement of $U_{a,I}$ in $\mathbb{R}_{4(n-1)}$

$$\operatorname{supp} r_I \subset \complement U_{a,I} = \bigcup_{\substack{i \in I \\ j \in X \setminus I}} \{x_i - x_j + 2ea \in \overline{V}_+\}. \tag{20}$$

Here $e = (1, 0, 0, 0)$ is the unit timelike vector and $\overline{V}_+$ is the closed forward light cone. Equation (20) becomes immediately clear if one draws a two-dimensional picture of the definitions (7) and (8). The proof of (19) is then left to the reader.

The relations (16″) and (18) imply

$$t_c * \alpha \in \mathcal{S}\left(\bigcup_{I \in \mathcal{K} \cup \mathcal{S}_+} \mathcal{U}_I\right) \tag{21}$$

if $\tilde{\alpha} \in \mathcal{D}(\Omega_{\mathcal{K}, \mathcal{S}_+})$.

If we define, following Hepp [2], the essential support of a C_∞ function as the complement of the open cone in which the function and all its derivatives vanish faster than any inverse power of the distance from the origin in the sense of Lemma 1, we can rewrite (21), using the formula (20) with $a = 0$, in the form

$$\operatorname{ess\,supp} t_c * \alpha = \bigcap_{I \in \mathcal{K} \cup \mathcal{S}_+} \bigcup_{\substack{i \in I \\ j \in X \setminus I}} \{x_i - x_j \in \overline{V}_+\} \tag{22}$$

if $\tilde{\alpha} \in \mathcal{D}(\Omega_{\mathcal{K}, \mathcal{S}_+})$.

The important feature of this formula is the fact that the essential support of $t_c * \alpha$ is a finite union of *convex proper* cones[6]). To exhibit this feature more clearly, let us define:

Definition: A choice is a map $I \to h(I)$ which associates to every proper subset I of X an element $h(I)$ of $\{1, \ldots, n\}$ contained in $I : h(I) \in I$.

---

[6]) We stick to the following definitions: *a* cone C in $\mathbb{R}_N$ is a set satisfying $\rho C = C$ for all $\rho > 0$; $x_0 + C$ is a cone with apex at the point x_0; a *proper* cone is a cone C whose closure $\overline{C}$ does not contain any linear subspace of $\mathbb{R}_N$ except the origin.

In other words, a choice picks out of every subset I an element contained in it. Of course there are a finite number of different choices if n is finite[7]). Let h, h' be two choices. Then (22) can be written as follows:

$$ess\,\mathrm{supp}\, t_c * \alpha = \bigcap_{I \in \mathscr{K} \cup \mathscr{S}_+} \bigcup_{h,h'} \{x_{h(I)} - x_{h'(x \setminus I)} \in \overline{V}_+\}$$

$$= \bigcup_{h,h'} \bigcap_{I \in \mathscr{K} \cup \mathscr{S}_+} \{x_{h(I)} - x_{h'(X \setminus I)} \in \overline{V}_+\} \equiv \bigcup_{h,h'} C_{hh'} \tag{23}$$

where the union runs independently over the pair of all possible choices h and h'. [Since with I the class of subsets $\mathscr{K}$ contains also $X \setminus I$ according to (12b), we were forced to introduce two independent choices in order to be able to interchange the intersection and the union.] The cones $C_{hh'}$ being a finite intersection of the closed convex cones:

$$\tilde{K}_{ij} = \{x \in \mathbb{R}_{4(n-1)} : x_i - x_j \in \overline{V}_+\} \tag{24}$$

are themselves convex and closed. The proof that the cones $C_{hh'}$ are also proper is left for the Appendix.

The decomposition (23) of the essential support into the convex cones $C_{hh'}$ is far from unique: as a more detailed investigation shows, some cones of the family are contained in other members of the family. By denoting with C_r, $r = 1, \ldots, N$, the uniquely determined maximal elements with respect to the partial order of inclusion, we shall write formula (23) in the form

$$ess\,\mathrm{supp}\, t_c * \alpha = \bigcup_{r=1}^{N} C_r. \tag{25}$$

At the end of this section we shall determine an upper bound for the cones C_r, when the hypercell $(K, \mathscr{S}_+)$ is such that $\Omega_{\mathscr{K},\mathscr{S}_+}$ intersects the mass shell, while in section 4 they will be explicitly calculated for some special cases.

Given the decomposition (25), $t_c * \alpha$ can be represented as a sum

$$t_c * \alpha = \sum_{r=1}^{N} f_{r,\epsilon} + s_\epsilon, \quad \mathrm{supp} f_{r,\epsilon} \subset C_{r,\epsilon}, \quad r = 1, \ldots, N,$$

$$C_r \setminus \{0\} \subset C_{r,\epsilon}, s_\epsilon \in \mathscr{S}(\mathbb{R}_{4(n-1)}) \tag{26}$$

where the functions $f_{r,\epsilon}$ are infinitely differentiable, of at most polynomial increase at infinity and have their support in the cones $C_{r,\epsilon}$, while s_ϵ is in $\mathscr{S}$. Here $C_{r,\epsilon}$ is any open cone containing $C_r \setminus \{0\}$. $C_{r,\epsilon}$ should be thought of as a '$\epsilon/2$ neighbourhood' of the cone C_r in the sense just indicated. s_ϵ has its support outside, say, an '$\epsilon/2$ neighbourhood' of $\bigcup_1^N C_r$ and is equal to $t_c * \alpha$ outside $\bigcup_1^N C_{r,\epsilon}$, and hence of fast decrease at infinity. It is clear that (26) can be achieved by an appropriate partition of unity (some care is needed near the origin). The decomposition (26) is of course not unique: apart from the ϵ neighbourhood question, $C_r \cap C_s$ will be in general $\neq \varnothing$.

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[7]) There are precisely

$$\sum_{\nu=1}^{n-1} \nu \binom{n}{\nu} = n(2^n - 1)$$

different choices.

After a Fourier transformation, (26) becomes:

$$\tilde{t}_c(p) = \sum_{r=1}^{N} \tilde{f}_{r,\epsilon}(p) + \tilde{s}_\epsilon(p) \tag{27}$$

for all $p \in K \subset \Omega_{\mathscr{K},\mathscr{S}_+}$.

Here we have taken advantage of the fact that α can be chosen such that $\tilde{\alpha}(p) = 1$ in any compact set K contained in $\Omega_{\mathscr{K},\mathscr{S}_+}$. Equation (27) is the main result of this section. For, by the Laplace transform theorem, the $\tilde{f}_{r,\epsilon}(p)$ are boundary values of functions

$$\tilde{f}_{r,\epsilon}(k), \quad k = p + iq$$

analytic in the tubes

$$\mathscr{T}_{r,\epsilon} = \{p + iq \in \mathbb{R}_{4(n-1)} : q \in \tilde{C}_{r,\epsilon}\} \tag{28}$$

where

$$\tilde{C}_{r,\epsilon} = \{q \in \mathbb{R}_{4(n-1)} : qx > 0 \quad \text{for all } x \in C_{r,\epsilon}\} \tag{29}$$

are the dual cones of $C_{r,\epsilon}$ [8]), while $\tilde{s}_\epsilon \in \mathscr{S}(\mathbb{R}_{4(n-1)})$. The cones $\tilde{C}_{r,\epsilon}$ are contained in the cones $\tilde{C}_r$, the dual cones of C_r, but can be chosen arbitrarily close to them. Since $\tilde{\tilde{C}}_r =$ convex hull of $C_r = C_r$ if C_r is convex, we see why the decomposition into convex cones is so important. A decomposition into non-convex cones would mean a loss of information in momentum space.

Let us concentrate now on points p near the mass shell. Denote by

$$\mathscr{M}^c = \{k = (k_1, \ldots, k_n) : k_1 + \cdots + k_n = 0, k_i^2 = m^2, i = 1, \ldots, n\} \tag{30}$$

the complex mass shell manifold. It is an analytic manifold. One of the main results of this paper is that all the tubes $\mathscr{T}_{r,\epsilon}$ for 'ϵ small enough' have a non-empty intersection with $\mathscr{M}^c$ near all its real points:

$$\mathscr{M} = \{p = (p_1, \ldots, p_n) : p_1 + \cdots + p_n = 0, p_i^2 = m^2, i = 1, \ldots, n\} \tag{31}$$

provided no two incoming and no two outgoing momenta p_i are mutually parallel. For that purpose it is of course enough to investigate $\mathscr{T}_r \cap \mathscr{M}^c$, $\mathscr{T}_r$ having as basis the cone $\tilde{C}_r$, since $\mathscr{T}_{r,\epsilon}$ can be chosen arbitrarily close to $\mathscr{T}_r$ and both are open.

Let us first clarify that the above purely geometrical statement implies restrictibility to the mass shell. $\mathscr{M}^c$ intersects $\mathscr{T}_{r,\epsilon}$ near a real point $p \in \mathscr{M}$ means more precisely the following: given a (real) point $p \in \mathscr{M}$ we consider the $4(n-1) - n$ complex dimensional tangent plane $\mathscr{P}^c(p)$ to $\mathscr{M}^c$ at p given by:

$$\mathscr{P}^c(p) = \left\{\zeta = \xi + i\eta \in \mathbb{C}_{4(n-1)} : \sum_1^n \zeta_i = 0, p_i \zeta_i = 0, i = 1, \ldots, n\right\} \tag{32}$$

and require that $\mathscr{T}_{r,\epsilon} \cap \mathscr{P}^c(p) \neq \varnothing$. $\mathscr{T}_{r,\epsilon} \cap \mathscr{P}^c(p)$ is a $3n - 4$ dimensional tube having as basis the cone

$$\tilde{C}_{r,\epsilon} \cap \mathscr{P}(p) \neq \varnothing \quad \text{with} \quad \mathscr{P}(p) = \left\{q \in \mathbb{R}_{4(n-1)} : \sum_1^n q_i = 0, p_i q_i = 0, i = 1, \ldots, n\right\} \tag{33}$$

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8) In the above formula $qx = q_1 x_1 + \cdots + q_n x_n$ and $q_1 + \cdots + q_n = 0$. It is at this point that the properness of the cones C_r is crucial: $\tilde{C}_r$ is open and non-empty if and only if C_r is proper.

Clearly if (33) holds at given point p, the same will be true for all the points in a sufficiently small real neighbourhood $\omega(p)$ of p. Condition (33) implies that $\mathscr{T}_{r,\epsilon} \cap \mathscr{M}^c$ is non-empty, has $\omega(p) \cap \mathscr{M}$ as boundary points and has the local structure of a tube there. Under these conditions the restriction $f_{r,\epsilon}(k)/\mathscr{M}^c$ of the analytic function $f_{r,\epsilon}(k)$ to $\mathscr{M}^c$ will be analytic near the considered real points and we can define the restriction of the distribution $f_{r,\epsilon}(p)$ to the mass shell as

$$f_{r,\epsilon}(p)|_{\mathscr{M}\cap\omega} = \lim_{\substack{q\to 0\\ q\in C_{r,\epsilon}}} f_{r,\epsilon}(p+iq)|_{\mathscr{M}^c}. \tag{34}$$

Although the notation is somewhat sloppy, the precise meaning of (34) is provided by a slight generalization of the following two well-known theorems (see, e.g., [17]):

Theorem 1. *Let the tempered distribution* $t \in \mathscr{S}'(\mathbb{R}_N)$ *have its support contained in a cone* C. *Then its Laplace-Fourier transform* $\tilde{t}(k) = \int e^{ikx} t(x) d^N x$, $k = p + iq \in \mathbb{C}_N$, *is analytic in the tube* $\mathscr{T} = \mathbb{R}_N + i\tilde{C}$ *and is bounded there by*

$$|\tilde{t}(k)| \leqslant L \frac{(1+|b|)^N}{d(q, \partial\tilde{C})^M}$$

with L, M, N *some positive constants,* $|p|$ *the Euclidean norm in* $\mathbb{R}_N$ *and* $d(q,\partial\tilde{C})$ *the Euclidean distance of the point* q *to the boundary of* $\tilde{C}$.

What matters for us it that $\tilde{t}(k)$ does not increase faster than an inverse power of the distance to the boundary.

