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Phase Transitions in Reservoir-Driven Open Systems with Applications to Lasers and Superconductors

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Abstract. We present a class of mean field model Hamiltonians consisting of a small, but macroscopic system S of N components interacting with a large reservoir R. The Dicke model of a laser is a particular example of S. Both R and S are fully quantum mechanical. The exact equations of motion are studied, and it is shown that it is possible to eliminate the reservoir variables and, in the limit $N \to \infty$, to derive closed equations for the extensive variables of S. These equations are classical, and the effect of the reservoir is to provide damping and driving (or pumping) terms. As the parameters of the system are varied, S can undergo phase transitions in the sense that its equilibrium orbits bifurcate. To the next order, $N^{1/2}$, the reservoir drives the fluctuation observables of S linearly with Markovian, Gaussian random forces. Within the context of our class of models our results are rigorous and are obtained without any approximation.

1. Introduction

The Dicke–Haken–Lax model of the finite mode laser [D1], [H1], [L2] is interesting both for equilibrium and non-equilibrium statistical mechanics. Not only is it supposed to have some relevance to the real world, but from the theoretical point of view it provides an exactly soluble model with non-trivial consequences. The structure of the model without losses is such that to leading order in N (the number of atoms) the equations of motion for the expectation values of extensive observables are classical, while to the next order, $N^{1/2}$, quantum effects become manifest. It was because of this clean separation of quantum and classical effects that we were able to deduce exactly the equilibrium thermodynamic properties of the 1-mode laser in the rotating wave approximation [H4] and were later able to generalize those results by permitting several photon modes, eliminating the rotating wave approximation and even dispensing with the necessity of spatial homogeneity of the atoms [H5], [H7], [W1].

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In this paper we take the next step which is to elucidate the properties of a laser coupled to a reservoir. On the basis of our previous work we expect that the primary effect of the reservoir will be to provide a simple damping for the laser and at the same time to drive the extensive quantities of the laser to new stationary, non-equilibrium values, when the coupling or pumping constant exceeds a critical value. Again, to leading order in N, the new equations for the extensive observables should be classical and the residual quantum effects of the reservoir should be manifest only to order $N^{1/2}$ as fluctuation forces which should drive the laser fluctuation observables linearly about their mean values. If one assumes this to be true, then one has precisely the theory of the laser developed by others, except near the critical point, where the limit $N \to \infty$ is not interchangeable with the limit time $\to \infty$. The problem is to validate the above assumptions, and it is to this to which this paper is devoted. If one looks at the equations of motion for the laser and the reservoir operators, the above-mentioned N-dependence is at once clearly seen to be consistent with those equations but, because the reservoir is huge compared to the laser, it is quite another matter to *prove* that the effect of the reservoir is really as tame as it would appear to be. Since different reservoirs can produce different stationary states for the laser, the reservoir is of paramount importance to the laser.

One might turn the question around by saying that the problem is to find a large system which, when coupled to the laser, behaves the way a reservoir is intuitively expected to behave according to the phenomenological equations of semi-classical theory. There are some analogies to the construction of a sensible macroscopic measuring apparatus in the theory of quantum mechanical measurement [H3]. In this paper we actually construct a proper reservoir. It is built of fermions and some objects not readily obtainable from the laboratory shelf, but it shows that the phenomenological equations are consistent with at least one microscopic model. In the future we hope to be able to provide more realistic reservoirs.

