Statistical description of elementary processes. I, Single particle theory

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Statistical Description of Elementary Processes[1]) I. Single Particle Theory

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Abstract. As a basis for the formulation of a statistical description of elementary processes, it is assumed that the measurements set up to characterize the quantum state of certain systems may not form (non-trivially) a complete set. The consequences of this assumption for the structure of quantum mechanical states and their evolution in time is investigated, and some aspects of scattering theory are discussed. A representation in terms of an overcomplete family of states is suggested by this structure, and its time evolution is studied. The phenomena of induced dispersion and broken symmetries are discussed.

1. Introduction

In recent years, there has been a significant effort devoted to the construction of statistical models for elementary processes, such as scattering and particle production [1–4]. The models proposed by Yang and co-workers [5, 6] and Feynman [7] also appear to have considerable statistical content. A fundamental basis for the statistical description of elementary processes exists within the structure of ordinary quantum mechanics; it is the purpose of this paper to formulate this idea and to indicate its consequences for scattering theory. In a succeeding paper, the many body problem will be treated, and its consequences for quantum field theory indicated.

In principle, a quantum mechanical state is specified by the measurements (ideal and of the first kind) of a complete set of commuting observables. For phenomena for which a complete theory is not available (e.g., involving strong interactions), it is conceivable that the measurements set up to characterize the quantum state of a system do not form (non-trivially) a complete set. We shall assume that they do not, and investigate the consequences of this assumption, for the structure of quantum mechanical states and their evolution in time, in the next section. In the third section some aspects of scattering theory are discussed, and in section 4 the algebraic implications of our assumption of incomplete measurement is shown to suggest a representation in terms of an overcomplete family of states [8] bearing some analogy to coherent states. The time evolution of these states is also studied. In the last section, the phenomena of induced dispersion and broken symmetries are discussed. It is suggested that there

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may exist, in principle, a set of measurements (not of the type ordinarily made) for which apparently broken symmetries, such as $SU(3)$, become exact.

2. States and Their Evolution in Time

To make our formulation precise, we assume that there exists a *special* complete set of commuting observables which can be divided into two independent subsets, α and β, of which α is not observed but β is. In the absence of a complete theory, the identity of this special set is to be considered as not known. We shall, however, make use of it, mathematically, to establish the structure of representations which are experimentally accessible. In general, other choices of complete commuting sets of observables have the property that each element is a function of the entire set $\{\alpha,\beta\}$. There will, therefore, be a restriction on the experimenter's ability to measure an arbitrary complete set with absolute precision; he can, in fact, only define a mixed state. This situation, of course, always prevails in an actual experiment. It is generally considered to be simply a practical limitation. What we wish to show, however, is that in certain classes of phenomena this limitation can be rather serious, influencing strongly our view of the fundamental dynamical mechanisms.

Let ρ correspond to the density matrix characterizing the state which could, in principle, be constructed with the help of a complete set of measurements (this includes the possibility that ρ could be a one-dimensional orthogonal projection corresponding to a pure state). Represented in terms of the eigenvectors of the special set $\{\alpha,\beta\}$, ρ has the form

$$\rho = \sum_{\alpha',\alpha'',\beta',\beta''} |\alpha'\beta'\rangle\langle\alpha'\beta'|\rho|\alpha''\beta''\rangle\langle\alpha''\beta''|. \tag{2.1}$$

The subset α cannot, according to our hypothesis, be subjected to measurement; assuming that the values of α are distributed uniformly and, for simplicity, finite (N_α) in number, the arbitrary density matrix (2.1) is reduced by incomplete measurement to the form [9]

$$\check{\rho} = \frac{1}{N_\alpha} \sum_{\alpha',\alpha'',\beta',\beta''} |\alpha'\beta'\rangle\langle\alpha''\beta'|\rho|\alpha''\beta''\rangle\langle\alpha'\beta''|. \tag{2.2}$$

In obtaining (2.2), we have taken advantage of the fact that α and β are independent; hence the representation is of direct product form, and we can carry out the trace over the α factor of the product space, normalize, and restore the definition of the contracted density matrix to the original Hilbert space by making it proportional to the unit matrix in the α factor.

It has also been assumed that the distribution over the spectrum of unmeasured variables is uniform. In their analysis (in some ways formally quite similar to ours) of a system immersed in a thermal bath, Favre and Martin [10] point out that the uniform distribution in unmeasured variables corresponds to infinite temperature. They further remark that it is only in this case that the operator in Liouville space (to be discussed below) mapping density operators of the form (2.1) into contracted form is projective. If there were reason, in some application, to assume some other distribution, it can be introduced in a natural way, as will be discussed later in this section and in section 4.

To eliminate explicit reference to the (presumably unknown) set of observables $\{\alpha, \beta\}$, we express ρ in terms of an arbitrary complete set of observables A, with eigenvalues a', and obtain

$$\langle a'''|\check{\rho}|a^{\mathrm{IV}}\rangle = \sum_{a'a''} \langle a'|\rho|a''\rangle\, \mathfrak{p}^{a'a''a'''a^{\mathrm{IV}}}, \tag{2.3}$$

where

$$\mathfrak{p}^{a'a''a'''a^{\mathrm{IV}}} = \frac{1}{N_\alpha} \sum_{\alpha',\alpha'',\beta',\beta''} \langle\alpha'\,\beta'|a'\rangle\langle\alpha'\,\beta''|a''\rangle^*\langle\alpha''\,\beta'|a'''\rangle^*\langle\alpha''\,\beta''|a^{\mathrm{IV}}\rangle. \tag{2.4}$$

Taking a convention of right multiplication, $\mathfrak{p}$ appears, in (2.3), to act as a linear operator on the vector space $\mathfrak{L}$ (to be called Liouville space [11])spanned by the positive unity-trace operators ρ. (The elements of the Liouville space are the Hilbert–Schmidt type operators acting on the original Hilbert space.) We may therefore write

$$\check{\rho} = \rho\mathfrak{p} \tag{2.5}$$

for equation (2.3). It is easily seen from (2.4) that $\mathfrak{p}$ has the following properties:

$$\sum_{a''} \mathfrak{p}^{a'a''a''a'''} = \delta(a', a''), \tag{2.6a}$$

$$(\mathfrak{p}^{a'a''a'''a^{\mathrm{IV}}})^* = \mathfrak{p}^{a'''a^{\mathrm{IV}}a'a''}, \tag{2.6b}$$

$$\sum_{a'''a^{\mathrm{IV}}} \mathfrak{p}^{a'a''a'''a^{\mathrm{IV}}}\, \mathfrak{p}^{a'''a^{\mathrm{IV}}a^{\mathrm{V}}a^{\mathrm{VI}}} = \mathfrak{p}^{a'a''a^{\mathrm{V}}a^{\mathrm{VI}}}, \tag{2.6c}$$

and

$$\mathfrak{p}^{a'a''a'''a^{\mathrm{IV}}} = \mathfrak{p}^{a^{\mathrm{IV}}a'''a''a'}. \tag{2.6d}$$

The first of (2.6) assures that $\mathrm{Tr}\check{\rho} = \mathrm{Tr}\rho$, and the second that $\mathfrak{p}$ is formally self-adjoint, i.e., that $\mathrm{Tr}(A^\dagger(B\mathfrak{p})) = \mathrm{Tr}((A\mathfrak{p})^\dagger B)$, for A and B Hilbert–Schmidt operators on the original Hilbert space. Equation (2.6b) together with (2.6c) implies that $\mathfrak{p}$ is an orthogonal projection onto a subspace of $\mathfrak{L}$; we shall call this subspace $\mathfrak{L}\mathfrak{p}$. The dimensionality of $\mathfrak{L}\mathfrak{p}$ is $\mathrm{Tr}_{\mathfrak{L}}\,\mathfrak{p} = N_\beta^2$. Given the property (2.6b), (2.6d) is necessary and sufficient for $(A\mathfrak{p})^\dagger = A^\dagger\mathfrak{p}$. If ρ is positive, it follows by construction (and explicitly from (2.4)) that $\check{\rho}$ is positive also.