Theorem 2. *Let* $F(k)$, $k = p + iq \in \mathbb{C}_N$ *be analytic in the local tube* $\mathscr{L} = \Omega + iB$, *where* Ω *is an open set in* $\mathbb{R}_N$ *and* $B = C \cap B_\epsilon$, C *being an open cone in* $\mathbb{R}_N$ *and* B_ϵ *the open ball* $|q| < \epsilon$. *If in* $\mathscr{L}$, F *satisfies the bound of Theorem* 1 *with* $\tilde{C}$ *replaced by* B_ϵ *the limit*

$$\lim_{\substack{q\to 0\\ q\in\Gamma}} \int F(p+iq)\rho(p) d^N p = \langle F, \rho\rangle$$

exists for every test function $\varphi \in \mathscr{D}(\Omega)$ *and every closed cone* Γ *such that* $\Gamma\backslash\{0\} \subset C$ *and defines a continuous linear functional in* $\mathscr{D}'(\Omega)$.

Moreover, let e be any vector contained in the cone C *and choose the co-ordinate system in* $\mathbb{C}_N$ *so that* $k = p + iq = (p_0 + iq_0 = (k,e), \vec{p} + i\vec{q})$. *Then*

$$\lim_{q_0\to +0} \int F(p_0' + iq_0, \vec{p}) f(p_0 - p_0') dp_0'$$

exist for every $f(p_0) \in \mathscr{D}(|p_0| < a)$, a *sufficiently small, and every* $p \in \Omega_a = \{(p_0, \vec{p}) \in \Omega : (p_0 \pm a, \vec{p}) \in \Omega\}$. *The limit thus defined is infinitely differentiable for all* $p \in \Omega_a$ *and is a linear functional in* f *continuous in the topology of* $\mathscr{D}(|p_0| < a)$.

The second part of this theorem says that the limiting distribution $F(p)$ if regularized in only one direction contained in the cone C becomes infinitely differentiable in all variables.

These two theorems guarantee the restrictibility of $f_{r,\epsilon}$ to $\mathscr{M}$ in a neighbourhood of any $p \in \mathscr{M}$ provided $C_{r,\epsilon} \cap \mathscr{P}(p) \neq \varnothing$: due to Theorem 1 $f_{r,\epsilon}(k)/\mathscr{M}^c$ will locally fulfil the conditions of Theorem 2. Actually, a slight generalization of Theorem 2 is needed here since $\mathscr{M}^c$ is not a linear space, but the problem is easily settled by introducing appropriate local co-ordinates.

Thus, provided $C_r \cap \mathscr{P}(p) \neq \varnothing$ for all r, the restrictibility to the mass shell has been re-obtained, since the last term $\tilde{s}_\epsilon$ in the decomposition (30) is C_∞. This is at the same time a refinement of the Hepp result: depending on the geometry of the situation, the smearing out of the Green's function only along a few directions contained in $\mathscr{M}$ is needed in order to obtain a function infinitely differentiable in all the variables. Thus for example, if for a given $p \in \mathscr{M}$ it turns out that $\mathscr{P}(p) \bigcap_{r=1}^{N} C_r \neq \varnothing$ only one regularization along a direction contained in this intersection will suffice. That such configurations do indeed occur in the many-particle case will be illustrated in section 4. They correspond to situations where the scattering amplitude is the boundary value of a single analytic function.

In the above proof we could have avoided any reference to analyticity. Instead of Theorems 1 and 2, we could use a slight modification of a simple theorem by H. Borchers (see [2], p. 180). In view of what follows, we choose, however, the more complicated way over Theorems 1 and 2.

It is clear that if the C^∞ function $\tilde{s}_\epsilon$ in the decomposition (30) could be shown to be analytic in a complex neighbourhood of the real points considered, formula (D) of the Introduction would be proved. In a nutshell, this is the object of the next section.

We still owe the proof of the following:

Lemma 2. *The tangent plane $\mathscr{P}(p)$ (33) to the mass shell intersects any of the cones $\tilde{C}_r$ provided no pair of incoming and no pair of outgoing four-momenta are parallel.*

In order to prove this lemma, we make some estimates on the essential support of t_c when p belongs to one of the regions $\Omega_{\mathscr{X},\mathscr{S}_+}$ attached to the point $p = (p_1, \ldots, p_n)$ when $p \in \mathscr{M}$ For that purpose, let us introduce the following notations: let $X_1 = \{1,2, \ldots, \nu\}$, $X_2 = \{\nu+1, \nu+2, \ldots, n\}$ so that $X_1 \cup X_2 = X = \{1, \ldots, n\}$. Let $\vec{I} = (i_1, i_2, \ldots, i_p)$ denote the ordered sequence of p distinct elements $i_r \in X$, while the same letter I is reserved for the corresponding set $I = \{i_1, \ldots, i_p\} \subset X$ and $|I| = p$ is the number of elements in I. Let now $\vec{I}, \vec{J}, \vec{R}, \vec{S}$ be four ordered sequences such that $I, J \subset X_1$ and $R, S \subset X_2$, $I \cap J = \varnothing$, $R \cap S = \varnothing$, $|I| = |J| > 0$ (and) $|R| = |S| > 0$. Then we claim that the essential support of t_c is contained in the union of the following closed, convex and pointed cones:

$$\begin{aligned} C_{\vec{I},\vec{J},\vec{R},\vec{S};h,h'} = \{x = (x_1, \ldots, x_n) : x_{i_1} = x_{j_1}, \ldots, x_{i_p} = x_{j_p};\\ x_{r_1} = x_{s_1}, \ldots, x_{r_q} = x_{s_q};\ x_{i_{h(l)}} - x_l \in \overline{V}_+ \text{ for all } l \in X_2\\ \cup [X_1 \setminus (I \cup J)];\ x_k - x_{r_{h'(k)}} \in \overline{V}_+ \text{ for all } k \in X_1 \cup [X_2 \setminus (R \cup S)]\}. \end{aligned} \tag{35}$$

Here h and h' are two 'choices': $h(l)$ takes its value in the set $\{1, \ldots, p\}$, $h'(k)$ in the set $\{1, \ldots, q\}$; $\vec{I} = (i_1, \ldots, i_p)$, $\vec{J} = (j_1, \ldots, j_p)$, $\vec{R} = (r_1, \ldots, r_q)$, $\vec{S} = (s_1, \ldots, s_q)$, $p \geqslant 1$, $q \geqslant 1$. In other words, the cone (35) can be described as follows: there are p pairs of points in X_1 and q pairs of points in X_2 which coincide; each point in X_2 (X_1) and each 'singleton' in X_1 (X_2) is in the closed past (future) of at least one coinciding pair in X_1 (X_2). In order to obtain a covering of the essential support, it is necessary to take the union of the cones (35) when $\vec{I}, \vec{J}, \vec{R}, \vec{S}, h$ and h' run over all the possible allowed values.

The above assertion is easy to prove. Consider the following space time configuration: suppose there is a single four-vector in X_1, say x_i, $i \in X_1$, which is 'maximal' with respect to the rest of X_1, i.e., for which

$$\{x_i\} \gtrsim [X_1 \backslash \{i\}] \tag{36}$$

That means that in $x_i + \bar{V}_+$ there is at most a group of space time points $X_2' \subset X_2$. In this configuration t_c will 'factorize' as follows

$$t_c = \langle T(\{i\} \cup X_2') T(\text{rest}) \rangle_c. \tag{37}$$

But the Fourier transform of the right-hand side vanishes in $\Omega_{\mathscr{K},\mathscr{S}_+}$ since $p_{\{i\} \cup X'_2} \notin \bar{V}_+(4m)$; it is even $\notin V_+$ when $X_2' \neq \varnothing$. Therefore the configuration (36) is outside the essential support of t_c and all the 'maximal' elements of X_1 have to be at least 'double points'. By reversing the future and the past, we conclude that in the essential support of t_c all the 'minimal' elements of X_2 have to coincide at least in pairs. Similarly, we see that any x_l, $l \in X_2$, must contain at least a point $\in X_1$ in its closed future cone since otherwise we would have the factorization

$$t_c = \langle T(X_2') T(X \backslash X_2') \rangle_c \quad \text{with} \quad l \in X_2' \subset X_2. \tag{38}$$

Again the right-hand side of (38) has vanishing Fourier transform in $\Omega_{\mathscr{K},\mathscr{S}_+}$ because of $p_{X'_2} \in V_-$. The same reasoning shows that any x_k, $k \in X_1$, must contain at least an x_i, $i \in X_2$, in its closed past shadow. By combining these four conditions, we conclude that any point of $\operatorname{ess\,supp} t_c$ is contained in at least one of the cones (35). Remark that (35) includes also the cases where more than just pairs of points coincide: $\bar{V}_+$ contains also the origin![9]) Remark also that the union of the cones (35) represents in general only an upper bound of the essential support. In the proof of (35) we have used only the fact that $\{i\} \in \mathscr{K}$ for $i = 1, \ldots, n$, $I \in \mathscr{S}_+$ for $I \subset X_1$ and $|I| > 1$, $I \in \mathscr{S}_-$ for $I \subset X_2$ and $|I| > 1$, $\{i\} \cup I \notin \mathscr{S}_+$ for $\{i\} \subset X_1$ and $I \subset X_2$, and finally $\{i\} \cup I \notin \mathscr{S}_-$ for $\{i\} \subset X_2$ and $I \in \mathscr{S}_+$. This fact can be formulated also in the following form: Equation (35) is a decomposition into convex cones of the essential support of t_c attached to the open set:

$$\begin{aligned} \Omega_{\mathscr{M}} = \{p \in \mathbb{R}_{4(n-1)}: p_i^2 < 4m^2 \quad & \text{for } i = 1, \ldots, n;\ p_I \in \complement \bar{V}(m, 2m) \\ \text{for all } I \subset X_1,\ |I| > 1;\ p_I \in \complement \bar{V}_+(m, 2m) \quad & \text{for all } I \subset X_2, |I| > 1; \\ p_{\{i\} \cup I} \in \complement \bar{V}_+(m, 2m) \quad & \text{for all } i \in X_1, I \subset X_2, |I| \geqslant 1; \\ p_{\{i\} \cup I} \in \complement \bar{V}_-(m, 2m) \quad & \text{for all } i \in X_2, I \subset X_1, |I| \geqslant 1\}. \end{aligned} \tag{39}$$

We have $\mathscr{M} \subset \Omega_{\mathscr{M}}$ and $\Omega_{\mathscr{K},\mathscr{S}_+} \subset \Omega_{\mathscr{M}}$ for all $\mathscr{K}, \mathscr{S}_+$ such that $\mathscr{M} \cap \Omega_{\mathscr{K},\mathscr{S}_+} \neq \varnothing$. The choice of the $\Omega_{\mathscr{K},\mathscr{S}_+}$ will in general depend on the position of $p \in \mathscr{M}$ and this more precise information will result in general in a further splitting of the cones (35) into smaller subcones. Only in the case $|X_2| = 2$ or $|X_1| = 2$, as shown in the last section, does (35) represent the best possible result.

We are now in a position to prove Lemma 2. Since, as we have just discussed, each C_r is contained in some $C_{\mathscr{A}}$, where $\mathscr{A} = (\vec{I}, \vec{J}; \vec{R}, \vec{S}; h, h')$, it is enough to show that

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[9]) Strictly speaking, the above argument is valid only in the case $a = 0$. When $a > 0$, replace in the configurations considered $(x_1, \ldots, x_n)$ by $\rho(x_1, \ldots, x_n)$, $\rho > 0$. When ρ is sufficiently big, the factorizations (37) and (38) will be valid, which is all that is needed in the computation of the essential support.

$\mathscr{P}(p) \cap \tilde{C}_{\mathscr{A}} \neq \varnothing$ for all $\mathscr{A}$. [Notice that $C_r \subset C_{\mathscr{A}}$ implies $\tilde{C}_{\mathscr{A}} \subset \tilde{C}_r$.] We shall use the following geometrical

Lemma 3. *Let C a proper convex closed cone in $\mathbb{R}_N$ and $\tilde{C}$ its dual in $\tilde{\mathbb{R}}_N$ with respect to the non-degenerate bilinear form px, $p \in \tilde{\mathbb{R}}_N$, $x \in \mathbb{R}_N$:*

$$\tilde{C} = \{p \in \tilde{\mathbb{R}}_N : px > 0 \quad \textit{for all } x \in C \backslash \{0\}\}$$

Then the open convex cone $\tilde{C}$ intersects the linear manifold $\mathscr{P} \subset \tilde{\mathbb{R}}_N$ if and only if

$$\tilde{\mathscr{P}} \cap C = \{0\}$$

where $\tilde{\mathscr{P}}$ is the dual of $\mathscr{P}$ defined by

$$\tilde{\mathscr{P}} = \{x \in \mathbb{R}_N : px = 0 \quad \textit{for all } p \in \mathscr{P}\}.$$

This lemma is a consequence of the Hahn-Banach theorem: if $\tilde{\mathscr{P}} \cap C = \{0\}$ then there exists a linear form $p_0 x$ such that $\tilde{\mathscr{P}} \subset \{x : p_0 x = 0\}$ and $p_0 x > 0$ for all x in $C \backslash \{0\}$, which means $\mathscr{P} \cap \tilde{C} \neq \varnothing$ since $p_0 \neq 0$ belongs, by the definition of duals, both to $\tilde{\tilde{\mathscr{P}}} = \mathscr{P}$ and to C. On the other hand, if $\tilde{\mathscr{P}} \cap C \neq \{0\}$ there exists a $x_0 \neq 0$ belonging to both $\tilde{\mathscr{P}}$ and C and the set $\mathscr{A} = \{p \in \tilde{\mathbb{R}}_N : px_0 = 0\}$ contains evidently $\tilde{\tilde{\mathscr{P}}} = \mathscr{P}$, but is not contained in $\tilde{C}$, which means $\mathscr{P} \cap \tilde{C} = \varnothing$, q.e.d.