A typical Hamiltonian for the laser system S that one might wish to consider is

$$H^{S}_{(N)} = \nu a^{*} a + \epsilon S^{3}_{(N)} + \lambda N^{-1/2}(S^{+}_{(N)} a + S^{-}_{(N)} a^{*}) + \mu N^{-1/2}(S^{+}_{(N)} a^{*} + S^{-}_{(N)} a). \qquad (1.1)$$

If $\mu = 0$, (1.1) is the 1-mode Dicke model in the rotating wave approximation. $a^{\#}$ are the creation and annihilation operators for a single photon mode, and $S^{i}_{(N)} = \sum_{n=1}^{N} S^{i}_{n}$, $i = 1, 2, 3$, are the total 'spin' operators for N 2-level atoms ($\underline{S}_n$ is a spin $\frac{1}{2}$ operator). The physical interpretations of $S^{3}_{(N)}$ is that of the total atomic inversion operator, while $S^{\pm}_{(N)} = S^{1}_{(N)} < iS^{2}_{(N)}$ are proportional to the total polarization operators. The inclusion of more than one photon mode and of multi-level atoms without spatially dependent coupling constants to the radiation field does not vitiate our methods, it only makes the resulting equations and their properties more complicated. The closed system properties of such a Hamiltonian (1.1) have been discussed by us previously [H4], [H5] and by many others [S2], [S3], [T1], [W1].

The reservoirs we shall consider consist of separate systems for each atom and each photon mode. A desirable photon reservoir would itself consist of bosons [S5], i.e.

$$H^{P} = \int dw E(w)\, A^{*}_{w} A_{w} + \int dw (k(w)\, A^{*}_{w} a + \text{h.c.}), \qquad (1.2)$$

where A_w, $w \in \mathbf{R}$, is a boson field with $[A_w, A^{*}_{w'}] = \delta(w - w')$, and the initial state of the reservoir is, for instance, the vacuum. The reservoir for a single atom could be imagined to consist of a large number of 2-level atoms [S1] in a state of inverted population which, by a suitable coupling, has the effect of pumping the laser atoms. It is well known that the excitations of such an ensemble of atoms are boson-like, so we may as well eliminate a needless difficulty by assuming *ab initio* that the atom reservoir does indeed consist of bosons, i.e. for the nth atom

$$H_n^A = \int dw F(w)[B_{nw}^* B_{nw} + C_{nw}^* C_{nw}] + \int dw[g_B(w) B_{nw}^* S_n^- + g_C(w) C_{nw}^* S_n^+ + \text{h.c.}], \tag{1.3}$$

where $B_{nw}^\#$ and $C_{nw}^\#$ are two boson fields. The B-field pumps the atom down and the C-field pumps it up. Actually, the reservoirs we shall consider in Section 5 are similar to (1.3), but are mathematically more tractable.

The full Hamiltonian [H1], [L2] is

$$H_{(N)} = H_{(N)}^S + H_{(N)}^R, \quad H_{(N)}^R = H^P + \sum_{n=1}^{N} H_n^A. \tag{1.4}$$

For any operator, A, $A(t) = \exp(iHt) A \exp(-iHt)$. (We drop the subscript N, whenever it is not absolutely necessary.) The equations of motion for the A-, B- and C-field operators are easy to derive and are linear, and can be solved in terms of the $(t=0)$-operators and the photon and atom operators. Inserting these solutions into the equations of motion for the latter operators, one finds (with $\dot{A} = dA/dt$)

$$\dot{a}(t) = -i\nu a(t) - i\lambda N^{-1/2} S_{(N)}^-(t) - i\mu N^{-1/2} S_{(N)}^+(t) - \int_0^t ds \kappa(t-s)\, a(s) + f(t), \tag{1.5}$$

$$\begin{aligned}\dot{S}_n^+(t) &= i\epsilon S_n^+(t) - 2i\lambda N^{-1/2} S_n^3(t)\, a^*(t) - 2i\mu N^{-1/2} S_n^3(t)\, a(t)\\ &\quad + 2\int_0^t ds \gamma_B(t-s)\, S_n^+(s)\, S_n^3(t) - 2\int_0^t ds \gamma_C(t-s)\, S_n^3(t)\, S_n^+(s) + G_n^+(t),\end{aligned} \tag{1.6}$$