Using the symmetry property (2.6d), we can define left multiplication

$$(\mathfrak{p}\rho)_{a'''a^{\mathrm{IV}}} = \sum_{a'a''} \mathfrak{p}^{a^{\mathrm{IV}}a'''a''a'}\, \rho_{a'a''} \tag{2.7}$$

so that $\mathfrak{p}\rho = \rho\mathfrak{p}$, and $\mathrm{Tr}(A(\mathfrak{p}B)) = \mathrm{Tr}((A\mathfrak{p})\,B)$. In general, however, $A(\mathfrak{p}B) \neq (A\mathfrak{p})\,B$.

With the representation (2.5) for density matrices corresponding to states which can be determined (or prepared) by incomplete measurement, we can discuss the time evolution of $\check{\rho}$ in a simple way. To do this, we first remark that $\mathfrak{p}$ has the geometrical significance of projecting the positive convex subset $\mathfrak{L}'$ (states) of $\mathfrak{L}$ into $\mathfrak{L}'\,\mathfrak{p}$ (a positive convex subset of $\mathfrak{L}'$); it is independent of time. In a Hamiltonian formalism, the density matrix ρ changes with time in the usual way, i.e.,

$$i\frac{d\rho}{dt} = [H, \rho]. \tag{2.8}$$

Defining the Liouville operator

$$\mathfrak{H}^{a'a''a'''a^{\mathrm{IV}}} = -\mathfrak{H}^{a^{\mathrm{IV}}a'''a''a'} = (\mathfrak{H}^{a''a^{\mathrm{IV}}a'a'''})^*, \tag{2.9}$$

such that

$$\rho\mathfrak{H} = [H, \rho], \tag{2.10}$$

we obtain

$$i\frac{d}{dt}\check{\rho} = \rho\mathfrak{H}\mathfrak{p}. \tag{2.11}$$

This equation is of a form familiar in non-equilibrium statistical mechanics, where the corresponding projection is often taken to be to the diagonal of ρ. In our case, $\mathfrak{p}$ is effectively a projection to diagonal form in the α factor (of the direct product space spanned by $\{|\alpha'\beta'\rangle\}$). The interpretation of (2.11) is very similar to that of a subsystem immersed in a thermal bath [10], but we emphasize that in our case, the thermal bath is provided by unmeasured degrees of freedom of an 'elementary' quantum mechanical system.

Introducing the unit operator in $\mathfrak{L}$,

$$I^{a'a''a'''a^{\mathrm{IV}}} = \delta(a', a''')\,\delta(a'', a^{\mathrm{IV}})$$

we may write $\bar{\mathfrak{p}} = I - \mathfrak{p}$ and obtain

$$i\frac{d}{dt}\rho\mathfrak{p} = \rho\mathfrak{p}\mathfrak{H}\mathfrak{p} + \rho\bar{\mathfrak{p}}\mathfrak{H}\mathfrak{p} \tag{2.12}$$

$$i\frac{d}{dt}\rho\bar{\mathfrak{p}} = \rho\mathfrak{p}\mathfrak{H}\bar{\mathfrak{p}} + \rho\bar{\mathfrak{p}}\mathfrak{H}\bar{\mathfrak{p}}. \tag{2.13}$$

Combining (2.12) and (2.13), one obtains the master equation (see also [10])

$$i\frac{d}{dt}\check{\rho}(t) = \check{\rho}(t)\,\mathfrak{p}\mathfrak{H}\mathfrak{p} + \rho(0)\,\bar{\mathfrak{p}}\exp(-i\bar{\mathfrak{p}}\mathfrak{H}\bar{\mathfrak{p}}t)\,\mathfrak{H}\mathfrak{p}$$
$$-i\int_0^t d\tau\check{\rho}(\tau)\,\mathfrak{p}\mathfrak{H}\bar{\mathfrak{p}}\exp(-i\bar{\mathfrak{p}}\mathfrak{H}\bar{\mathfrak{p}}(t-\tau))\,\mathfrak{H}\mathfrak{p}, \tag{2.14}$$

where we have taken initial conditions at $t = 0$. It should be recognized that $\rho(0)$ cannot be prepared in an arbitrary way, but that it must also be of the form $\rho(0)\,.\,\mathfrak{p}$.

In this case, the second term of (2.14) automatically vanishes. The formal solution of (2.14) (with $\rho(0) = \rho(0)\,\mathfrak{p}$),

$$\breve{\rho}(t) = (U(t)\,\rho(0)\,U^{+}(t))\,\mathfrak{p} = \rho(0)\,\mathfrak{U}(t)\,\mathfrak{p}$$
$$= \rho(0)\,\mathfrak{p}\mathfrak{U}(t)\,\mathfrak{p}, \tag{2.15}$$

corresponds to an evolution of the system undisturbed by measurement except at time $t = 0$ (to prepare the state) and at time t. If at time t, $\breve{\rho}(t)$ is taken as the initial state for further undisturbed evolution, for example, to t', then the state at t' will be

$$\rho(0)\,\mathfrak{p}\mathfrak{U}(t)\,\mathfrak{p}\mathfrak{U}(t'-t)\,\mathfrak{p}.$$

In general, the reduced motion $\mathfrak{p}\mathfrak{U}(t)\,\mathfrak{p}$ does not have the semigroup property, i.e.,

$$\mathfrak{p}\mathfrak{U}(t)\,\mathfrak{p}\mathfrak{U}(t'-t)\,\mathfrak{p} \neq \mathfrak{p}\mathfrak{U}(t')\,\mathfrak{p}.$$

The analogous situation for reduced motions in the original Hilbert space (often studied as a model for decay systems) has been discussed by Horwitz and Marchand [12].

Equation (2.14) is of non-Schrödinger type; there is, in general, no unitary transformation that will account for the evolution of $\breve{\rho}(t)$. Since $\mathrm{Tr}(\rho\mathfrak{H}\mathfrak{p}) = \mathrm{Tr}(\rho\mathfrak{H}) = 0$, $\mathrm{Tr}\,\breve{\rho} = 1$ is conserved, but the *entropy* of the state $\breve{\rho}$ is not.

For the entropy, we take the convex function [13]

$$S = -k \ln \mathrm{Tr}\,\rho'^2 = -k \ln \mathrm{Tr}(\rho\mathfrak{p}\rho). \tag{2.16}$$

Conservation of the entropy is equivalent to the vanishing of

$$\frac{d}{dt}\,\mathrm{Tr}(\rho\mathfrak{p}\rho).$$

However,

$$i\frac{d}{dt}\,\mathrm{Tr}(\rho\mathfrak{p}\rho) = \mathrm{Tr}(\rho\mathfrak{H}\mathfrak{p}\rho + \rho\mathfrak{p}(\rho\mathfrak{H}))$$
$$= \mathrm{Tr}(\rho(\mathfrak{H}\mathfrak{p} - \mathfrak{p}\mathfrak{H})\rho), \tag{2.17}$$

which is not, in general, zero if the Liouville space operators $\mathfrak{H}$ and $\mathfrak{p}$ do not commute.