All we have to do now is to compute $\tilde{\mathscr{P}}(p)$ and to show that $\tilde{\mathscr{P}}(p) \cap C_{\mathscr{A}} = \{0\}$. An elementary calculation gives

$$\tilde{\mathscr{P}}(p) = \{x = (x, \ldots, x_n) : x_i - x_j = \lambda_i p_i - \lambda_j p_j \quad \text{for all } i < j,$$

$$i, j = 1, \ldots, n \quad \text{and all } (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_n\} \tag{40}$$

For any $x \in \tilde{\mathscr{P}} \cap C_{\mathscr{A}}$ we must have:

$$x_i - x_j = \lambda_i p_i - \lambda_j p_j = 0$$

for any coinciding pair $\{i, j\} \subset X_1$ or X_2, which implies $\lambda_i = \lambda_j = 0$ since no two four-vectors in X_1 or X_2 are supposed to be parallel. If l does not belong to one of the pairs in (35) then it is 'sandwiched' between two pairs:

$$x_i = x_j,\ x_i - x_l = -\lambda_l p_l \in \overline{V}_+,\ x_l - x_r = \lambda_l p_l \in \overline{V}_+,\ x_r = x_s$$

But this is possible only if $\lambda_l = 0$ also. Hence $\tilde{\mathscr{P}}(p) \cap C_{\mathscr{A}} = \{0\}$ and Lemma 2 is proved.

3. Analyticity Properties of $\tilde{t}_c$ Near the Mass Shell

In this section we want to prove the announced generalization of the decomposition (27).

We shall proceed as follows: due to the support property (20) of the amplitude r_I we can represent this amplitude in the form

$$r_I = \sum_{\substack{i \in I \\ j \in X \backslash I}} f_{ij}^I \text{ with } f_{ij}^I \in \mathscr{S}'(\mathbb{R}_{4(n-1)}),\ \operatorname{supp} f_{ij}^I \subset \tilde{K}_{ij}^a =$$

$$= \{x \in \mathbb{R}_{4(n-1)} : x_i - x_j + 2ea \in \overline{V}_+\}. \tag{41}$$

If r_I were a bounded, or more generally, a measurable function, this decomposition could be trivially obtained by a partition of unity into appropriate step functions. Since we are dealing with distributions, we have to make appeal to a well-known theorem by Lojasiewicz (cf., [18]) which says that any (tempered) distribution T having its support in the closed set $A = A_1 \cup A_2$ can be written in the form $T = T_1 + T_2$, where the (tempered) distributions $T_{1,2}$ have their support in the sets $A_{1,2}$ provided these sets meet some very mild conditions, which are certainly satisfied in our case.

Now, although the (displaced) cones $\tilde{K}_{ij}$ are not proper cones—they are of the form $\mathbb{R}_4 \times \cdots \times \mathbb{R}_4 \times (\overline{V}_+ - 2ea)$ if we choose $\xi_r = x_r - x_j$, $r = 1, \ldots, n$, $r \neq j$, as independant co-ordinates in $\mathrm{R}_{4(n-1)}$—the Fourier transforms of f_{ij}^I are never the less boundary values of functions analytic in lower dimensional tubes: the integral

$$\tilde{f}_{ij}^I(p) = \int f_{ij}^I(\xi) e^{ip_1\xi_1 + \ldots + ip_i\xi_i + \ldots + ip_n\xi_n} d^{4(n-1)}\xi \tag{42}$$

can obviously be extended to complex values k_i of the variable p_i provided $\operatorname{Im} k_i \in V_+$. Thus (42) is the boundary value of a function $\tilde{f}_{ij}^I(p_1, \ldots, k_i, \ldots, p_n)$ analytic in $\operatorname{Im} k_i \in V_+$ and distribution valued with respect to the rest of the variables. A more precise formulation of this statement is rather obvious and it is also clear that with appropriate changes Theorems 1 and 2, quoted in the previous section, will apply to this slightly more general situation (cf., [3]). With an abuse of language, we shall simply say that the $\tilde{f}_{ij}^I$ are boundary values of functions analytic in the 'flat' tubes

$$\mathscr{T}_{ij} = \{k = (k_1, \ldots, k_n) : k_1 + \cdots + k_n = 0, \operatorname{Im} k \in K_{ij}\}, \quad i \neq j, \tag{43}$$

with

$$K_{ij} = \{q : q_1 + \cdots + q_n = 0;\ q_r = 0, \quad r \neq i, j; \quad q_i = -q_j \in V_+\}$$

Thus relation (16) becomes

$$\tilde{t}_c(p) = \sum_{\substack{i \in I \\ j \in X \setminus I}} \tilde{f}_{ij}^I(p) \quad \text{for} \quad p \in \Omega_{\mathscr{K}, \mathscr{S}_+} \quad \text{and all } I \in \mathscr{K} \cup \mathscr{S}_+ \tag{16a}$$

which trivially implies the set of equations

$$\sum_{\substack{i \in I \\ j \in X \setminus I}} \tilde{f}_{ij}^I(p) - \sum_{\substack{i \in J \\ j \in X \setminus J}} \tilde{f}_{ij}^J(p) = 0 \quad \text{for} \quad p \in \Omega_{\mathscr{K}, \mathscr{S}_+} \quad \text{and all } I, J \in \mathscr{K} \cup \mathscr{S}_+ \tag{44}$$

to which we apply the fundamental theorem:

Theorem 3. *The generalized edge-of-the-wedge theorem (local version)*: let $f_i(k)$, $i = 1, \ldots, l$ *be* l *functions analytic in the 'localized tubes'*

$$\mathscr{T}_{B_i, S} = \{k = p + iq \in \mathbb{C}_N : p \in S, q \in B_i\}, i = 1, \ldots, l$$

where S *is the unit ball* $\{p \in \mathbb{R}_N : |p| < 1\}$, $|p|$ *the Euclidean norm in* $\mathbb{R}_N$, *and* B_i *the star-shaped set ('basis')*:

$$B_i = \{q \in \mathbb{R}_N : 0 < |q| < r_i(\omega) \leqslant 1, \omega = q/|q|, \text{ if } r_i(\omega) \neq 0, |q| = 0 \text{ if } r_i(\omega) = 0\}$$

(see the comments below). Let the boundary values

$$\lim_{\substack{q \to 0 \\ q \in B_i}} f_i(p + iq) = f_i(p), \quad p \in S, \quad i = 1, \ldots, l$$

exist in the sense of distributions $\mathcal{D}'(S)$ *(cf., Theorem 2) and satisfy the identity:*

$$\sum_{i=1}^{l} f_i(p) = 0, \quad p \in S \tag{45}$$

Then there exists a real constant λ, $0 < \lambda < 1$, *depending only on* N, *and* $l(l-1)/2$ *functions* $f_{ij}(k)$ *satisfying*

$$f_{ij} = -f_{ji}, \; i, j = 1, \ldots, n \tag{46}$$

analytic in the tubes $\mathcal{T}_{\lambda B_{ij}, \lambda S}$, *where*

$$B_{ij} = \operatorname{conv}(B_i \cup B_j)$$

is the convex envelope of the set $B_i \cup B_j$, *and such that*

$$f_i(k) = \sum_{j=1}^{l} f_{ij}(k), \quad i = 1, \ldots, l, \quad k \in \mathcal{F}_{\lambda B_i, \lambda S} \equiv \lambda \mathcal{F}_{B_i, S} \tag{47}$$

Moreover, the boundary values

$$\lim_{\substack{q \to 0 \\ q \in \lambda B_{ij}}} f_{ij}(p + iq) = f_{ij}(p), \quad p \in \lambda S$$

exist in the sense of distribution $\mathcal{D}'(\lambda S)$, *so that in this sense* (47) *is valid also for* $k = p \in \lambda S$.

In case the unit ball S is replaced by the whole of $\mathbb{R}_N$ the localized tubes $\mathcal{T}_{B,S}$ (localized in S) become ordinary tubes with basis B, and Theorem 3 (with $\lambda = 1$) can be proved rather trivially by studying the essential support of the Fourier transforms of the functions $f_i(p + iq)$ with respect to the variable p (cf., [11]). Notice that λS is the homothetic sphere $|p| < \lambda$ and similarly for the other sets. The functions $r_i(\omega)$ defined on the unit sphere $|q| = 1$ have to be such that B_i is either an open set in $\mathbb{R}_N$ (r_i is then semicontinuous from below on all of $|q| = 1$) or an open set in a lower dimensional linear subspace $\mathbb{R}_\nu$ or $\mathbb{R}_N$ (the case of a localized 'flat' tube; r_i has to be semicontinuous from below in $\mathbb{R}_\nu \cap \{|q| = 1\}$ and 0 otherwise). If $\inf r_i(\omega) > 0$ the point $q = 0$ has to be added to the basis B_i. The origin $q = 0$ is always a boundary point of B_i. In our application the bases B_i will be to start with the truncated cones:

$$K_{ij}^T = K_{ij} \cap \{|q| < 1\}, \tag{48}$$

where K_{ij} are the 'flat' cones (43) and $|q|$ is the Euclidean norm in some arbitrarily chosen co-ordinate system in $\mathbb{R}_{4(n-1)}$, e.g.,

$$|q|^2 = \sum_{1}^{n-1} |q_r|^2$$

with

$$|q_r|^2 = \sum_{\mu=0}^{3} (q_r^\mu)^2.$$

Notice that by a real translation and a change of scale, the ball S can be replaced by the ball $p_0 + bS$ with centre at p_0 and radius b provided the bases B_i are replaced also by bB_i. The fact that the functions f_i might be analytic beyond the ball bS in purely imaginary directions, as is the case with (44), gets typically lost in the above theorem.

Also the contraction of the domain due to the scale factor λ is unavoidable in the above theorem[10]). A global version of this theorem will be discussed later.

The case $l = 2$ of the above theorem is the ordinary edge-of-the-wedge theorem ([3] or [19]), the subcase of two opposed cones $B_i = C_i$, $C_1 = -C_2$ goes back to N. N. Bogoliubov and co-workers (for a survey and references, cf., [3] or [2]). As to the general case, see the text and footnotes in the Introduction. The global version of this theorem given in [11] and [13] will be explained later (Theorem 3′).

In order to apply Theorem 3 to the identity (44), take any fixed $p_0 \in \Omega_{\mathscr{K},\mathscr{S}_+}$, take a $b > 0$ such that the sphere $bS + p_0 \subset \Omega_{\mathscr{K},\mathscr{S}_+}$, fix any pair of indices $I \neq J$ and rewrite the relation (44) in the form:

$$\sum_{r \in A_I} \tilde{f}_r^I(p) - \sum_{s \in A_J} \tilde{f}_s^J(p) \equiv \sum_1^l f_i(p) = 0 \tag{49}$$

for all $p \in p_0 + bS$.