$$\begin{aligned}\dot{S}_n^3(t) &= (-i\lambda N^{-1/2} S_n^+(t)\, a(t) - i\mu N^{-1/2} S_n^+(t)\, a^*(t) + \text{h.c.})\\ &\quad + \int_0^t ds[\gamma_C(t-s)\, S_n^-(t)\, S_n^+(s) - \gamma_B(t-s)\, S_n^+(s)\, S_n^-(t) + \text{h.c.}] + G_n^3(t)\end{aligned} \tag{1.7}$$

and

$$f(t) = -i\int dw k^*(w)\, A_w \exp(-iE(w)\, t), \tag{1.8}$$

$$\begin{aligned}G_n^+(t) &= -2i\int dw g_B(w)\, B_{nw}^*\, S_n^3(t) \exp(iF(w)\, t)\\ &\quad -2i\int dw g_C^*(w)\, S_n^3(t)\, C_{nw} \exp(-iF(w)\, t),\\ G_n^3(t) &= i\int dw g_B(w)\, B_{nw}^* S_n^-(t) \exp(iF(w)\, t)\\ &\quad -i\int dw g_C(w)\, C_{nw}^* S_n^+(t) \exp(iF(w)\, t) + \text{h.c.}\end{aligned} \tag{1.9}$$

In the above

$$\kappa(t) = \int dw |k(w)|^2 \exp(-iE(w)\, t), \tag{1.10}$$

$$\gamma_B(t) = \int dw |g_B(w)|^2 \exp(iF(w)\, t), \tag{1.11}$$

$$\gamma_C(t) = \int dw |g_C(w)|^2 \exp(-iF(w)\, t). \tag{1.12}$$

In (1.5) and (1.6), the operators $f(t)$ and $G_n(t)$ have the property that their (partial) expectation value in the reservoir vacuum is zero. For that reason we call them *fluctuation forces*. f is explicitly given and is of $O(1)$, and the main problem is to show that $g^i_{(N)}(t) = N^{-1/2} \sum G^i_n(t)$ is also $O(1)$, where the precise meaning of $O(1)$ for these unbounded operators will be explained later. If f is truly $O(1)$ and if $S^-_{(N)}$ is $O(N)$ and a is $O(N^{1/2})$, then there is no objection to dropping f in (1.5). However, in (1.6) G^+_n is $O(1)$, and it has a profound influence on S^+_n, which is also $O(1)$. In fact, it is an essential ingredient in preserving the norms and the commutation relations of the $S^i_n(t)$. The hope would be that one could find an equation for the total $S^i_{(N)}(t)$, in which the total fluctuation force $N^{1/2} g^i_{(N)}$, which is $O(N^{1/2})$, can be dropped. This suggests summing (1.6) over n and dividing by N. Unfortunately this does not yield a closed equation for $S^i_{(N)}(t)$; it would do so, if γ_B and γ_C were Dirac delta functions. If γ_B and γ_C are *not* delta functions (which case we shall term the *regular* reservoir) a more subtle procedure is required to obtain an equation for $S^i_{(N)}(t)$. In a subsequent paper we shall deal with that question. For now, we suppose that γ_B, γ_C and κ are indeed proportional to delta functions. This requires choosing

$$E(w) = F(w) = w, \quad g_B(w) = g_B, \quad g_C(w) = g_C \quad \text{and} \quad k(w) = k.$$

Such a reservoir is called *singular*, because its energy has neither an upper nor a lower bound and the Hamiltonian is in general only defined as a bilinear form and not as an operator. In Section **3** we shall give a meaning to the time evolution of systems coupled to singular reservoirs.