It is easy to show that the $\mathfrak{H}_0$ corresponding to

$$H_0 = H_\alpha \otimes I_\beta + I_\alpha \otimes H_\beta \tag{2.18}$$

commutes with $\mathfrak{p}$. The general Hamiltonian will have an additional term, V, which is not of the form (2.18). We might expect, however, that V will be small, since it couples the action of H on α and β factors. If V were too strong, the measurement apparatus could not be sensitive to the β variables alone. It is in this sense that the α variables can play the role of a thermal bath for the β variables, and an analysis from the point of view of statistical mechanics can be applicable. As Favre and Martin [10] have pointed out, a very large number of degrees of freedom of the unmeasured variables can lead to irreversible behaviour of the measured subsystem. In our case, this can be interpreted as resonance or particle decay.

As an example of such a phenomenon, consider the system composed of a charged particle and its radiation field, i.e., the vacuum. It is possible to have an infinite number of very low energy photons (corresponding to 'fluctuations') in this vacuum, and a complete measurement to specify the state of the system cannot be constructed. Only certain characteristics of the particle can be determined experimentally, but it is difficult to separate, theoretically, the particle from its radiation field. Neither the particle nor its associated vacuum can be considered as isolated systems, and in equilibrium the vacuum should be described by a canonical (or grand canonical) ensemble. Recent treatments of the infrared problem of electrodynamics suggest a treatment from this point of view [14]. The problem is not appreciably different for fermions, such as neutrinos, because arbitrarily many pairs can be created with very little energy.

This argument can be extended to strongly interacting systems. In the neighbourhood of a proton, for example, a large number of (virtual) pions can be expected as vacuum fluctuations. This large number of degrees of freedom can act as a thermal bath for the proton subsystem, where the separation into proton and physical vacuum is effectively the process of renormalization. In a high energy collision, the temperature of the thermal bath can be raised, resulting in a 'boiling off' of real particles, as in Hagedorn's model [2].

For the unstable particles, such as the ρ or K° (or even the neutron), we may think of an evolution of the state of the system for which equilibrium cannot be achieved for the 'single particle' subsystem; during a finite period of its evolution, however, the constraints imposed by some particle definition can be satisfied with reasonable likelihood. As in the case of high energy collisions, the subsystem appears to be placed in an environment that is too hot to preserve its integrity, and a phase transition takes place. The equations of time evolution given in this section [(2.14), for example] appear suitable for the description of such irreversible processes. We have so far, however, considered only the case of 'infinite temperature'.

To ascribe a finite temperature T to the system, we can assume [for a large number of degrees of freedom in the α variables and only weak coupling to be added to (2.18)] that the α 'system' is in thermal equilibrium, with a distribution

$$\rho_\alpha(T) = e^{-H\alpha/kT} Z_\alpha^{-1}, \tag{2.19}$$

where

$$Z_\alpha = \mathrm{Tr}_\alpha\, e^{-H_\alpha/kT}. \tag{2.20}$$

The appropriate contraction would then be

$$\check{\rho} = \sum_{\alpha'\ldots\beta''} |\alpha'\,\beta'\rangle\langle\alpha''\,\beta'|\rho|\alpha''\,\beta''\rangle\langle\alpha'|\rho_\alpha(T)|\alpha'''\rangle\langle\alpha'''\beta''|; \tag{2.21}$$

the corresponding contraction operator in Liouville space is given by

$$\mathfrak{p}(T)^{a'a''a'''a^{\mathrm{IV}}} = \sum_{\alpha'\ldots\beta''} \langle\alpha'\,\beta'|a'\rangle\langle\alpha'\,\beta''|a''\rangle^*\langle\alpha''\,\beta'|a'''\rangle^*\,\langle\alpha''|\rho_\alpha(T)|\alpha'''\rangle\langle\alpha'''\,\beta''|a^{\mathrm{IV}}\rangle. \tag{2.22}$$

Since $\mathrm{Tr}_\alpha \rho_\alpha(T) = 1$, $\mathfrak{p}(T)^2 = \mathfrak{p}(T)$, but (2.22) is not a self-adjoint operator in the Liouville space unless ρ_α is the uniform distribution $1/N_\alpha$ [this follows immediately from (2.22) by choosing the α basis such that ρ_α is diagonal]. The symmetry property (2.6d) is also valid only for $T \to \infty$. We shall restrict ourselves, in most of what follows, to the limit of uniform distribution in α.

3. Scattering

A more complete discussion of the asymptotic conditions for scattering will be given elsewhere. In this section, a brief outline is given of the general features of a scattering theory in the context of incomplete measurement.

We shall consider a non-relativistic scattering theory, where

$$H = H_0 + V \tag{3.1}$$

The interaction picture evolution, for time independent H, is formally defined as

$$U_I(t, t_0) = e^{iH_0 t}\, e^{-iH(t-t_0)}\, e^{-iH_0 t_0}, \tag{3.2}$$

so that

$$\rho(t) = \rho(0)\, \mathfrak{U}_I(t, 0)\, \mathfrak{U}_0(t). \tag{3.3}$$

If $\rho(0) \in \mathfrak{L}\mathfrak{p}$, as a state prepared initially at $t = 0$, the contracted motion is given by

$$\breve{\rho}(t) = \rho(t)\, \mathfrak{p} = \rho(0)\, \mathfrak{p}\mathfrak{U}_I(t, 0)\, \mathfrak{U}_0(t)\, \mathfrak{p}. \tag{3.4}$$

Since the asymptotic in and out states in our case of mixed states are to be accessible to measurement, they must belong to $\mathfrak{L}\mathfrak{p}$; they must, moreover, evolve with the unperturbed Hamiltonian. If a measurement is made at some arbitrary time t, the state will be found as predicted by (3.4), and for $t \to \mp\infty$, we will want convergence to zero, in some sense, of

$$\rho(0)\, \mathfrak{p}\mathfrak{U}_I(t, 0)\, \mathfrak{U}_0(t)\, \mathfrak{p} - \rho^{\substack{\mathrm{in}\\ \mathrm{out}}}\, \mathfrak{U}_0(t)\, \mathfrak{p}, \tag{3.5}$$

where $\rho^{\substack{\mathrm{in}\\ \mathrm{out}}} = \rho^{\substack{\mathrm{in}\\ \mathrm{out}}}\mathfrak{p}$.

If the wave operators exist for the V of (3.1) in the usual strong limit [15], then $\mathfrak{U}_I(t, 0)$ converges strongly for $t \to \mp\infty$. This does not, however, guarantee that $\lim \rho(0)\, \mathfrak{p}\mathfrak{U}_I(t, 0)$ will lie in $\mathfrak{L}\mathfrak{p}$; if it does not, there will be, in general, no convergence of (3.5) to zero. From this point of view, the existence of asymptotic states in $\mathfrak{L}\mathfrak{p}$ implies a stronger restriction on the admissible class of interactions than the requirement that the wave operator exist.

On the other hand, the notion of an asymptotic state for a stable system suggests an entropy conserving evolution, i.e., of Schrödinger type. In this case, $\mathfrak{U}_0(t)$ must commute with $\mathfrak{p}$, and the unperturbed Hamiltonian H_0 must be of the form (2.18), without coupling between the α and β factors. It is not necessary, however, to choose H_0 in this special way. If $\mathfrak{U}_0(t)$ and $\mathfrak{p}$ do not commute, the convergence to zero of (3.5) implies an evolution, asymptotically, of the form $\rho^{\substack{\mathrm{in}\\ \mathrm{out}}}\, \mathfrak{p}\mathfrak{U}_0(t)\, \mathfrak{p}$, with which one might construct a model for the production and decay of resonant states (unstable particles).