Here r and s are a shorthand notation for the pair of indices ij, A_I and A_J are the sets over which they run. We then evidently have: $\tilde{f}_t^I(k)$, respectively $\tilde{f}_t^J(k)$ analytic in $p_0 + b\mathscr{T}_{K_t^T,S}$, so that (47) becomes:

$$\tilde{f}_r^I = \sum_{r' \in A_I} \tilde{f}_{rr'}^{IJ} + \sum_{s' \in A_J} \tilde{f}_{rs'}^{IJ}, \quad -\tilde{f}_s^J = -\sum_{r' \in A_I} \tilde{f}_{r's}^{IJ} + \sum_{s' \in A_J} \tilde{f}_{ss'}^{IJ} \tag{50}$$

where the functions $\tilde{f}_{\alpha\beta}^{IJ}$ are analytic in $\lambda b\mathscr{T}_{K_\alpha^T \cdot K_\beta^T,S} + p_0$ and antisymmetric in the pair of indices rr' and ss'. The antisymmetry in the other two combinations of indices is taken care of by the $-$ sign in the second part of (50). Here $A \cdot B$ is a shorthand notation for $\operatorname{conv}(A \cup B)$. This antisymmetry entails

$$\sum_{r \in A_I} \tilde{f}_r^I(p) = \sum_{s \in A_J} \tilde{f}_s^J(p) = \sum_{\substack{r \in A_I \\ s \in A_J}} \tilde{f}_{rs}^{IJ}(p) = \tilde{t}_c(p) \text{ in } p_0 + \lambda bS.$$

By fixing a third index $K \neq I, J$ we reapply Theorem 3 to the identity

$$\sum \tilde{f}_{rs}^{IJ}(p) - \sum_{t \in A_K} \tilde{f}_t^k(p) = 0 \text{ in } p_0 + \lambda bS$$

and after $\nu - 1$ steps, ν being the number of elements in $\mathscr{K} \cup \mathscr{S}_+$, we arrive at the representation:

$$\tilde{t}_c(p) = \sum \tilde{f}_{r_1 \cdots r_\nu}(p) \quad \text{in } p_0 + \lambda^{\nu-1} bS \tag{51}$$

where the sum runs over all $r_i \in A_{I_i}$, $i = 1, \ldots, \nu$ and where the $\tilde{f}_{r_1,\ldots,r_\nu}(p)$ are boundary values of functions $\tilde{f}_{r_1,\ldots,r_\nu}(k)$ analytic in

$$p_0 + \lambda^{\nu-1} b\mathscr{T}_{K_{r_1}^T \cdots K_{r_\nu}^T, S}.$$

Here we have used the fact that $(\lambda A) \cdot (\lambda B) = \lambda(A \cdot B)$. Note also that the formation of convex envelopes is an associative and commutative operation. Using the notation of equation (23), (51) can be written as follows:

$$\tilde{t}_c(p) = \sum_{h,h'} f_{hh'}(p), \quad p \text{ in } p_0 + \lambda^{\nu-1} bS, \tag{52}$$

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10) The optimal value of λ is unknown to the authors.

$\tilde{f}_{hh'}(k)$ analytic in

$$p_0 + \lambda^{\nu-1} b\mathcal{T}_{B_{hh'},S}$$

where

$$B_{hh'} = \operatorname{conv}\left\{\bigcup_{I\in\mathcal{K}\cup\mathcal{S}_+} K^T_{h(I),h'(X\setminus I)}\right\}. \tag{53}$$

Now, if in the last formula, we replaced the truncated cone K^T_{ij} (48), by the untruncated cone (43), we would evidently get $B_{hh'} = \tilde{C}_{hh'}$, where $\tilde{C}_{hh'}$ is the dual of the cone $C_{hh'}$ (23) of last section (the intersection goes over into convex union by duality!). We claim therefore that (53) is of the form:

$$B_{hh'} = \tilde{C}_{hh'} \cap A \tag{54}$$

where A is a certain open convex neighbourhood of the origin in $\mathbb{R}_{4(n-1)}$. Property (54) can be inferred from the following explicit representation of $B_{hh'}$:

$$B_{hh'} = \left\{q = (q_1, \ldots, q_n) : q_i = \sum_{I\in\mathcal{K}\cup\mathcal{S}_+} \epsilon_i^{h(I),h'(X\setminus I)} \eta_{h(I),h'(X\setminus I)}\, \rho_{h(I),h'(X\setminus I)},\ i = 1, \ldots, n\right\}$$

where $\epsilon_i^{rs} = +1$ for $i = r$, $= -1$ for $i = s$, $= 0$ for $i \neq r,s$, where the η_{rs} are four vectors varying independently over $V_+^T = V_+ \cap \{|q| < 1\}$ and ρ_{rs} are the parameters of convex completion: $\rho_{rs} \geqslant 0$ and $\sum_I \rho_{hh'} = 1$[11]).

As it was pointed out in the discussion after formula (23), we will in general have $\tilde{C}_{hh'} \subset \tilde{C}_{h_1 h'_1}$ for certain pairs of the family of cones in the decomposition (52). Denoting by $\tilde{C}_r$, $r = 1, \ldots, N$, the set of minimal elements with respect to the partial order of inclusion in this family, formula (52) can be conveniently rewritten (in general in a non-unique way) in a form analogous to the decomposition (27). Thus, in view of Lemma 2, we have proved:

Theorem 4 (local version): *given any $p_0 \in \Omega_{\mathcal{K}.\mathcal{S}_+}$ there exists a complex neighbourhood $\mathcal{N}(p_0)$ of the real point p_0 such that*

$$\tilde{t}_c(p) = \sum_{r=1}^{N} \tilde{f}_r(p) \quad \text{for } p \in \mathcal{N}(p_0) \cap \mathbb{R}_{4(n-1)} \subset \Omega_{\mathcal{K}.\mathcal{S}_+} \tag{55}$$

where the $\tilde{f}_r(p)$ are boundary values in the sense of distributions in $\mathcal{D}'(\mathbb{R}_{4(n-1)} \cap \mathcal{N}(p_0))$, of functions $\tilde{f}_r(k)$ analytic in the localized tubes $\mathcal{T}^l_r = \mathcal{T}_r \cap \mathcal{N}(p_0)$, $\mathcal{T}_r = \{k = p + iq : q \in \tilde{C}_r\}$, $r = 1, \ldots, N$. Furthermore, if $p_0 \in \mathcal{M} \cap \Omega_{\mathcal{K}.\mathcal{S}_+}$ is such that no pair of incoming and no pair of outgoing momenta are mutually parallel, $\mathcal{T}^l_r \cap \mathcal{M}_c \neq \varnothing$ for $r = 1, \ldots, N$ and the restriction of $\tilde{t}_c$ to the mass shell in $\mathbb{R}_{4(n-1)} \cap \mathcal{N}(p_0)$ is well defined in the sense of Theorem 2.

Let us now describe briefly the global version of Theorems 3 and 4. For details, the reader is referred to the paper [13]. In order to study the analyticity properties of a distribution $\tilde{f}(p) \in \mathcal{S}'(\mathbb{R}_N)$ in a complex neighbourhood of an open set $\Omega \subset \mathbb{R}_N$ it is very

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[11]) Formula (54) needs a proof since $\operatorname{conv}\{(A_1 \cap C_1) \cup (A_2 \cap C_2)\}$ is in general not of the form $A_3 \cap \operatorname{conv}\{C_1 \cap C_2\}$ if $C_{1,2}$ are any two cones and A_i, $i = 1, 2, 3$ convex neighbourhoods of the origin.

convenient, as first suggested in the paper [9], to investigate its generalized Fourier transform

$$f(x,x_0) = \int e^{-ipx - x_0\phi(p)} \tilde{f}(p) d^N p \tag{56}$$

where ϕ is an auxiliary function having the following properties:

a) $\phi(k)$, $k = p + iq \in \mathbb{C}_N$, is analytic in a complex neighbourhood $\mathcal{N}$ of $\overline{\Omega} \in \mathbb{R}_N$, the closure of Ω in $\mathbb{R}_N$, and satisfies there $\phi(\bar{k}) = \overline{\phi(k)}$.

b) The set of real points $0 \leqslant \phi(p) < 1$ is equal to the open set Ω, which is supposed to be bounded.

c) The origin $p = 0$ belongs to Ω and is a critical point for ϕ $(\nabla\phi(0) = 0)$; moreover, it is assumed that ϕ has no other critical points inside Ω so that the set of level surfaces $\phi(p) = c$ $(0 \leqslant c \leqslant 1)$ is topologically equivalent to the set of nested spheres with equations $p^2 = \sum_1^N p_i^2 = c$; in particular $\phi(p) = 0$ implies $p = 0$.

Thus the domains Ω considered are limited to open bounded sets containing the origin given by the equation $0 \leqslant \phi(p) < 1$ for some ϕ satisfying the above conditions. But any bounded domain homeomorphic to a sphere can be approximated arbitrarily closely by such an Ω. The simplest example (sufficient for the proof of Theorems 3 and 4), is the choice $\phi(p) = p^2$, corresponding to the unit sphere $\Omega = \{|p| < 1\}$.

In order to give the integral (56) a meaning, it is supposed that $\tilde{f}$ has its support contained in $\mathcal{N} \cap \mathbb{R}_N$ [outside of this set $\phi(p)$ is undefined]. For $x_0 = 0$, (56) reduces to the usual Fourier integral of the function $\tilde{f}$. Moreover, $f(x,x_0)$ satisfies the equation:

$$\left\{\frac{\partial}{\partial x_0} + \phi\left(i\frac{\partial}{\partial x}\right)\right\} f(x,x_0) = 0 \tag{57}$$

which for $\phi = p^2$ reduces to the heat equation. Thus $f(x,x_0)$ will be uniquely determined by its 'initial value' $f(x,0)$. Now, if in Ω $\tilde{f}$ is the boundary value of a function analytic in a tube (or, more generally, in a flat tube), the part of the integral (56) extending over Ω can be deformed into the complex and will result in exponential decrease properties of $f(x,x_0)$ in directions within $\mathbb{R}_N \times \mathbb{R}_+$ outside a certain *essential support* determined by the behaviour of $\mathrm{Re}(-ikx - x_0\phi(k)) = qx - x_0 \mathrm{Re}\,\phi(p + iq)$ over the deformed integration contour. The domains of analyticity, which play the same role with respect to the generalized Fourier transform (56) as the ordinary tubes do with respect to the ordinary Fourier transform, are the *local tubes* $T_{B\phi}$ with basis B described as follows[12]).

Let B be a bounded star-shaped set in an auxiliary N dimensional space $\mathbb{R}_N$ given by the inequalities:

$$B = \{\xi \in \mathbb{R}_N : 0 < |\xi| < r(\omega) < r_\phi(\omega) \quad \text{if } r(\omega) > 0,\ |\xi| = 0$$
$$\text{if } r(\omega) = 0,\ \omega = \xi/|\xi|\}. \tag{58}$$

B is required to have exactly the same properties as the sets B_i introduced in Theorem 4, except that the upper bound 1 is replaced by the strictly positive bounded function $r_\phi(\omega)$ defined uniquely in terms of the 'localizing function' ϕ as described in detail in [13]. Given B, consider the set $\mathscr{E}_{B\phi}$ of points $k = p + iq \in \mathbb{C}_N$ in the domain of analyticity of $\phi(k)$ such that:

$$|q| + r(\omega)[\mathrm{Re}\,\phi(p + iq) - 1] < 0$$

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[12]) These domains are different in character from the *localized* tubes used in Theorems 3 and 4.

We notice that the open set Ω always belongs to $\mathscr{E}_{B\phi}$ [$|q| = 0$ implies $\phi(p) < 1$ since we suppose $r(\omega) \not\equiv 0$]. If the connected component of $\mathscr{E}_{B\phi}$ which contains Ω is bounded and has a compact closure inside the domain of analyticity of ϕ, we define the local tube $T_{B\phi}$ as just that component of $\mathscr{E}$, more precisely

$$\mathscr{T}_{B\phi} = \text{conn. comp. of}\{p + iq \in \mathbb{C}_N : 0 < |q| < r(\omega)[1 - \operatorname{Re}\phi(p + iq)]$$
$$\text{if } r(\omega) > 0,\ |q| = 0 \text{ if } r(\omega) = 0,\ \omega = q/|q|\}. \tag{59}$$

The only real points which belong to the closure of $T_{B\phi}$ are those of $\overline{\Omega}$ and this is why one calls ϕ a 'localizing function' in the open set Ω. If B is an open set containing the origin—i.e., $\inf r(\omega) > 0$—we drop the condition $|q| > 0$ in the definition (59), so that then $\Omega \subset T_{B\phi}$.

In order to better visualize the domain $T_{B\phi}$ let us write $q = |q| \cdot \omega$ and resolve for ω and $p \in \Omega$ fixed the inequality appearing in (59) with respect to $|q|$. The local tube $T_{B\phi}$ is expected to be of the form

$$\mathscr{T}_{B\phi} = \{p + iq \in \mathbb{C}_N : p \in \Omega,\ 0 < |q| < R(\omega,p) \text{ if } R(\omega,p) > 0,$$
$$|q| = 0 \text{ if } R(\omega,p) = 0,\ \omega = q/|q|\}. \tag{59a}$$

Indeed, when $r(\omega)$ is small enough, it is easily seen that $R(\omega,p) = F(\omega,p,r(\omega))$ where $F(\omega,p,r)$ is a continuous function of $p \in \Omega$, $\omega \in \{|\omega| = 1\}$ and r, increasing in r such that $r > 0$ implies $F(\omega,p,r) > 0$ for all $p \in \Omega$ and $F(\omega,p,0) = 0$. The upper bound $r_\phi(\omega)$ such that this condition be satisfied for all $r < r_\phi(\omega)$ and a fixed ω is precisely the function $r_\phi(\omega)$ appearing in the definition (58) of B. [For a more precise definition, cf., Ref. [13], equation (6).] $F(p,\omega,r)$ will tend to 0 when p approaches the boundary of Ω. This is all illustrated by the case $\phi = p^2$, where, as it is easily computed

$$r_\phi(\omega) = \tfrac{1}{2},\ F(\omega,p,r) = \frac{(1 - p^2)r}{1 + \sqrt{1 - 4r^2(1 - p^2)}},\ |p| < 1, r < \tfrac{1}{2}. \tag{60}$$

Thus $\mathscr{T}_{B\phi}$ can be best visualized as a tube localized in Ω whose (bounded) basis depends on the position of the real point p in Ω.