With this assumption we can now sum (1.6) over n and, recalling that $S^3_n(t) S^+_n(t) = S^+_n(t)/2$, and introducing the intensive observables

$$\begin{aligned} \alpha_{(N)}(t) &= N^{-1/2} a_{(N)}(t), \\ \sigma^i_{(N)}(t) &= N^{-1} S^i_{(N)}(t), \end{aligned} \tag{1.13}$$

one finds

$$\dot\alpha_{(N)}(t) = -(i\nu + \kappa)\, \alpha_{(N)}(t) - i\lambda \sigma^-_{(N)}(t) - i\mu \sigma^+_{(N)}(t) + \varphi_{(N)}(t), \tag{1.14}$$

$$\dot\sigma^+_{(N)}(t) = (i\epsilon - \gamma)\, \sigma^+_{(N)}(t) - 2i\lambda \sigma^3_{(N)}(t)\, \alpha^*_{(N)}(t) - 2i\mu \sigma^3_{(N)}(t)\, \alpha_{(N)}(t) + \chi^+_{(N)}(t), \tag{1.15}$$

$$\dot\sigma^3_{(N)}(t) = -2\gamma(\sigma^3_{(N)}(t) - \eta) + \chi^3_{(N)}(t) + [i\lambda \sigma^-_{(N)}(t)\, \alpha^+_{(N)}(t) + i\mu \sigma^-_{(N)}(t)\, \alpha_{(N)}(t) + \text{h.c.}], \tag{1.16}$$

where

$$\eta = [|g_C|^2 - |g_B|^2]/2[|g_C|^2 + |g_B|^2] \tag{1.17}$$

is the pumping rate (which can vary between $-\frac{1}{2}$ and $\frac{1}{2}$), and

$$\gamma = \pi[|g_C|^2 + |g_B|^2], \tag{1.18}$$

$$\kappa = \pi |k|^2 \tag{1.19}$$

are damping constants.

Equations (1.14)–(1.16) are the desired equations of motion for the intensive observables. As the commutators of $\sigma^i_{(N)}$ and $\alpha^\#_{(N)}$ are $O(N^{-1})$, and as the fluctuation forces $\varphi_{(N)}, \chi^i_{(N)}$ are hopefully $O(N^{-1/2})$, we expect that (1.14)–(1.16) go over into classical equations in suitable states (which we shall call 'classical' in Section 2).

The nature of the solutions of this non-linear equation (1.14)–(1.16) (in the classical limit) can be expected to depend sensitively on the parameters (damping constants and pumping rate), and we shall give some physical examples in the last section. In particular, the laser threshold will be an example of a phase transition in a stationary, but non-equilibrium state. This interpretation of the laser threshold has been given before but without rigorous justification [D2], [G2], [S4].

We have written the above equations (1.1)–(1.19) to outline the steps one would like to follow from a microscopic Hamiltonian to a classical differential equation. It is plain that our justification, such as it was, for the various steps was at best circular and imprecise. It is to give a rigorous proof of the above expansion in powers of $N^{-1/2}$ that the rest of this paper is devoted. In so doing we shall give a non-trivial example of a microscopic quantum mechanical theory of macroscopic irreversibility and self-oscillatory behaviour, in a sense a model of a quantum mechanical steam engine. To be perfectly precise, however, the model we shall treat in the sequel is not exactly the same as (1.1)–(1.3), although it is hardly less physical and the classical equations are essentially the same as (1.14)–(1.16). The reason for the alteration is a technical one having to do with the unbounded nature of boson fields. In a later paper [H6] we shall return to the model (1.1)–(1.3), but here we study fermion reservoirs and introduce an arbitrarily large photon number cutoff proportional to N.

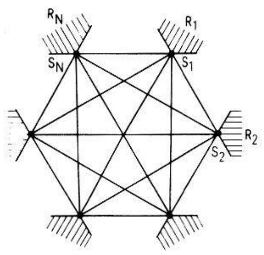

Figure 1
Structure of a system $S = \{S_1, \ldots S_N\}$ with mean field interaction and independent reservoirs $R = \{R_1, \ldots R_N\}$.