Jauch, Misra and Gibson [16] have shown that a physically justified metric between states is given by the trace norm, defined for compact A as $\|A\|_1 = \mathrm{Tr}\sqrt{A^\dagger A}$ (this is equivalent to the operator norm $\|A\|$). If $\mathfrak{U}_0$ commutes with $\mathfrak{p}$, the convergence of the trace norm of (3.5) to zero implies that $\mathfrak{p}\mathfrak{U}_I(t,0)\,\mathfrak{p}$ converges; we define, for convergence in this norm,

$$\check{\Omega}_{\pm} = \mathrm{tr} - \lim_{t \to \mp\infty} \mathfrak{p}\mathfrak{U}_I(0,t)\,\mathfrak{p}, \tag{3.6}$$

and $\check{\Omega}_{\pm}^{\dagger}$ as the Liouville operator inducing the conjugate transformation (it is the formal conjugate of $\check{\Omega}_{\pm}$ in $\mathfrak{L}$ also). It therefore follows, in the sense of trace norm equivalence, that

$$\rho^{\substack{\mathrm{in}\\ \mathrm{out}}} = \rho(0)\,\check{\Omega}_{\pm}^{\dagger} \tag{3.7}$$

The transition probability is given by

$$\mathrm{Tr}(\rho^{\mathrm{out}}\,\rho^{\mathrm{in}}) = \mathrm{Tr}[(\rho'(0)\,\check{\Omega}_{-}^{\dagger}\,\check{\Omega}_{+})\,\rho(0)], \tag{3.8}$$

but although $\mathrm{Tr}\,\rho^{\substack{\mathrm{in}\\ \mathrm{out}}} = \mathrm{Tr}\,\rho(0)$, $\check{\Omega}_{-}^{\dagger}\,\check{\Omega}_{+}$ is not unitary. The operator $\check{\Omega}_{-}^{\dagger}\,\check{\Omega}_{+} = \mathfrak{p}\check{\Omega}_{-}^{\dagger}\,\mathfrak{p}\check{\Omega}_{+}\mathfrak{p}$ is not, in general, simply related to the S matrix for the corresponding pure state scattering problem.

We finally remark that the condition that $\mathfrak{p}\mathfrak{U}_I(t,0)\,\mathfrak{p}$ converge for $t \to \mp\infty$ can be weakened in a natural way. The operator $\check{\rho}(t)$ corresponds to a state evolving in time which is of an essentially statistical nature. As in statistical mechanics, we are interested in the values of observables which at large times assume stationary values *on the average*. Hence we consider $\langle \mathrm{Tr}(\check{\rho}(t)0)\rangle_T$, a suitable average over time of the expectation value of an observable which is expected to have macroscopic utility asymptotically. We therefore obtain a condition of the form

$$\langle \check{\rho}(t) - \rho^{\substack{\mathrm{in}\\ \mathrm{out}}}\,\mathfrak{U}_0(t)\,\mathfrak{p}\rangle_T \to 0. \tag{3.9}$$

Let us assume for definiteness that the time average is of the ergodic form, and that (assuming that $\mathfrak{U}_0(t)$ and $\mathfrak{p}$ commute) $\rho^{\substack{\mathrm{in}\\ \mathrm{out}}}$ can be chosen an eigenstate in unperturbed energy. Then (3.9) becomes, for example, at $T \to \infty$ [17]

$$\lim_{T \to +\infty} \left\| \frac{1}{T} \int_0^T \rho(0)\,\mathfrak{U}(t)\,\mathfrak{p}\,dt - \rho^{\mathrm{out}} \right\|_1 = 0, \tag{3.10}$$

where $\mathfrak{U}(t)$ is the full evolution. Let

$$\check{R}_+(z, T) = -i \int_0^T \mathfrak{p}\mathfrak{U}(t)\,\mathfrak{p}e^{izt}\,dt. \tag{3.11}$$

The function (3.11) is analytic for $\mathrm{Im}\,z > 0$ when $T \to +\infty$, and corresponds to the resolvent of the Liouville operator $\mathfrak{H}$, contracted to the subspace $\mathfrak{L}\mathfrak{p}$,

$$\lim_{T \to \infty} \check{R}_+(z, T) = \check{R}_+(z) = \mathfrak{p}\,\frac{1}{z - \mathfrak{H}}\,\mathfrak{p}. \tag{3.12}$$

In trace norm, (3.10), implies

$$\lim_{T \to \infty}\ \lim_{z \to 0\downarrow} \frac{i}{T}\,\rho(0)\,\check{R}_+(z, T) = \rho^{\mathrm{out}}; \tag{3.13}$$

an interchange of limits would, however, result in 0. It therefore follows that (in trace norm or operator norm), the limiting process is not uniformly convergent. If this non-uniform convergence can be accounted for by a singularity in (3.12), then it is just this singularity of the resolvent $\check{R}_+(z)$, at $z \to 0$ from above, that contains ρ^{out} in its range. This result is in agreement with the idea, expressed by Prigogine, George and Henin [18], that the approach to equilibrium occurs in the subspace of the Liouville space in the range of the singularity of the resolvent of the evolution operator at $z = 0$.

4. Overcomplete Families of States

In this section, we discuss the algebraic implications of incomplete measurement for the structure of the quantum mechanical Hilbert space.

Performing the contraction operation $\mathfrak{p}$ on pure states, one obtains a set of states that are minimally mixed. These correspond to the best that can be done in preparing a state with the assumed limitation on the measuring apparatus available. The algebra of these special density matrices (corresponding to the action of repeated measurement) can be closed by enlarging it in a simple way, and its factored representation [19] yields a set of vectors in a linear vector space closely related to the original Hilbert space. These vectors form overcomplete families of states which are analogous to the generalized coherent states defined by Klauder [8].

Equation (2.2) can be put into a form which is more convenient than (2.5) for algebraic study. Defining

$$\bar{M}(\alpha', \alpha'') = \sum_{\beta'} |\alpha' \beta'\rangle \langle \alpha'' \beta'|, \tag{4.1}$$

(2.2) becomes

$$\check{\rho} = \rho \mathfrak{p} = \frac{1}{N_\alpha} \sum_{\alpha'\alpha''} \bar{M}(\alpha', \alpha'')\, \rho \bar{M}(\alpha'', \alpha'). \tag{4.2}$$

If ρ is a pure state, it can be represented as $M(a') = |a'\rangle\langle a'|$, where we follow Schwinger's notation [19] for measurements of the first kind. In Schwinger's development of quantum mechanics, the closed algebra of measurements of the first kind as well as the most general type, measurements of the second kind $M(a', b')$, is found to be represented by the direct product of Hilbert space vectors and dual vectors, thus giving rise to the Hilbert space of states. The algebra of minimally mixed states,

$$M(a')\,\mathfrak{p} = \frac{1}{N_\alpha} \sum_{\alpha'\alpha''} \bar{M}(\alpha', \alpha'')\, M(a')\, \bar{M}(\alpha'', \alpha'), \tag{4.3}$$

however, does not close. The product of two incomplete measurements is (using $\bar{M}(\alpha', \alpha'')\,\bar{M}(\alpha''', \alpha^{\mathrm{IV}}) = \delta(\alpha'', \alpha''')\,\bar{M}(\alpha', \alpha^{\mathrm{IV}})$)

$$\begin{aligned}(M(a')\,\mathfrak{p})(M(a'')\mathfrak{p}) &= \frac{1}{N_\alpha^2} \sum_{\alpha'\alpha''\alpha'''} \bar{M}(\alpha', \alpha'')\, M(a')\, \bar{M}(\alpha'', \alpha''')\, M(a'')\, \bar{M}(\alpha''', \alpha') \\ &= \frac{1}{N_\alpha^2} \sum_{\alpha'\alpha''\alpha'''} \langle a' | \bar{M}(\alpha'', \alpha''') | a''\rangle\, M(\alpha', \alpha'')\, \bar{M}(a', a'')\, \bar{M}(\alpha''', \alpha'). \end{aligned} \tag{4.4}$$