Now, if $\tilde{f}(p)$ is the boundary value of a function $\tilde{f}(k)$ analytic in a local tube $\mathscr{T}_{B\phi}$, the generalized Fourier transform $f(x,x_0)$ can be shown to have its essential support contained in the convex cone S_B with apex at the origin in (x,x_0) space, whose section $x_0 = 1$ is the (closed convex) polar set $\tilde{B}$ of the set B defined by:
$\tilde{B} = \{x \in \mathbb{R}_N : x\xi + 1 \geqslant 0 \text{ for all } \xi \in B\}$.
This is the analogue of the notion of essential support of tempered distributions discussed in section 2. [The essential support S_B can be also defined directly as follows: $S_B = \{(x,x_0) \in \mathbb{R}_N \times \mathbb{R}_+ : x\xi + x_0 \geqslant 0 \text{ for all } \xi \in B\}$.] $f(x,x_0)$, which is an entire function in x for each fixed x_0 due to the compact region of integration in (56), satisfies in addition some more precise growth properties at infinity for x_0 fixed, which are due to the behaviour of $\tilde{f}(k)$ near the boundary of $\mathscr{T}_{B\phi}$, i.e., to the distribution character of $\tilde{f}(p)$, for which the reader is referred to [13] and [14]. The converse is also true: by using a generalization of the Parseval formula, it is shown that every solution of the equation (57) having its essential support in a convex cone S_B with section $\tilde{B}$ and the mentioned growth properties, is the generalized Fourier transform of a distribution $\tilde{f}(p)$, which is

the boundary value of a function analytic in $\mathscr{T}_{B\phi}$ with $B = \tilde{\tilde{B}}$[13]). $\tilde{f}(p)$ is defined only modulo a distribution vanishing in Ω.

Since $\tilde{\tilde{B}} = \operatorname{conv} B \equiv \hat{B}$, one finds as a first consequence of the above theory that every function analytic in a local tube $\mathscr{T}_{B\phi}$ can be analytically continued to the local tube $\mathscr{T}_{\hat{B}\phi}$, which is, as can be inferred from the inverse generalized Fourier formula, a natural domain of holomorphy. This is a generalization of the usual tube theorem. Almost as immediate a consequence is the

Theorem 3′. *The generalized edge-of-the-wedge theorem (global version): let the distributions $f_i(p)$, $i = 1, \ldots, l$ having their support in $\mathscr{N} \cap \mathbb{R}_N$ satisfy the identity*

$$\sum_{1}^{l} f_i(p) = 0$$

Let in Ω the $f_i(p)$ be the boundary values of functions $f_i(k)$ analytic in local tubes $\mathscr{T}_{B_i\phi}$. Then there exist $l(l-1)/2$ functions $f_{ij}(k) = -f_{ji}(k)$, $i,j = 1, \ldots, l$, analytic in the local tubes $\mathscr{T}_{B_{ij}\phi}$ with $B_{ij} = \operatorname{conv}(B_i \cup B_j)$ such that

$$f_i(k) = \sum_{j=1}^{l} f_{ij}(k), \quad k \in \mathscr{T}_{B_i\phi}, \quad i = 1, \ldots, l \tag{61}$$

The f_{ij} have boundary values $f_{ij}(p)$ in the sense of distributions extendable to all $\mathbb{R}_N$, so that (61) is valid also for $k = p \in \mathbb{R}_N$.

Theorem 3 is a simple corollary of Theorem 3′. To show it, take $\phi = p^2$, observe that the function F (60) attached to this ϕ satisfies the following inequalities.

$$(1-p^2)\frac{r}{2} \leqslant F(\omega, p, r) \leqslant (1-p^2)r \quad \text{for all } |\omega| = 1, p \in S, 0 \leqslant r \leqslant \tfrac{1}{2}.$$

According to the last part of this inequality we certainly have $\mathscr{T}_{\frac{1}{2}B_i\phi} \subset \mathscr{T}_{B_i,S}$ where B_i and $\mathscr{T}_{B_i,S}$ are the sets defined in Theorem 3, while the first part of the inequality implies $\lambda\mathscr{T}_{B_i,S} \subset \mathscr{T}_{\frac{1}{2}B_i\phi}$ if $0 < \lambda \leqslant \sqrt{2} - 1$. Thus Theorem 3 with $\lambda = \sqrt{2} - 1$ follows.

It is also clear now how a generalization of Theorem 4 is to be achieved. Take any point $p_0 \in \Omega_{\mathscr{K},\mathscr{S}_+}$ and any open set Ω with localizing function ϕ such that $p_0 + \Omega \Subset \Omega_{\mathscr{K},\mathscr{S}_+}$. Take $B_{ij} = K_{ij} \cap B_\phi$, where $B_\phi = \{q \in \mathbb{R}_{4(n-1)} : 0 \leqslant |q| < r_\phi(\omega)$, $\omega = q/|q|\}$ and K_{ij} are the flat cones (43) and where $|q|$ is a Euclidean norm in $\mathbb{R}_{4(n-1)}$ as explained after formula (48). It follows then from the previous definitions that the local tubes $\mathscr{T}_{B_{ij}\phi}$ will be contained in the flat tubes $\mathscr{T}_{ij}$ (43). Thus Theorem 3′ implies

Theorem 4′ (*semi-global version*): *let the bounded open set $p_0 + \Omega$ with localizing function ϕ be compactly contained in $\Omega_{\mathscr{K},\mathscr{S}_+}$. Then there exist functions $\tilde{f}_{hh'}(k)$ analytic in the open local tubes $\mathscr{T}_{B_{hh'}\phi}$ with basis*

$$B_{hh'} = \operatorname{conv}\left\{ \bigcup_{I \in \mathscr{K} \cup \mathscr{S}_+} B_{h(I), h'(X \setminus I)} \right\}$$

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[13]) In [13] only the case of infinitely differentiable functions $f(p)$ and open B's has been treated, while the case of distributions and flat B's has been worked out in [14].

such that the boundary values $\tilde{f}_{hh'}(p)$ *exist in the sense of distributions in* $\mathscr{D}'(\Omega)$ *and satisfy*

$$\tilde{t}_c(p) = \sum_{hh'} \tilde{f}_{hh'}(p), \quad p \text{ in } p_0 + \Omega. \tag{62}$$

Here the indices hh' run independently over all the choices as in equation (52). *The sets* B_{ij} *were introduced above. Furthermore, if a point* $p \in \mathscr{M} \cap \Omega_{\mathscr{K},\mathscr{S}_+}$ *is such that no pair of incoming and outgoing momenta are mutually parallel, then* $\mathscr{M}_c \cap \mathscr{T}_{B_{hh'}\phi} \neq \varnothing$ *within every sufficiently small neighbourhood of that point.*

The decomposition (62) can be again recast into the form (55) of Theorem 4 since $B_1 \subset B_2$ obviously implies $\mathscr{T}_{B_1\phi} \subset \mathscr{T}_{B_2\phi}$.

The advantage of Theorems 3′ and 4′ over the corresponding Theorems 3 and 4 is obvious: they allow the computation of rather big domains of analyticity of the functions $\tilde{f}_r$ in the decomposition (55). In the immediate neighbourhood of a given real point, these domains coincide, however, exactly with the corresponding domains given by Theorems 3 and 4. This is clearly implied by what has been said in connection with the representation (59a) of a local tube. Notice also that the decompositions of Theorem 4 are attached to *bounded* subsets $p_0 + \Omega$ of $\Omega_{\mathscr{K},\mathscr{S}_+}$, $\Omega_{\mathscr{K},\mathscr{S}_+}$ itself being unbounded (it is invariant under real Lorentz transformations!) is not of this form. This is why we call Theorem 4′ the semi-global version of Theorem 4. It is of course hoped that an appropriate extension of the theory of the generalized Fourier transformation will permit to construct a decomposition (55) valid all over $\Omega_{\mathscr{K},\mathscr{S}_+}$. At this point we remind again the reader that only part of the causality of the theory [cf., conditions (C) and (C′) of section 2] has been used so far. As it will be indicated in the next section on the example of the five-point function, better local results can be expected from a better exploitation of the causality condition.

Theorem 3′ allows to answer the question about the uniqueness of a decomposition (62). Suppose we have two sets of functions $\tilde{f}_r$, respectively, $\tilde{f}'_r$, $r = 1, \ldots, N$ such that

$$\tilde{t}_c(p) = \sum_1^N \tilde{f}_r(p) = \sum_1^N \tilde{f}'_r(p), \quad p \text{ in } p_0 + \Omega, \tag{63}$$

$\tilde{f}_r(k)$ and $\tilde{f}'_r(k)$ analytic in $\mathscr{T}_{B_r\phi}$, $r = 1, \ldots, N$. Equation (63) implies that

$$\sum_1^N (\tilde{f}'_r - \tilde{f}_r)(p) = 0$$

and Theorem 3′ tells us then that

$$\tilde{f}'_r(k) = \tilde{f}_r(k) + \sum_{S=1}^N \tilde{f}_{rs}(k)$$

where the functions $\tilde{f}_{rs}$ are analytic in the local tubes $\mathscr{T}_{B_{rs}\phi}$, $B_{rs} = \operatorname{conv}(B_r \cup B_s)$. Only when $N = 1$, i.e., when $\tilde{t}_c$ is itself the boundary value of a single analytic function, is the decomposition unique by a well-known theorem on analytic continuation. Theorem 3′ permits also to answer the problem of 'gluing together' decompositions of $\tilde{t}_c$ attached to two different real regions $p_0 + \Omega$ and $p'_0 + \Omega'$ having a non-empty intersection: only when $N = 1$ are the functions $\tilde{f}_r$ and $\tilde{f}'_r$ pertaining to these two regions, necessarily analytic continuations of each other, as the reader may easily verify himself.

Thus we have proved the decomposition (D) announced in the Introduction. We just have to set

$$F_c(p) = \tilde{t}_c(p)/_{\mathcal{M}}, \quad F_r(k) = \tilde{f}_r(k)/_{\mathcal{M}^c},$$
$$\mathcal{F}_r = \mathcal{T}^{\,!}_r \cap \mathcal{M}^c \quad \text{or} \quad \mathcal{T}_{B_r\phi} \cap \mathcal{M}^c,$$

where the restriction of $\tilde{t}_c$ to the real mass shell has to be understood in the sense of Theorem 2.

4. Examples and Comments

In the process two particles $\to (n-2)$ particles, the essential support attached to the scattering amplitude $\langle p_1, \ldots, p_{n-2}|S-1|-p_{n-2}, -p_n\rangle$ can be completely specified independently of the particular values of the incoming and outgoing momenta. Indeed, the collection of subsets $\mathcal{S}_+$ (section 2) consist always of precisely the following subsets:

$$\mathcal{S}_+ = \{I \in \mathcal{P}*(X) : I \subset X_1, |I| > 1\}$$

where $X_1 = \{1; 2; \ldots; n-2\}$, $X_2 = \{n-1; n\}$ and $X = X_1 \cup X_2 = \{1; 2; \ldots; n\}$. According to (12), the collection $\mathcal{S}_-$ consists of the complementary subsets $X \backslash I$ where $I \in \mathcal{S}_+$, while all the other subsets of X belong to the collection $\mathcal{K}$. In order to see this, let us first remark that obviously $p_I \in V_+(2m)$ when $|I| > 1$ and $I \subset X_1$, while $p_{X\backslash I} = -p_I \in \overline{V}_-(2m)$. It is also clear that the subsets $I = \{i\}$, $i = 1, 2, \ldots, n$, consisting of single elements, belong to the collection $\mathcal{K}$ since $p_i^2 = m^2 < 4m^2$. According to property (12a) the subsets I of length $|I| = n-1$ also belong to $\mathcal{K}$. What remains to be verified is that all the subsets of the form $I = \{n-1\} \cup X_1'$ and $I' = \{n\} \cup X_1'' \equiv X\backslash I$, where $X_1', X_1'' \subset X_1$, $X_1' \cup X_1'' = X_1$ and $X_1', X_2' \neq \varnothing$ belong to $\mathcal{K}$. Now for such an I we obviously have $p_I \notin V_-$ and $p_{I'} \notin V_-$. But from $p_I + p_{I'} = 0$ we conclude $p_I^2 \leqslant 0$, $p_{I'}^2 \leqslant 0$, which proves our assertion.