Sections 2, 3 and 4 give a systematic account of dissipative fermion systems with non-linear mean field type interaction of the symbolic structure depicted in Figure 1. The setting is somewhat abstract, but we believe that this has the advantage of clarifying the fundamental mathematical ideas behind the proofs. The main theorems have clear intuitive meaning, and they are not pushed to utmost generality. Chapter 5 gives the application of the general theory to various physical systems, namely to the laser and to a system consisting of two strongly coupled superconductors, the reservoir driven, mean-field Josephson effect.

2. Classical Limits in Mean Field Models

In this section we consider a large number, N, of identical subsystems coupled together by a mean field-like Hamiltonian. Clearly, the relevant operators do not have limits as $N \to \infty$ in a strong sense, but their expectation values do have limits in certain states (which might typically be the ground state or Gibbs states [H4], but not necessarily so). We shall characterize states in which a polynomial algebra of macroscopic observables have reasonable physical homogeneity properties, as classical with respect to these observables, and the precise definition of this concept is given below. The problem is to show that if the expectation values are classical at time zero, that they continue to be classical for all times, and that expectation values (Wightman or Green functions) obey simple classical equations. A similar problem arises, if one starts with initial conditions where the fluctuations of relevant observables around the classical values are normal. Do they continue to be normal, and are the resulting equations of motion linear as one expects from the theory of ordinary differential equations?

In the following, each $\mathscr{A}_n$ is a collection of operators which act on one of the identical subsystems, and it can be imagined to be all linear operators on a finite dimensional complex vector space. This space might, for example, be the internal states of several multi-level atoms with a fixed number ot electrons.

Let $\mathscr{A}_n$, $n \in \mathbf{Z}$ (all integers) be a countable family of C^*-algebras, where each $\mathscr{A}_n \simeq \mathscr{B}(\mathbf{C}^K)$ is isomorphic to the set of all linear operators on a fixed $\mathbf{C}^K$. For $m \neq n$, $[\mathscr{A}_m, \mathscr{A}_n] = 0$. Let $\mathscr{A}$ be the quasilocal C^*-algebra generated by $\bigcup \mathscr{A}_n$[R2]. (There are obvious generalizations to fancier quasilocal structures.) Let τ be a representation of $\mathbf{Z}$ as translation automorphisms of $\mathscr{A}$. Let $A \in \mathscr{A}_0$ and

$$\alpha_{(N)} = N^{-1} \sum_{n=1}^{N} \tau_n A. \tag{2.1}$$

$\alpha_{(N)}$ is called an intensive observable of first degree. (A more general class of averages has been studied by [K1], [R3].) Let $A^1, \dots A^L$ be a basis of $\mathscr{A}_0$ ($L = K^2$) and $\alpha^1_{(N)}, \dots \alpha^L_{(N)}$ the corresponding intensive observables. A polynomial $\beta_{(N)} = \beta(\alpha^1_{(N)}, \dots \alpha^L_{(N)})$ is called an intensive polynomial. If

$$\gamma_{(N)} = \gamma(\alpha^1_{(N)}, \dots \alpha^L_{(N)}) = n - \lim_{m \to \infty} \beta^m(\alpha^1_{(N)}, \dots \alpha^L_{(N)}) \tag{2.2}$$

uniformly in N with fixed intensive polynomials $\beta^m_{(N)}$, then $\gamma_{(N)}$ is called an intensive observable.

An extensive observable, $A_{(N)}$, is N times an intensive observable, $A_{(N)} = N\alpha_{(N)}$. For every $\alpha \in \mathbf{C}$

$$a_{(N)} = \sqrt{N}(\alpha_{(N)} - \alpha) \tag{2.3}$$

is called the fluctuation observable of $\alpha_{(N)}$ around α.

For intensive polynomials $\alpha_{(N)}$ and $\beta_{(N)}$ (and more generally for a large class of intensive observables of the type (2.2)), $[\alpha_{(N)}, \beta_{(N)}] = O(N^{-1})$ for $N \to \infty$. Hence these operators are candidates for classical observables, while the fluctuation operators could become boson operators with c-number commutators in the limit $N \to \infty$. The limit $N \to \infty$ cannot be discussed in $\mathscr{A}$, but it is meaningful in certain states of $\mathscr{A}$ which we now describe.