We are therefore led to define the generalised incomplete measurement of the second kind,

$$M_{\alpha'\alpha''}(a', a'') = \sum_{\alpha'''} \bar{M}(\alpha''', \alpha')\, M(a', a'')\, \bar{M}(\alpha'', \alpha''').\tag{4.5}$$

Since the $\{M(a' a'')\}$ form a basis for any operator, we define

$$K_{\alpha'\alpha''}(A) = \sum_{\alpha'''} \bar{M}(\alpha''', \alpha')\, A \bar{M}(\alpha'', \alpha''')\tag{4.6}$$

as the commutative kernel of A, i.e., the part of A which commutes with the subalgebra α (note that $\sum_{\alpha'} K_{\alpha'\alpha'}(A) = A\mathfrak{p}$). The operation implied by (4.6) on A is invertable:

$$A = \sum_{\alpha'\alpha''} K_{\alpha'\alpha''}(A)\, \bar{M}(\alpha', \alpha''),\tag{4.7}$$

so that the commutative kernel is a representation of the operator, somewhat analogous to the diagonal representation of an operator in a basis of coherent states [8]. The representation (4.7) is analogous to the usual Frobenius form [20]; implications for the structure of the associated Hilbert space have been studied in a particularly simple case [21].

The set of operators defined by (4.5) belongs to a closed algebra which contains

$$M(a')\mathfrak{p} = \frac{1}{N_\alpha} \sum_{\alpha'} M_{\alpha'\alpha'}(a', a');$$

the products of elements of this algebra have the form:

$$M_{\alpha'\alpha''}(a', a'') M_{\alpha'''\alpha^{\mathrm{IV}}}(a''' a^{\mathrm{IV}}) = \langle a'' | \bar{M}(\alpha'', \alpha''') | a''' \rangle\, M_{\alpha'\alpha^{\mathrm{IV}}}(a', a^{\mathrm{IV}}).\tag{4.8}$$

Following Schwinger's argument, we construct a representation of the generalized incomplete measurements of the second kind in terms of a vector space $\{|a'(\alpha'))\}$, with scalar product

$$(a'(\alpha') | a''(\alpha'')) = \langle a' | \bar{M}(\alpha', \alpha'') | a'' \rangle.\tag{4.9}$$

In terms of these vectors,

$$M_{\alpha'\alpha''}(a', a'') = |a'(\alpha'))(a''(\alpha'')|,\tag{4.10}$$

$$M(a')\, \mathfrak{p} = \frac{1}{N_\alpha} \sum_{\alpha'} |a'(\alpha'))(a'(\alpha')|,\tag{4.11}$$

and we note that

$$\begin{aligned}
\sum_{a'} |a'(\alpha'))(a'(\alpha')| &= \sum_{a'} M_{\alpha'\alpha'}(a', a') \\
&= \sum_{a'\alpha'''} \bar{M}(\alpha''', \alpha')\, M(a')\, \bar{M}(\alpha', \alpha''') \\
&= \sum_{\alpha'''} \bar{M}(\alpha''') = I,
\end{aligned}\tag{4.12}$$

so the vectors $\{|a'(\alpha'))\}$ form a complete set. This set is, in fact, *overcomplete*, since $\{|a'\rangle\}$ is a complete set itself. As pointed out by Klauder [8], such overcomplete sets are characterized by self-reproducing kernels; (4.9) is the appropriate kernel in our case:

$$\sum_{a''} \langle a'|\bar{M}(\alpha', \alpha'')|a''\rangle \langle a''|\bar{M}(\alpha''', \alpha^{\mathrm{IV}})|a'''\rangle$$
$$= \delta(\alpha'', \alpha''')\langle a'|\bar{M}(\alpha', \alpha^{\mathrm{IV}})|a'''\rangle \tag{4.13}$$

By completeness,

$$\sum_{a'} |a'(\alpha'))(a'(\alpha')|a''(\alpha'')) = |a''(\alpha''))$$

and hence

$$\sum_{a'} |a'(\alpha'))\, \langle a'|\bar{M}(\alpha', \alpha'')|a''\rangle = |a''(\alpha'')). \tag{4.14}$$

Since

$$\sum_{a'} M_{\alpha'\alpha''}(a', a') = \delta(\alpha', \alpha'')\,.I, \tag{4.15}$$

a more general form of (4.14) is

$$\sum_{a'} |a'(\alpha'))\langle \alpha'|\bar{M}(\alpha'', \alpha''')|a''\rangle = \delta(\alpha', \alpha'')|a''(\alpha''')). \tag{4.16}$$

To make explicit the structure of these vectors, we write (4.2) in a different way:

$$\check{\rho} = \frac{1}{N_\alpha} \sum_{\alpha' \ldots \beta''} |\alpha' \beta'\rangle \langle \alpha'' \beta'|\rho|\alpha'' \beta''\rangle \langle \alpha' \beta''|$$
$$= \sum_{\beta'\beta''} \bar{M}(\beta', \beta'')\langle \beta'\|\rho\|\beta''\rangle, \tag{4.17}$$

where

$$\langle \beta'\|\rho\|\beta''\rangle = \frac{1}{N_\alpha} \sum_{\alpha''} \langle \alpha'' \beta'|\rho|\alpha'' \beta''\rangle \tag{4.18}$$

is independent of unitary transformations on the α factor, and

$$\bar{M}(\beta', \beta'') = \sum_{\alpha'} M(\alpha' \beta', \alpha' \beta''). \tag{4.19}$$

The second-kind measurement operators $\bar{M}(\beta' \beta'')$ form a basis for the contracted states $\check{\rho}$, and satisfy an algebra identical in structure to the algebra of $M(a', a'')$:

$$\bar{M}(\beta', \beta'')\, \bar{M}(\beta''', \beta^{\mathrm{IV}}) = \delta(\beta'', \beta''')\, \bar{M}(\beta', \beta^{\mathrm{IV}})$$
$$\sum_{\beta'} \bar{M}(\beta') = I. \tag{4.20}$$

Hence we can construct a representation of $\bar{M}(\beta', \beta'')$ in the form

$$\bar{M}(\beta', \beta'') = |\beta')(\beta''|,$$
$$(\beta|\beta'') = \delta(\beta', \beta''). \tag{4.21}$$

According to the completeness property (4.20), an arbitrary operator B can be represented as

$$B = \sum_{\beta'\beta''} |\beta')(\beta'|B|\beta'')(\beta''|; \tag{4.22}$$

if $(\beta'|B|\beta'')$ were to be treated as a common number, then a comparison of (4.22) and (4.17) would imply the contradictory result that B is contracted, i.e., $B = B\mathfrak{p}$. Hence $(\beta'|B|\beta'')$ cannot, in general, commute with the vectors $\{|\beta')\}$. These vectors, in fact, form an 'algebraic' vector space. It has been shown [22] that such vector spaces (at least over finite algebras) can be closed as Hilbert spaces.