From the above we conclude that the cones (35) represent the best possible result in the case considered. The sequences $\vec{R}$ and $\vec{S}$ consist, or course, of the single points $(n-1)$ and (n), so that the essential support is the union of the cones

$$C_{\vec{I},\vec{J},h} = \{x = (x_1, \ldots, x_n) : x_{i_1} = x_{j_1}, \ldots, x_{i_p} = x_{j_p};\ x_{n-1} = x_n;$$
$$x_i - x_n \in \overline{V}_+ \text{ for all } i \in X_1;\ x_{i_{h(l)}} - x_l \in \overline{V}_+ \text{ for all } l \in X_1 \backslash (I \cup J)\} \qquad (64)$$

with $\vec{I} = (i_1, \ldots, i_p)$, $\vec{J} = (j_1, \ldots, j_p)$, h taking its value in $\{1, \ldots, p\}$.

As an illustration of the general theory, we will discuss in more detail the simplest cases $n = 4$ and $n = 5$. For $n = 4$ we will rediscover part of well-known results, while the case $n = 5$ will illustrate several claims made in the previous sections.

The four-point function

According to formula (64) the essential support of t_c relevant for the evaluation of $\langle p_1 p_2|S-1|-p_3, -p_4\rangle$ consists of the single cone

$$\tilde{C}_{12} = \{x = (x_1, \ldots, x_4) : x_1 = x_2,\ x_3 = x_4,\ x_1 - x_4 \in \overline{V}_+\}. \qquad (65)$$

Its dual cone is given by

$$C_{12} = \{p = (p_1, \ldots, p_4) : p_1 + \cdots + p_4 = 0,\ p_1 + p_2 \in V_+\} \qquad (66)$$

as it is easily computed by noticing that $\sum_1^4 p_i x_i = (p_1 + p_2, x_1 - x_4)$ when $x \in C_{12}$ and $p_1 + \cdots + p_4 = 0$. Therefore the amplitude t_c is the boundary value of a single function analytic in the localized tube

$$\mathscr{T}^{\,l}_{12} = \{k = p + iq : p \in \Omega_{12},\ q \in \tilde{C}_{12} \cap A(p)\}$$

where

$$\Omega_{12} \equiv \Omega_{\mathscr{K},\,\mathscr{S}_+} = \{p : p_i^2 < M_i^2,\ i = 1, \ldots, 4;\ (p_r + p_s)^2 < M_{rs}^2 \text{ and } \neq m_{rs}^2 \text{ for } (r,s) = (2,3) \text{ or } (3,1);\ p_1 + p_2 \in \mathbb{C}\overline{V}_-(m_{12}, M_{12})\} \tag{67}$$

and $A(p)$ is an open convex set containing the origin and depending on $p \in \Omega_{12}$. For the sake of completeness, we have considered the general case of particles with different masses $m_r > 0$, $r = 1, \ldots, 4$, to which evidently our theory, mutatis mutandis, applies also. m_I denotes the (positive) discrete mass in the channel I, while M_I is the threshold mass of the continuum.

Let us verify explicitly that the complex mass shell $k_i^2 = m_i^2$, $i = 1, \ldots, 4$, intersects $\mathscr{T}^{\,l}_{12}$ in the neighbourhood of the real points $\mathscr{M} = \{p_{1,2} \in \overline{V}_+(m_{1,2}), p_{3,4} \in \overline{V}_-(m_{3,4})\}$. For that purpose, it is sufficient, as we have seen earlier, to verify that the real tangent plane to $\mathscr{M}$ at a given point p

$$\mathscr{P}(p) = \{q : p_i q_i = 0,\ i = 1, \ldots, 4,\ \sum_1^4 q_i = 0\} \tag{68}$$

intersects $\tilde{C}_{12}$. In other words, one has to show that there exist non-trivial solutions of the system (68) such that $q_1 + q_2 = -q_3 - q_4 \in V_+$. But this is always trivially possible, provided the two four-vector couples p_1, p_2 and p_3, p_4 are not parallel, i.e., provided we are away from the thresholds. When we approach the threshold, the intersection in question will, however, shrink to nothing.

On the other hand, it is known ([20], [21]) that the envelope of holomorphy of the n point amplitude is automatically invariant under complex Lorentz transformations. Therefore the analyticity region $\mathscr{T}^{\,l}_{12}$ just computed will automatically extend to

$$\mathscr{T}^{\,l}_{12} = \bigcup_{\Lambda \in \mathscr{L}_+(C)} \Lambda\, \mathscr{T}^{\,l}_{12} \supset \{k = (k_1, \ldots, k_4) : Im(k_1 + k_2)^2 \equiv Im s > 0\} \cap \mathscr{N} \tag{69}$$

where $\mathscr{N}$ is a complex open set containing the real region Ω_{12}. The last assertion in (69) follows easily from the fact that the extended tube in one four-vector k is the whole of $\mathbb{C}_4$ minus the cut $k^2 = \rho \geqslant 0$. Thus we reobtain an old result [6]: in a complex neighbourhood of the real mass shell $\mathscr{M}$ the only singularity of the four-point function is the s cut (remember that $\mathscr{M} \subset \Omega_{\mathscr{K},\mathscr{S}_+}$, that $\mathscr{M}$ is closed and $\Omega_{\mathscr{K},\mathscr{S}_+}$ open !). Therefore also at the threshold the scattering amplitude is the boundary value of an analytic function.

It is instructive to indicate explicitly how these stronger results are due to an exploitation of the full causality condition (C). Instead of the 'retarded' linear combinations $r_I = \langle T(X)\rangle_c - \langle T(I')T(I)\rangle$, (18), one can introduce the generalized retarded functions

$$r_{\mathscr{S}}(X) = \langle T(X)\rangle_c + \sum_{\nu=2}^{n} (-)^{\nu-1} \sum_{J_1, \ldots, J_\nu \in \mathbb{K}^\nu_{\mathscr{S}}} \langle T(J_1)T(J_2) \cdots T(J_\nu)\rangle_c,\quad n = |X|, \tag{70}$$

where the inner sum runs over the following 'chain' $\mathbb{K}^\nu_{\mathscr{S}}$ of proper subsets $J_r \subset X$: $J_1 \cup \ldots \cup J_\nu = X$, $J_j \cap J_k = \varnothing$ for all $j \neq k$, $J_j \neq \varnothing$ for all $j = 1, \ldots, \nu$, and

$J_1 \in \mathscr{S}', \ldots, J_1 \cup \ldots \cup J_r \in \mathscr{S}'$ for all $r \leqslant \nu - 1$. Here $\mathscr{S}$ and $\mathscr{S}'$ are 'cells', i.e., collections of proper subsets of X similar to the couple of 'hypercells' $\mathscr{S}_+$ and $\mathscr{S}_-$ introduced earlier and defined as follows: consider the n dimensional real space $\mathbb{R}_n$ consisting of n-tuples $(s_1, s_2, \ldots, s_n)$ and in it the $n-1$ dimensional hyperplane $\sum_{n-1}: s_1 + s_2 + \cdots + s_n = 0$. The hyperplanes $s_I = \sum_{i \in I} s_i = 0, I \in \mathscr{P}_* (X)$ divide $\sum_{n-1}$ into a certain number of conical polyhedra ('geometrical cells') defined by $s_I > 0$ or <0 for all $I \in \mathscr{P}_* (X)$. A (set theoretical) cell is the collection of the $I \subset X$ such that $s_I > 0$ in a geometrical cell, while the 'opposed cell' $\mathscr{S}'$ consists of all $I \subset X$ such that $s_I < 0$. Evidently $\mathscr{S}'$ consists of the complementary sets in $\mathscr{S}$ and every $I \in \mathscr{P}_* (X)$ belongs either to $\mathscr{S}$ or to $\mathscr{S}'$. $\mathscr{S}$ and $\mathscr{S}'$ enjoy the properties (12c)–(12e) of the couple $\mathscr{S}_+$ and $\mathscr{S}_-$ of section 3 and correspond to the special case $\mathscr{K} = \varnothing$ of a hypercell.

From the causality property (C) follows the support property

$$r_{\mathscr{S}} (X) = 0 \quad \text{if} \quad [I]_a \lesssim [X \backslash I]_a \quad \text{and} \quad I \in \mathscr{S}. \tag{71}$$

It can be seen that the support of $r_{\mathscr{S}}$ is a cone $C_{\mathscr{S}}$, in general non-convex, whose convex closure is a *pointed* cone (displaced away from the origin if $a \neq 0$). The dual cone of $C_{\mathscr{S}}$ is given by [14])

$$\tilde{C}_{\mathscr{S}} = \{p = (p_1, \ldots, p_n) : \sum_1^n p_i = 0, p_I \in V_+ \text{ for all } I \in \mathscr{S}\}. \tag{72}$$

Therefore the Fourier transform $\tilde{r}_{\mathscr{S}}(p)$ of $r_{\mathscr{S}}$ is the boundary value (in the sense of tempered distributions) of a function $\tilde{r}_{\mathscr{S}}(k)$ analytic in the tube $\mathscr{T}_{\mathscr{S}}$ having as basis cone $\tilde{C}_{\mathscr{S}}$:

$$\mathscr{T}_{\mathscr{S}} = \{k = p + iq : q \in \tilde{C}_{\mathscr{S}}\}. \tag{73}$$

The proof of (71) and (72) is surprisingly cumbersome and lengthy for such a simple geometrical problem, and is contained in an unpublished paper by two of the authors [22][15]). What is also important to us is the coincidence of $\tilde{t}_c$ and $r_{\mathscr{S}}$ in certain portions of momentum space. In analogy with (19), it namely follows from the definition (70) and the spectral condition:

$$\tilde{r}_{\mathscr{S}} (p) = \tilde{t}_c (p) \text{ for } p \in \Omega_{\mathscr{S}} \tag{74}$$

where $\Omega_{\mathscr{S}}$ [cf., the definition (16) of $\Omega_{\mathscr{K}, \mathscr{S}_+}$] is the following open set

$$\Omega_{\mathscr{S}} = \{p \in \mathbb{R}_{4(n-1)} : p_I \in \complement \overline{V}_- (M_I) \text{ for all } I \in \mathscr{S}\}. \tag{75}$$

The sets $\Omega_{\mathscr{S}}$ are therefore a subclass of the sets $\Omega_{\mathscr{K}, \mathscr{S}_+}$ corresponding to the case $\mathscr{K} = \varnothing$. In general a given $\Omega_{\mathscr{S}}$ will contain several different sets $\Omega_{\mathscr{K}, \mathscr{S}_+}$. If $m \neq 0$ it is easily seen that the collection of all $\Omega_{\mathscr{S}}$ forms an open covering of the whole of $\mathbb{R}_{4(n-1)}$ so that $\tilde{t}_c$ is everywhere the boundary value of at least one $\tilde{r}_{\mathscr{S}} (k)$. This result, which is due to Ruelle [25], is the usual starting point for the study of analyticity properties of the n point function.

[14]) Notice that some of the conditions defining $\tilde{C}_{\mathscr{S}}$ are redundant: if $I = I_1 \cup I_2$ with $I_1 \cap I_2 = \phi$ and $I_{1,2} \in \mathscr{S}$, $p_I \in V_+$ is a consequence of $p_{I_{1,2}} \in V_+$.

[15]) The first to introduce generalized retarded functions was to our knowledge Polkinghorne [23], the systematic study of a subclass of these functions is due to Steinman [24] while Ruelle [25] treated them in full generality. The first proof of (72) appeared in [26]. The definition (70) in terms of T products appears to our knowledge for the first time here and is extracted from [22].

Let us indicate how the domain $\mathscr{T}_{12}^{l}$ can be reobtained starting from the generalized retarded functions with the help of the special and rather elementary edge-of-the-wedge theorem ($n = 2$). From (75), we deduce:

$$\tilde{t}_c(p) = \tilde{r}_{\mathscr{S}}(p) \tag{76}$$

in Ω_{12} for $\mathscr{S}$ such that $\Omega_{12} \subset \Omega_\rho$.

There are therefore several different functions $\tilde{r}_{\mathscr{S}}(k)$ analytic in different tubes $\mathscr{T}_{\mathscr{S}}$ the boundary values of which coincide on the real open set Ω_{12}. An elementary calculation—it was performed in [6]—shows that there are 16 different cells satisfying (76) and that the convex envelope of the corresponding cones $\tilde{C}_{\mathscr{S}}$ is precisely $\tilde{C}_{12}$. Therefore the successive application of the ordinary edge-of-the-wedge to different pairs of the functions $\tilde{r}_{\mathscr{S}}$ yields analyticity in the local tube $\mathscr{T}_{12}^{l}$ obtained above.

That the inclusion of general retarded functions gives more information can be seen as follows. It is clear that the r_I have a much larger support in x space than the $r_{\mathscr{S}}$: as it is easily seen, the convex hull of supp r_I equals the whole space $\mathbb{R}_{4(n-1)}$ if $n > 2$, so that the Fourier transform $\tilde{r}_I$ is *not* the boundary value of a single analytic function, in contrast to the $\tilde{r}_{\mathscr{S}}$. If we consider the set of all I in a given cell $\mathscr{S}$, we will have $\tilde{r}_I = \tilde{t}_c$ in $\Omega_{\mathscr{S}}$ for all $I \subset \mathscr{S}$. The application of the generalized edge-of-the-wedge procedure to this set shows that in $\Omega_{\mathscr{S}}$ $\tilde{t}_c$ is the boundary value of a single function analytic only in a *localized* tube $\mathscr{T}_{\mathscr{S}}^{l}$, while $\tilde{t}_c = \tilde{r}_{\mathscr{S}}$ gives us analyticity in the whole of $\mathscr{T}_{\mathscr{S}}$. Now, in order to show the invariance of the domain of analyticity under complex Lorentz transformations—not to speak of the proof of the crossing theorem—global methods of analytic completion are needed, in which analyticity near the complex infinity within the tubes $\mathscr{T}_{\mathscr{S}}$ play an essential role, as it can be inferred from the corresponding proofs in the papers [20] and [21].

The role of the generalized retarded functions for $n > 4$ will be discussed after the following example.

The five-point function

The essential support of the amplitude t_c pertaining to the region $\Omega_{\mathscr{K},\mathscr{S}_+}$ relevant for the computation of the matrix element $\langle p_1 p_2 p_3 | S - 1 | -p_4, -p_5 \rangle$ consists, according to formula (64), of the following three cones:

$$C_r = \{x = (x_1, \ldots, x_5) : x_s = x_t, x_4 = x_5, x_r - x_5 \in \overline{V}_+, x_s - x_r \in \overline{V}_+\} \tag{77}$$

where (r, s, t) is a cyclic permutation of $(1, 2, 3)$. Notice that the conditions $x_s - x_5 \in \overline{V}_+$ and $x_t - x_5 \in \overline{V}_+$ are implied by the conditions written down in (77) and so they can be omitted. The fact that

$$\sum_1^5 p_i x_i = (p_s + p_t)(x_s - x_r) - (p_4 + p_5)(x_r - x_5)$$

when $\sum_1^5 p_i = 0$ and $x \in C_r$ immediately yields the dual cones

$$\tilde{C}_r = \left\{ p : \sum_1^5 p_i = 0, p_s + p_t \in V_+, p_1 + p_2 + p_3 = -p_4 - p_5 \in V_+ \right\} \tag{78}$$

We have therefore a decomposition into three 'partial amplitudes':

$$\tilde{t}_c(p) = \sum_1^3 \tilde{f}_r(p), \quad p \in \Omega_{\mathscr{K},\mathscr{S}_+} \tag{79}$$

where each $\tilde{f}_r$ can be analytically continued into the localized tube $\mathscr{T}^l_r$ attached to the cone $\tilde{C}_r$.

The linear combination (79) is analytic in the localized tube $\mathscr{T}^l = \mathscr{T}^l_1 \cap \mathscr{T}^l_2 \cap \mathscr{T}^l_3$ attached to the cone

$$\tilde{C} = \bigcap_1^3 \tilde{C}_r = \left\{p : \sum_1^5 p_i = 0, p_1 + p_2, p_2 + p_3, p_3 + p_1 \in V_+\right\} \tag{80}$$

[since $p_4 + p_5 \in V_-$ is implied by the three conditions (80), it is omitted]. The question we want to answer is: over which real physical points p is $\mathscr{M}_c \cap \mathscr{T}^l \neq \varnothing$, i.e., $\tilde{C} \cap \mathscr{M}$ non-empty?

According to Lemma 3, section 2, a necessary and sufficient condition for that is that the linear manifold

$$\begin{aligned}\tilde{\mathscr{P}}(p) = \{x = (x_1, \ldots, x_5) : x_i - x_j = \lambda_i p_i - \lambda_j p_j,\ i < j,\ i, j = 1, \ldots, 5,\\ (\lambda_1, \ldots, \lambda_5) \in \mathbb{R}_5, p \in \mathscr{M}\}\end{aligned} \tag{81}$$

do not intersect $C \backslash \{0\}$, where C is the closure of the dual cone to $\tilde{C}$. C is best computed by introducing the variable transformation (invertible in view of $\sum_1^5 p_i = 0$)

$$\sigma_r = p_s + p_t, (r, s, t) = \text{cycl.}\ (1, 2, 3), p_4 - p_5 = \tfrac{1}{2} v \tag{82}$$

in terms of which $\tilde{C}$ becomes

$$\tilde{C} = \{(\sigma_1, \sigma_2, \sigma_3, v) : \sigma_r \in V_+, r = 1, 2, 3, v \text{ arbitrary}\}$$

and

$$\sum_1^5 p_i x_i = \sum_{\text{cycl.}} \tfrac{1}{2}(-x_r + x_s + x_t - x_4 - x_5)\sigma_r + v(x_4 - x_5).$$

This implies immediately:

$$C = \{x : x_4 = x_5, \tfrac{1}{2}[-(x_r - x_4) + (x_s - x_4) + (x_t - x_4)] \in V_+, \text{cycl.}\ (1, 2, 3)\} \tag{83}$$

Now the linear manifold $\mathscr{P}(p)$ will *not intersect* the cone C only if $0 = x_4 - x_5 = \lambda_4 p_4 - \lambda_5 p_5$, which implies $\lambda_4 = \lambda_5$, since we suppose the four-vectors p_4, p_5 non-collinear, and if

$$\tfrac{1}{2}(-\lambda_r p_r + \lambda_s p_s + \lambda_t p_t) \equiv \eta_r \in \overline{V}_+, (r, s, t) = \text{cycl.}\ (1, 2, 3) \tag{84}$$

for some $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$. Condition (84) involves only the three out-going momenta. We can resolve the system (84) with respect to the momenta p_r. The mass shell condition $p_r^2 = m_r^2$, $p_r \in V_+$—we again treat the case of different mass particles for the sake of generality—fixes then uniquely the value of the parameters λ_r. We get

$$p_r = m_r \frac{\eta_s + \eta_t}{\sqrt{(\eta_s + \eta_t)^2}}, \quad \eta_r \in \overline{V}_+, \quad (r, s, t) = \text{cycl.}\ (1, 2, 3). \tag{85}$$

Thus $\tilde{C} \cap \mathscr{M}$ will be empty at a point p if and only if the three outgoing momenta are presentable in the parametric form (85) in terms of three four-vectors η_r in $\overline{V}_+$. In order to see that there is an abundance of physical points not representable in the above form, let us compute the square of the three-dimensional space-time volume subtended by the three four-vectors p_r (85), which is equal to the Gram determinant of the 3×3 matrix $(p_r p_s)$ formed by the scalar products $p_r p_s$. We find

$$0 \leqslant \det(p_r p_s) = \frac{4 m_1^2 m_2^2 m_3^2}{\prod\limits_{i<j} (\eta_i + \eta_j)^2} \det(\eta_r \eta_s) \leqslant m_1^2 m_2^2 m_3^2. \tag{86}$$

The factor 4 comes from the determinant of the linear transformation $y_r = x_s + x_t$ which has the value 2, while the last inequality stems from the fact that the factor multiplying $(m_1 m_2 m_3)^2$ in the second equality varies between 0 and 1 when the three vectors η_r vary over $\overline{V}_+$. In order to prove the last inequality, we compute by brute force the expression:

$$\prod_{i<j} (\eta_i + \eta_j)^2 - 4 \det(\eta_r \eta_s) = \sum_{\text{cycl.}} \{\tfrac{1}{3} \eta_r^2 [4 \eta_s^2 \eta_t^2 + 3(\eta_s^2 - \eta_t^2)^2]$$

$$+ 2(\eta_r^2 + \eta_s^2)(\eta_s + \eta_t)^2 (\eta_t, \eta_r) + 4 \eta_r^2 (\eta_s, \eta_t)^2\}. \tag{87}$$

Since all the $\eta_r \eta_s$ are $\geqslant 0$ for $\eta_r \in \overline{V}_+$, $r = 1,2,3$, we see that (87) is always $\geqslant 0$; it vanishes only when all three $\eta_r^2 = 0$. Therefore the last equality sign in (86) is attained if and only if all the three η_r are light-like. Thus we have proved that the S matrix elements $\langle p_1 p_2 p_3 | S - 1 | -p_4, -p_5 \rangle$ are boundary values of a single function analytic in $\mathscr{M}_c \cap \mathscr{T}^l$ provided that the outgoing momenta satisfy the inequality

$$\det(p_r p_s) - m_1^2 m_2^2 m_3^2 \equiv 2(p_1 p_2)(p_2 p_3)(p_3 p_1) - \sum_{\text{cycl.}} m_r^2 (p_s p_t)^2 > 0. \tag{88}$$

Since for physical values of the outgoing momenta the determinant (86) can take any value $\geqslant 0$, we see that the condition (88) excludes only a relatively thin layer containing the thresholds. The 'region of analyticity' (88) can be best visualized in terms of the Dalitz plot, where the final state configuration is described in terms of three independent parameters, the total centre-of-mass energy $M = \sqrt{p^2}$, where $p = p_1 + p_2 + p_3 = (M, \vec{0})$, and the three centre-of-mass energies $E_r = (p_r, p) M^{-1} = x_r M$ linked by the relation $E_1 + E_2 + E_3 = M$, respectively the relation $x_1 + x_2 + x_3 = 1$. For fixed $M \geqslant m_1 + m_2 + m_3$ the physical region is the region $D \geqslant 0$ contained in the triangle $E_r \geqslant m_r$, $r = 1,2,3$, bounded by the third degree curve $D = 0$, homeomorphic to a circle, where D is the Gram determinant (86). If M is above a certain limiting value, the domain (88) $D > m_1^2 m_2^2 m_3^2$ will appear in the Dalitz plot. In the limit $M \to \infty$ this domain will rather quickly converge to the whole of $D > 0$. In the equal mass case $m_r = m$, $r = 1,2,3$, the maximal value of D for fixed M is easily calculated to be

$$D_{\max} = \tfrac{3}{4} M^2 \left(\frac{M^2}{9} - m^2\right)^2 \tag{89}$$

so that for $M > \sqrt{4/3}\, 3m$ the 'region of analyticity' (88) will start to appear. This value is not too far from the threshold energy $M = 3m$.

The results (80) to (88) can be obtained with the help of the ordinary edge-of-the-wedge theorem ($n = 2$) via the generalized retarded functions by exactly the same method as for the four-point function [see (76)]. They were therefore known to the authors for quite some time. Let us mention that also in the general case 2 particles $\to$ $(n - 2)$ particles there exist physical points where the scattering amplitude is the boundary value of a single analytic function.

A comparison of the localized tube $\mathscr{T}_{12}^{l}$ (67) with the 'extended localized tube' $\mathscr{T}_{12}'^{l}$ (69) in the case of the four-point function, leads naturally to the supposition that the full use of causality will lead for arbitrary n to the generalization

$$\tilde{t}_c = \sum_{r=1}^{N} f_r \tag{90}$$

$\tilde{f}_r$ analytic in $\mathscr{T}_r'^{l}$, $r = 1, \ldots, N$, where

$$\mathscr{T}_r'^{l} = \left\{ \bigcup_{\Lambda \in \mathscr{L}_+(\mathbb{C})} \Lambda \mathscr{T}_r^{l} \right\} \cap \mathscr{N}. \tag{91}$$

$\mathscr{N}$ being a sufficiently small complex neighbourhood of the real point considered. Indeed, this was proved in [7] for the case $n = 5$. It was achieved through the study of the set of Steinmann relations satisfied by appropriate groups of generalized retarded functions: the Steinmann relations were resolved through a repeated use of the generalized edge-of-the-wedge theorem and the invariance of the domain of analyticity of the functions $\tilde{r}_{\mathscr{S}}$ under complex Lorentz transformations was incorporated. It is hoped that a general proof of (90) and (91) will be possible by using the geometrically simpler method of this paper.