It follows from (4.19) that

$$|\beta')_{\alpha'} = |\alpha' \beta'\rangle \tag{4.23}$$

and that

$$|\beta')(\beta''| = \sum_{\alpha'} |\beta')_{\alpha'\alpha'}(\beta'|. \tag{4.24}$$

From (4.5), we see that

$$M_{\alpha'\alpha''}(a', a'') = \sum_{\alpha''', \beta', \beta''} |\alpha''' \beta'\rangle \langle \alpha' \beta'|a'\rangle \langle a''|\alpha'' \beta''\rangle \langle \alpha''' \beta''|,$$

so that (within an arbitrary phase),

$$|a'(\alpha')) = \sum_{\beta'} |\beta')\langle \alpha' \beta'|a'\rangle, \tag{4.25}$$

i.e.,

$$|a'(\alpha'))_{\alpha''} = \bar{M}(\alpha'', \alpha')|a'\rangle. \tag{4.26}$$

It is evident that a 'pure state' in this algebraic vector space appears (for example, $\bar{M}(\beta')$) as a mixed state in the original Hilbert space.

The 'scalars' of the algebraic vector space are the elements of the α subalgebra. Suppose $\alpha = \mathcal{O}_\alpha \otimes \mathbb{1}_\beta$ is an element of the α subalgebra; then

$$\alpha|a'(\alpha'))_{\alpha''} = \sum_{\alpha'''\beta'} |\alpha''' \beta'\rangle \langle \alpha''' \beta'|\alpha|\alpha'' \beta'\rangle \langle \alpha' \beta'|a'\rangle$$
$$= \sum_{\alpha'''} |a'(\alpha'))_{\alpha'''} (\boldsymbol{\alpha})_{\alpha'''\alpha''},$$

or

$$\alpha|a'(\alpha')) = |a'(\alpha'))\,\boldsymbol{\alpha}, \tag{4.27}$$

where $(\boldsymbol{\alpha})_{\alpha'''\alpha''} = \langle \alpha''' \beta'|\alpha|\alpha'' \beta'\rangle$ is independent of β'.

A Hilbert space spanned by vectors of the form $\{|a'(\alpha'))\}$ over 'scalars' belonging to the α subalgebra as in (4.27) has superselection rules in accordance with the representations of operators of the form α, provided that we accept as observables only those operators which commute with the α subalgebra [23]. Contracted operators, such as $\check{\rho}$, and the commutative kernels of arbitrary operators satisfy this condition, but we wish to admit a wider class of self-adjoint operators as observables. The superselection rules of the algebraic linear vector space may therefore be broken. As in the discussion of time dependence in the previous section, very strong breaking by an observable would not be consistent with the separation between α and β variables on the basis of measurability.

The effect of this breaking can be illustrated for the time evolution of the states $\{|a'(\alpha'))\}$. Let us consider a system in the pure state $|a',t\rangle = e^{-iHt}|a'\rangle$. The corresponding vector for the incompletely measured state is

$$|a'(\alpha')),t)_{\alpha''} = \bar{M}(\alpha'',\alpha')|a',t\rangle, \tag{4.28}$$

with the inversion

$$|a',t\rangle = \sum_{\alpha'} \bar{M}(\alpha',\alpha'')|a'(\alpha'),t)_{\alpha''}. \tag{4.29}$$

Provided that the α variables do not depend explicitly on time,

$$i\frac{\partial}{\partial t}|a'(\alpha'),t)_{\alpha''} = \bar{M}(\alpha'',\alpha')\,H|a',t\rangle,$$

or, with the help of (4.29) and the representation (4.7),

$$i\frac{\partial}{\partial t}|a'(\alpha'),t) = \sum_{\alpha''} K_{\alpha'\alpha''}(H)|a'(\alpha''),t). \tag{4.30}$$

Since

$$\sum_{\alpha''} K_{\alpha'\alpha''}(A)\,K_{\alpha''\alpha'''}(B) = K_{\alpha'\alpha'''}(AB),$$

it follows from (4.30) that

$$|a'(\alpha'),t) = \sum_{\alpha''} K_{\alpha'\alpha''}(e^{-iHt})|a'(\alpha'')). \tag{4.31}$$

and these states form a good representation for translations in time, i.e.,

$$\begin{aligned} e^{\tau(\partial/\partial t)}|a'(\alpha'),t) &= \sum K_{\alpha'\alpha''}(e^{-iH\tau})|a'(\alpha''),t) \\ &= \sum_{\alpha''} K_{\alpha'\alpha''}(e^{-iH(t+\tau)})|a'(\alpha'')). \end{aligned} \tag{4.32}$$

According to (4.30), different 'layers' of the one-dimensional subspace, corresponding to $|a',t\rangle$, generating the overcomplete set $|a'(\alpha'),t)$, will mix in time, unless H is diagonal in the α subalgebra. The evolution will not be generated by H unless it commutes with the α subalgebra[3]).

The scalar product

$$(a'(\alpha')|a'(\alpha'))_{\alpha''\alpha'''} = \delta_{\alpha''\alpha'''} \langle a'|\bar{M}(\alpha')|a'\rangle$$

or

$$(a'(\alpha')|a'(\alpha')) = \langle a'|\bar{M}(\alpha')|a'\rangle . \mathbb{1}_{\alpha}, \tag{4.33}$$

enables us to define a positive norm bounded by unity. The diagonal element of $(a'(\alpha'),t|a'(\alpha'),t)$ is

$$\sum_{\alpha''\alpha'''} \langle a'|K_{\alpha''\alpha'}(e^{iHt})\, K_{\alpha'\alpha'''}(e^{-iHt})\, \bar{M}(\alpha'',\alpha''')|a'\rangle,$$

and the norm (4.33) is therefore, in general, not conserved.

According to (4.33),

$$\sum_{\alpha'} (a'(\alpha')|a'(\alpha')) = \mathbb{1}_{\alpha},$$

and, in fact, with the help of (4.26),

$$\sum_{\alpha'} (a'(\alpha')|a''(\alpha')) = \delta(a',a'')\, \mathbb{1}_{\alpha}. \tag{4.34}$$

Interpreting $K_{\alpha'\alpha''}$ as an operator on a direct integral space $|a') = \{|a'(\alpha'))\}$, we see that (4.31) can be written as

$$|a',t) = K(e^{-iHt})|a') = e^{-iK(H)t}|a'). \tag{4.35}$$

Although $K(H)$ is self-adjoint in the complete direct integral space, with respect to the scalar product (4.44), it is not a self-adjoint operator in a component Hilbert space labelled by α'. The projective quality of $\bar{M}(\alpha'',\alpha')$ in (4.28) leads to the possibility of decay [12].

At finite temperature, i.e., when (2.21) is valid, (4.3) must be replaced by

$$M(a')\,\mathfrak{p}(T) = \sum_{\alpha'\alpha''\alpha'''} \bar{M}(\alpha',\alpha'')|a'\rangle\langle a'|\,\langle\alpha'|\rho_{\alpha}|\alpha'''\rangle\, \bar{M}(\alpha'',\alpha'''),$$

and (4.11) is therefore replaced by

$$M(a')\,\mathfrak{p}(T) = \sum_{\alpha'} |a'(\alpha'))\, \boldsymbol{\rho}_{\alpha}\, (a'(\alpha')|, \tag{4.36}$$

where $(\boldsymbol{\rho}_{\alpha}(T))_{\alpha''\alpha'''} = \langle\alpha''|\rho_{\alpha}(T)|\alpha'''\rangle$ is the distribution function for the α 'bath'. The algebra of minimally mixed measurements with elements of the form (4.36) does not close on quantities of the form (4.5). Closure could be obtained by replacing (4.10) by

$$|a'(\alpha'))\, \boldsymbol{\rho}_{\alpha}\, (T)^n (a''(\alpha'')| \tag{4.37}$$

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[3]) This condition is stronger than that of the commutivity of $\mathfrak{H}$ and $\mathfrak{p}$, which would assure Schrödinger evolution for $M(a')\mathfrak{p}$. The 'states' (4.28) have their origin in the closure of the algebra of minimally incomplete measurements, and are therefore associated with correlations between successive measurements.

for $n = 1, 2, \ldots$ There is no new mathematical content in this algebra, however, since the states $\{|a'(\alpha'))\}$ have already been defined for the infinite temperature limit.