In order to exhibit explicitly the improvement due to (90) and (91) compared to (79), we shall calculate the domains $\mathscr{T}_r'^{l}$ corresponding to the cones (78) in the immediate neighbourhood of a given real point p. All we have to do is to compute the extended tubes

$$\mathscr{T}_r' = \bigcup_{\Lambda \in \mathscr{L}_+(\mathbb{C})} \Lambda \mathscr{T}_r \text{ with } \mathscr{T}_r = \{(\xi, \xi_r) \in \mathbb{C}_8 : \operatorname{Im} \xi \in V_+, \operatorname{Im} \xi_r \in V_+\} \tag{92}$$

where $\xi = k_1 + k_2 + k_3$, $\xi_r = k_s + k_t \equiv \xi - k_r$, $(r, s, t) = \text{cycl}(1,2,3)$ in the neighbourhood of the real point $\xi = p_1 + p_2 + p_3 \equiv p$, $\xi_r = p - p_r$, $p_i \in \overline{V}_+(m_i)$, $i = 1,2,3$. Now according to [27] in this region the extended tube (92) is given by the inequalities:

$$\operatorname{Im} z > 0,\ \operatorname{Im} z_r > 0,\ \operatorname{Im} t_r^{\epsilon} > 0,\ \epsilon = \pm 1, \text{ where}$$

$$z = \xi^2,\ z_r = \xi_r^2,\ t_r^{\epsilon} = w_r + \epsilon\sqrt{w_r^2 - z z_r},\ w_r = (\xi, \xi_r). \tag{93}$$ [16])

In the vicinity of the real point p we are allowed to approximate the domain (93) by its tangent planes, i.e., to put $z = \mathring{z} + \delta z$, $z_r = \mathring{z}_r + \delta z_r$, $w_r = w_r^0 + \delta w_r$ and develop the inequalities (93) up to first order in δz, etc. Since at p all quantities in (93) are real, we obtain

$$\operatorname{Im} \delta z > 0, \operatorname{Im} \delta z_r > 0, \frac{1}{2\sqrt{\lambda_r}} \operatorname{Im}[2(\sqrt{\lambda_r} + \epsilon w_r)\delta w_r - \epsilon z \delta z_r - \epsilon z_r \delta z] > 0, \epsilon = \pm 1,$$

$$\text{where } z = p^2, z_r = (p - p_r)^2, w = (p - p_r, p), \lambda_r = (p - p_r, p)^2 - (p - p_r)^2 p^2. \tag{93.a}$$

This is the sought local description of $\mathscr{T}_r'^{l}$. It ceases to be valid only at points where the determinant λ_r vanishes, i.e., when the vectors p and p_r become parallel. In (93a) we have to insert

$$\delta z = \delta \left(\sum_1^3 k_r \right)^2 = \sum_1^3 \delta k_r^2 + 2 \sum_{r<s} \delta(k_r, k_s)$$

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[16]) There is another part of the boundary of the extended tube in two four-vectors—the so-called S curve. This is, however, very far from the points $\xi_r = p - p_r \in V_+$, $\xi = p \in V_+$ considered.

and similarly for δz_r and δw_r. Since in this linear approximation the complex mass shell $\mathcal{M}_c$ is given simply by $\delta k_r^2 = 0$, $r = 1,2,3$, $\delta(k_s, k_t) \equiv u_r + iv_r$ arbitrary, $(r,s,t) =$ cycl$(1,2,3)$, we find after an elementary calculation that $\mathcal{T}_r'^l \cap \mathcal{M}_c$ is locally described by the tube:

$$l_r^\epsilon(v, v_r) \equiv (\sqrt{x_r^2 - \mu_r^2} - \epsilon x_r) v_r + [\sqrt{x_r^2 - \mu_r^2} + \epsilon(x_r - \mu_r^2)] v > 0,$$

$$\epsilon = \pm 1,\ v_r > 0,\ v \equiv v_1 + v_2 + v_3 > 0. \tag{94}$$

Here we have used the variables of the Dalitz plot already described:

$$p^2 = M^2, (p_r, p) = E_r M = x_r M^2, x_1 + x_2 + x_3 = 1 \text{ and } \mu_r = \frac{m_r}{M}. \tag{95}$$

Note that (95) and $p_r \in \overline{V}_+(m_r)$ imply the inequalities:

$$0 < \mu_r \leqslant x_r < \tfrac{1}{2}(1 + \mu_r^2 - \mu_s^2 - \mu_t^2),\ (r, s, t) = \text{cycl. } (1, 2, 3)$$

$$\text{and } \sum_1^3 \mu_r \leqslant 1. \tag{96}$$

The cone (94) can be drawn in the two-dimensional plane of the variables v_r and v: it is the sector between the two straight lines $l_r^+ = 0$, $l_r^- = 0$ contained in the first quadrant $v_r > 0$, $v > 0$. For $x_r = \mu_r$ the two straight lines $l_r^\epsilon = 0$ degenerate into $v_r - (1 - \mu_r) v = 0$. When $x_r > \mu_r$ the set

$$v = \rho > 0, \quad v_r = (1 - \mu_r)\rho > 0 \tag{97}$$

[note that (96) implies $1 - \mu_r > 0$ when $x_r > \mu_r$] is always contained in (94) since

$$l_r^\epsilon(1, 1 - \mu_r) = (2 - \mu_r)\sqrt{x_r^2 - \mu_r^2} + \epsilon \mu_r (x - \mu_r) \geqslant (2 - \mu_r)\sqrt{x_r^2 - \mu^2}$$

$$- \mu_r (x - \mu_r) \geqslant 2(1 - \mu_r)\sqrt{x_r^2 - \mu_r^2} > 0 \text{ if } x_r - \mu_r > 0. \tag{98}$$

Here we have used the inequality $\sqrt{x_r^2 - \mu_r^2} \geqslant x_r - \mu_r$ and (96). Therefore, the open cone (94) is always non-empty, excepting the case $x_r = \mu_r$. But this case corresponds exactly to $\lambda_r = 0$, when the linearized description of $\mathcal{T}_r'^l \cap \mathcal{M}_c$ ceases to be valid. Now $x_r = \mu_r$, i.e., $E_r = m_r$ corresponds to the configuration:

$$p_r = (m_r, \vec{0}),\ p_{s,t} = (\sqrt{\vec{p}^2 + m_{r,s}^2}, \pm\vec{p})$$

in the centre-of-mass system $p = (M, \vec{0})$. In this configuration no two vectors are parallel, except when $\vec{p} = 0$. So from Lemma 2 of section 2, we conclude that $\mathcal{T}_r'^l \cap \mathcal{M}_c$ is always non-empty, except when $p_r = m_r e$, $r = 1,2,3$, $e^2 = 1$, $e \in V_+$, i.e., except when all the three four-vectors p_r are parallel. This is certainly an improvement compared to (78), (79).

Let us consider now the intersection $\mathcal{T}_r'^l \cap \mathcal{T}_s'^l \cap \mathcal{M}_c$, $r \neq s$. We have to look at the intersection of two sets (94), say $r = 1$ and $r = 2$. By choosing v_1, v_2 and v as independent variables, we immediately see from the inequalities (98), $r = 1,2$, that the set

$$v = \rho > 0, \quad v_r = (1 - \mu_r)\rho > 0, \quad r = 1,2$$

is contained in $\mathcal{T}_1'^l \cap \mathcal{T}_2'^l \cap \mathcal{M}_c$ provided $x_r > \mu_r$ for $r = 1,2$. As before we conclude that $\mathcal{T}_r'^l \cap \mathcal{T}_s'^l \cap \mathcal{M}_c$, $r \neq s$ is always non-empty with the exception of the case when all three vectors p_r are parallel. This is interpreted by saying that the five-point scattering amplitude can be decomposed everywhere only into two partial amplitudes each of which is restrictible to the mass shell. The only exceptional point is the threshold $p_r = m_r e$, $r = 1, 2, 3$.

We are still left with the case of the intersection $\mathcal{T}_1'^{l} \cap \mathcal{T}_2'^{l} \cap \mathcal{T}_3^{l} \cap \mathcal{M}_c$. It is described locally by the inequalities:

$$l_r^{\epsilon}(v, v_r) > 0, \quad v_r > 0, \quad \epsilon = \pm 1, \quad r = 1, 2, 3 \tag{99}$$

[the inequality $v > 0$ is dropped since it is already implied by (99)]. As it is readily seen, the set (99) is empty when the point p is in a neighbourhood of the threshold $p_r = m_r e$, $r = 1, 2, 3$. Now, when the masses are equal ($m_r = m$) it can be proved by a chain of ingenious inequalities, which will not be reproduced here, that (99) is non-empty whenever

$$M > 4{,}8m \tag{100}$$

where M is the total centre-of-mass energy[17]). This shows that with the exception of a rather small set around the threshold $p_r = me$, $r = 1, 2, 3$, the five-point scattering amplitude is the limiting value of a single analytic function, which is a palpable improvement of (88).

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Appendix

Proof that $\tilde{C}_{hh'}$ has interior points

$$C_{hh'} = \{x\colon \text{for all } I \in \mathcal{S}_+ \cup \mathcal{K},\ x_{h(I)} - x_{h'(X \setminus I)} \in \overline{V}_+\}$$

We can picture the whole list of these conditions by drawing the following diagram: let $a_1, \ldots, a_n$ be distinct points in $\mathbb{R}_2$. If the condition $x_j - x_k \in \overline{V}_+$ appears in the list given above, we draw a line between a_j and a_k with an arrow in the direction $k \to j$. In this diagram there will be, *at most*, two lines joining a_j and a_k, namely one pointing from a_j to a_k and another pointing from a_k to a_j (even if, e.g., the condition $x_j - x_k \in \overline{V}_+$ appears many times). We claim that the graph obtained in this way is a connected graph. Indeed, if it were not, there would be two proper subsets I and I' of $1, \ldots, n$ such that no line connects $\{a_j\}_{j \in I}$ with $\{a_k\}_{k \in I'}$, and: $I \cup I' = \{1, \ldots, n\}$, $I \cap I' = \varnothing$. Then one of these subsets, say I, would be in $\mathcal{S}_+ \cup \mathcal{K}$; the condition $x_{h(I)} - x_{h'(I')} \in \overline{V}_+$ would be represented by a line joining $\{a_j\}_I$ to $\{a_k\}_{I'}$, in contradiction with our hypothesis.

Because this diagram is connected, it is possible to make it into a tree diagram by striking out a few lines. We claim that, if the corresponding conditions $x_j - x_k \in \overline{V}_+$ are

[17]) This estimate is due to A. Martin. The authors are very thankful for his generous help.

struck out from the list which defines $C_{hh'}$, the remaining conditions define a 'simplicial' cone, i.e., that the remaining conditions can be written $x_{u_1} - x_{v_1} \in \overline{V}_+, \ldots, x_{u_{n-1}} - x_{v_{n-1}} \in \overline{V}_+$ with the $x_{u_j} - x_{v_j}$ being independent variables. This is easily proved by induction on the number n of vertices of the tree: indeed a tree with $n-1$ vertices is obtained if one extremity of the n vertex tree, say x_{u_1}, is cut-off. The line thus severed corresponds to a condition $x_{u_1} - x_{v_1} \in V_\pm$. The rest of the conditions involve only variables where x_{u_1} does not appear and are linearly independent of $x_{u_1} - x_{v_1}$.

As a consequence, we see that $C_{hh'}$ is always contained in a 'simplicial' cone. Hence it is a proper cone and its dual has interior points.

Final Remark

The present work is a contribution to the study of the local analytic structure of the scattering amplitudes, from the point of view of the general principle of quantum field theory. It is an interesting problem to compare these results with those obtained in the framework of pure S matrix theory, as developed especially by Stapp and co-workers [8]–[10]. In this connection we draw the attention of the reader to a very recent investigation by Cahill and Stapp [28] about the links between the algebraic aspects of the two points of view.

It is, however, clear that the postulated cluster properties for the S matrix with *exponential* rates of decrease, imply a richer *local* analytic structure of the amplitudes near the physical regions, than the corresponding structure obtained on the basis of local field theory: this is because, under the name of 'macrocausal laws', the S matrix theory includes from the beginning the assumption of the short range character of strong interactions together with relaxation-type assumptions and the usual principle of causality.

Concerning the continuity properties of the scattering amplitudes, as discussed in section 2, it would be interesting to compare them with the analogous analysis by Williams [29], who approaches the problem with rather different methods.

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