For $T \to \infty$, the expectation value of an observable is given, for a minimally mixed state, by

$$\begin{aligned}\mathrm{Tr}(A\check{\rho}) &= \mathrm{Tr}(A(M(a')\,\mathfrak{p})) \\ &= \sum_{\alpha'} \mathrm{Tr}(A\,|a'(\alpha'))(a'(\alpha')|) \\ &= \sum_{\alpha'} \mathrm{tr}(a'(\alpha'|A|a'(\alpha')),\end{aligned}$$

where we write 'tr' to indicate a trace over the α subalgebra, i.e., the algebra to which scalar products among the $\{|a'(\alpha'))\}$ (and 'scalars' $\boldsymbol{\alpha}$) belong. For $T < \infty$,

$$\begin{aligned}\mathrm{Tr}(AM(a')\,\mathfrak{p}(T)) &= \mathrm{Tr} \sum_{\alpha'} (A\,|a'(\alpha'))\, \boldsymbol{\rho}_\alpha (a'(\alpha')|) \\ &= \sum_{\alpha'} \mathrm{tr}((a'(\alpha')|A|a'(\alpha'))\, \boldsymbol{\rho}_\alpha),\end{aligned} \tag{4.38}$$

so that $\boldsymbol{\rho}_\alpha$ appears as a density matrix in the α subalgebra $\sum_{\alpha'} (a'(\alpha')|A|a'(\alpha'))$ corresponds to the reduction of the operator A to its remaining degrees of freedom in this algebra.

Linear combinations of vectors in the original Hilbert space form new pure states with coefficients which have the interpretation of probability amplitudes. The vector space $\{|a'(\alpha'))\}$ is defined over coefficients which are non-trivial operators in the α factor, as we have seen in (4.27).

Let $c_{a'}$ belong to the α subalgebra. Then, we define

$$|c(\alpha')) = \sum_{a'} c_{a'}\,|a'(\alpha')) = \sum_{a'} |a'(\alpha'))\,\mathbf{c}_{a'},$$

and the value of an operator A in this new state is

$$\sum_{\alpha'} \mathrm{Tr}(A\,|c(\alpha'))(c(\alpha')|) = \sum_{\alpha',a'a''} \mathrm{tr}((a'(\alpha')|A|a''(\alpha'))\,\mathbf{c}^\dagger_{a'}\,\mathbf{c}_{\alpha''}). \tag{4.39}$$

The quantity $\mathbf{c}^\dagger_{a'}\,\mathbf{c}_{a''}$ appears in the place of $\boldsymbol{\rho}_\alpha$ in (4.48), and we therefore conclude that coefficients in the linear expansions of the states $\{|a'(\alpha'))\}$ induce statistical mixtures (in addition to transformations on the original Hilbert space).

5. Induced Dispersion and Broken Symmetries

Consider an observable A with spectral decomposition (we restrict our discussion to compact operators)

$$A = \sum_{a'} a'\, M(a'). \tag{5.1}$$

In the pure state represented by the density matrix $M(a')$, A is dispersionless. However, in the mixed state $M(a')\,\mathfrak{p}$, A will, in general, have non-vanishing dispersion

$$(\Delta A)^2 = \langle A^2 \rangle_{M(a')\mathfrak{p}} - \langle A \rangle^2_{M(a')\mathfrak{p}} \tag{5.2}$$

For an (almost) arbitrary function $f(a')$,

$$\mathrm{Tr}(f(A)(M(a')\,\mathfrak{p})) = \mathrm{Tr}((f(A)\,\mathfrak{p})\,M(a')) = \langle a'|f(A)\,\mathfrak{p}|a'\rangle. \tag{5.3}$$

Using the spectral decomposition (5.1), we find that

$$\begin{aligned}\langle f(A)\rangle_{M(a')\mathfrak{p}} &= \sum_{a''} f(a'')\langle a'|M(a'')\,\mathfrak{p}|a'\rangle \\ &= \frac{1}{N_\alpha}\sum_{a'',\alpha'\alpha''} f(a'')\langle a'|\bar{M}(\alpha'\,\alpha'')\,M(a'')\,\bar{M}(\alpha'',\alpha')|a'\rangle \\ &= \frac{1}{N_\alpha}\sum_{a'',\alpha'\alpha''} f(a'')|\langle a'|\bar{M}(\alpha'\,\alpha'')|a''\rangle|^2,\end{aligned} \tag{5.4}$$

so that the reproducing kernel (4.9) plays the role of a transition probability amplitude from the pure states to the minimally mixed states.

We may now prove the following theorem: *A has a dispersion free state (in the framework of incomplete measurement) if and only if there is an element $M(a')$ in its spectrum such that*

$$[M(a'),\bar{M}(\alpha',\alpha'')] = 0 \tag{5.5}$$

for all α', α'', and in this case the density matrix is $M(a')\,\mathfrak{p} = M(a')$. To prove this assertion, let $f(a'') = (a''-\bar{a})^2$, where $\bar{a} = \langle A\rangle_{M(a')\mathfrak{p}}$. Then, according to (5.4), $\langle f(A)\rangle_{M(a')\mathfrak{p}} = 0$ implies that

$$|\langle a'|\bar{M}(\alpha',\alpha'')|a''\rangle|^2 = 0 \tag{5.6}$$

for all $a'' \neq \bar{a}$. Taking $f(a'') = a''$, it follows, moreover, that (for $\bar{a}\neq 0$; addition of a multiple of the unit matrix to A avoids this case) there must exist an $a'' = \bar{a}$, and that

$$\frac{1}{N_\alpha}\sum_{\alpha'\alpha''} |\langle a'|\bar{M}(\alpha',\alpha'')|\bar{a}\rangle|^2 = 1. \tag{5.7}$$

Equation (5.7) is equivalent to

$$\langle a'|\frac{1}{N_\alpha}\sum_{\alpha'\alpha''} \bar{M}(\alpha'\,\alpha'')\,M(\bar{a})\,\bar{M}(\alpha''\,\alpha')|a'\rangle = 1. \tag{5.8}$$

Since for any vector $|f\rangle$,

$$\begin{aligned}&\langle f|\frac{1}{N_\alpha}\sum_{\alpha'\alpha''} \bar{M}(\alpha'\,\alpha'')\,M(\bar{a})\,\bar{M}(\alpha''\,\alpha')|f\rangle \\ &= \frac{1}{N_\alpha}\sum_{\alpha'\alpha''} \|M(\bar{a})\,\bar{M}(\alpha'',\alpha')|f\rangle\|^2 \\ &\leqslant \frac{1}{N_\alpha}\sum_{\alpha'\alpha''} \|\bar{M}(\alpha'',\alpha')|f\rangle\|^2 \\ &= \frac{1}{N_\alpha}\sum_{\alpha'\alpha''} \langle f|\bar{M}(\alpha')|f\rangle = 1,\end{aligned} \tag{5.9}$$

the operator in (5.8) reaches its upper bound on $|a'\rangle$. Identifying $|f\rangle$ with $|a'\rangle$ in (5.9), we remark that the equality is reached in the second to last relation. Hence,

$$\sum_{\alpha'\alpha''} \|(1 - M(\bar{a}))\, \bar{M}(\alpha'', \alpha')|a'\rangle\|^2 = 0,$$

or

$$M(\bar{a})\, \bar{M}(\alpha'', \alpha')|a'\rangle = \bar{M}(\alpha'', \alpha')|a'\rangle \tag{5.10}$$

for every α', α''. Setting $\alpha' = \alpha''$ and summing over α', one finds that $\bar{a} = a'$ and hence

$$M(a')\, \bar{M}(\alpha'', \alpha')|a'\rangle = \bar{M}(\alpha'', \alpha')|a'\rangle. \tag{5.11}$$

Finally, we multiply by $\langle a'|$ on the right, and compare

$$M(a')\, \bar{M}(\alpha'', \alpha')\, M(a') = M(\alpha'', \alpha')\, M(a') \tag{5.12}$$

with its conjugate

$$M(a')\, \bar{M}(\alpha', \alpha'')\, M(a') = \bar{M}(a')\, M(\alpha', \alpha''). \tag{5.13}$$

Interchanging α' and α'' in (5.13) and comparing with (5.12), we conclude that (5.5) follows, as was to be proven, Clearly. then

$$\begin{aligned} M(a')\,\mathfrak{p} &= \frac{1}{N_\alpha} \sum_{\alpha'\alpha''} \bar{M}(\alpha', \alpha'')\, M(a')\, \bar{M}(\alpha'', \alpha') \\ &= \frac{1}{N_\alpha} \sum_{\alpha'\alpha''} M(a')\, \bar{M}(\alpha') = M(a'), \end{aligned}$$

and the condition (5.5) is sufficient as well. ■

The commutative kernel of an $M(a')$ which satisfies (5.5) is

$$\begin{aligned} K_{\alpha'\alpha''}(M(a')) &= \sum_{\alpha'''} \bar{M}(\alpha''', \alpha')\, M(a')\, \bar{M}(\alpha'', \alpha''') \\ &= \delta(\alpha', \alpha'')\, M(a'), \end{aligned}$$

and $M(a')$ is, therefore, a nontrivial operator only in the β-factor; this is not necessarily true of the operator A.

We now turn to a discussion of symmetries, as a generalization of the treatment given in section 2 of the time evolution of states. Let us suppose that we have a physical compact Lie group $\{A\,|\,A \epsilon g\}$ with unitary irreducible representations $D^\Gamma(A)$ defined on physical states $\{|f^\Gamma\rangle\}$. Then

$$A(|f^\Gamma\rangle) = D^\Gamma(A)|f^\Gamma\rangle, \tag{5.14}$$

and on a density matrix constructed of these states *within the same irreducible representation*, we have

$$A(\rho^\Gamma) = D^\Gamma(A)\, \rho^\Gamma\, D^\Gamma(A)^\dagger. \tag{5.15}$$

In case the determination and preparation of the state is limited by incomplete measurement, $\rho^\Gamma \to \rho^\Gamma \mathfrak{p}$, and we have that

$$A(\rho^\Gamma \mathfrak{p}) = (D^\Gamma(A)\,\rho^\Gamma\, D^\Gamma(A)^\dagger)\,\mathfrak{p}$$

$$= \frac{1}{N_\alpha}\sum_{\alpha'\alpha''} \bar{M}(\alpha'\,\alpha'')\, D^\Gamma(A)\,\rho^\Gamma\, D^\Gamma(A)^\dagger\, \bar{M}(\alpha'', \alpha'), \tag{5.16}$$

where, as in the case of unitary time evolution, $\mathfrak{p}$ remains invariant. For an infinitesimal transformation, where $D^\Gamma(A) \sim 1 + i\epsilon G^\Gamma$, (5.16) becomes

$$\delta(\rho^\Gamma \mathfrak{p}) = i\epsilon[G^\Gamma, \rho^\Gamma \mathfrak{p}]$$

$$-\frac{i\epsilon}{N_\alpha}\sum_{\alpha'\alpha''} \{[G^\Gamma, \bar{M}(\alpha', \alpha'')]\,\rho^\Gamma\, \bar{M}(\alpha'', \alpha') + \bar{M}(\alpha', \alpha'')\,\rho^\Gamma[G^\Gamma, \bar{M}(\alpha'', \alpha')]\}, \tag{5.17}$$

so that $\rho^\Gamma \mathfrak{p}$ does not transform, in general, irreducibly under g. The second term of (5.17) provides corrections which may not vanish. If $g = g_\alpha \otimes g_\beta$ (a condition analogous to (2.18)), a direct product of groups on the α and β factors, then $\rho^\Gamma \mathfrak{p}$ does transform irreducibly, as can easily be seen by direct substitution into (5.16). 'Exact' symmetries observed experimentally would then correspond to groups of this type.

The group is characterized by a Lie algebra of operators on the Hilbert space with irreducible representation G^Γ, but, as we have seen, the physical states associated with incomplete measurement ($\rho^\Gamma \mathfrak{p}$) do not transform irreducibly under this group. To see the effect of this mathematical structure on a relation such as a mass formula, consider

$$\mathrm{Tr}(H(\rho^\Gamma \mathfrak{p})) = \mathrm{Tr}((H\mathfrak{p})\,\rho^\Gamma). \tag{5.18}$$

If H is, say, $SU(3)$ invariant, it is not necessarily true that $H\mathfrak{p}$ is:

$$H\mathfrak{p} = \frac{1}{N_\alpha}\sum_{\alpha'\alpha''} \bar{M}(\alpha', \alpha'')\, H\bar{M}(\alpha', \alpha'')$$

$$= H + \frac{1}{N_\alpha}\sum_{\alpha'\alpha''} [\bar{M}(\alpha', \alpha''), H]\, \bar{M}(\alpha', \alpha''), \tag{5.19}$$

where the second term could contribute the symmetry breaking terms proposed, for example, by Gell-Mann, Oakes and Renner [24]. Clearly, if $\{\alpha'\}$ transforms like an $SU(3)$ tensor, the extra terms in (5.19) remain scalar. Symmetry breaking implies that some directions in $SU(3)$ are not subject to direct measurement; the direction of electric charge, for example, certainly is.

The very simple mechanism for symmetry breaking proposed above suggests that the set of observables (β) one actually measures in defining a particle state are not adequate to exhibit the full symmetry of $SU(3)$, and that the limitation on our measurements is responsible for choosing the direction in which $SU(3)$ is broken. Cabibbo's program [25] of establishing the direction of strong symmetry breaking by putting in driving terms to the non-linear 'spontaneous breakdown' equations which are in the directions of weak and electromagnetic interactions is consistent with this point of view, since these are directions to which our experiments are sensitive.

It is difficult to believe that the remarkable results of the past decade which are based on $SU(3)$ are the purely accidental coincidence of an $SU(3)$ analysis and some other dynamical structure in nature which somehow approximates $SU(3)$ but has no fundamental relation to it. What has been somewhat puzzling is the fact that $SU(3)$ has been so good in some cases and so bad in others (for example, good relations among coupling constants but large mass splitting). The idea of incomplete measurements may provide a basis for understanding these apparently paradoxical observations. If the mechanism discussed here for symmetry breaking is correct, then, in principle, there exists a more complete set of measurements for which the $SU(3)$ symmetry is exact.

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