Let $\mathscr{L}$ be a self-adjoint Lie subalgebra of $\mathscr{A}_0$ with real basis $A^1, \dots A^l$. All future statements will be made relative to $\mathscr{L}$.

A state ω of $\mathscr{A}$ is called classical (with respect to $\mathscr{L}$) if, for *all* intensive polynomials $\beta_{(N)} = \beta(\alpha^1_{(N)}, \dots \alpha^l_{(N)})$,

$$\lim_{N \to \infty} \omega(\beta_{(N)}) = \langle \beta \rangle_\omega \tag{2.4}$$

exists. Then there exists [R3] a probability measure μ_ω on the 'phase space' $\mathbf{R}^l$ with support in $|\alpha^k| \leqslant \|A^k\|$, $1 \leqslant k \leqslant l$, such that for all monomials

$$\lim_{N \to \infty} \omega((\alpha^1_{(N)})^{m(1)} \dots (\alpha^l_{(N)})^{m(l)}) = \int \mu_\omega(d\alpha)(\alpha^1)^{m(1)} \dots (\alpha^l)^{m(l)}. \tag{2.5}$$

A state ω of $\mathscr{A}$ is called a pure classical state (with respect to $\mathscr{L}$), if μ_ω is concentrated on one point $\underline{\alpha} = (\alpha^1, \dots \alpha^l) \in \mathbf{R}^l$; alternatively, if for all monomials

$$\lim_{N\to\infty} \omega((\alpha^1_{(N)})^{m(1)} \dots (\alpha^l_{(N)})^{m(l)}) = \prod_{\lambda=1}^{l} (\alpha^\lambda)^{m(\lambda)}. \tag{2.6}$$

It is known [K1], [R3] that every translation invariant state ω on $\mathscr{A}$ is classical and ω is pure if it is an extremal point of all translation invariant states on $\mathscr{A}$.

A state ω on $\mathscr{A}$ has normal fluctuations with respect to $\underline{\alpha}_{(N)} = (\alpha^1_{(N)}, \dots \alpha^l_{(N)})$ around $\underline{\alpha} \in \mathbf{R}^l$, if, for all monomials in $a^k_{(N)} = \sqrt{N}(\alpha^k_{(N)} - \alpha^k)$, $1 \leqslant k \leqslant l$, $s = 1, 2, \dots$

$$\lim_{N\to\infty} \omega(a^{k(1)}_{(N)} \dots a^{k(s)}_{(N)}) = \langle a^{k(1)} \dots a^{k(s)} \rangle_\omega \tag{2.7}$$

exists. By the GNS construction there exists a Hilbert space $\mathscr{H}_\omega$ with a vector $\Omega_\omega \in \mathscr{H}_\omega$ which is cyclic with respect to the algebra of all polynomials in (possibly unbounded) linear operators $a^1_\omega, \dots a^l_\omega$ on $\mathscr{H}_\omega$, such that

$$\langle a^{k(1)} \dots a^{k(s)} \rangle_\omega = (\Omega_\omega, a^{k(1)}_\omega \dots a^{k(s)}_\omega \Omega_\omega). \tag{2.8}$$

It is easy to see that a classical state with normal fluctuations around $\underline{\alpha}$ is pure with μ_ω concentrated on $\underline{\alpha}$ and that for all $1 \leqslant i, j \leqslant l$

$$[A^i, A^j] = \sum_{k=1}^{l} C^{ij}_k A^k \Rightarrow [a^i_\omega, a^j_\omega] = \sum_{k=1}^{l} C^{ij}_k \alpha^k. \tag{2.9}$$

Of course, not every pure classical state has normal fluctuations. Necessary and sufficient conditions for purity and normality can be expressed in terms of the truncated ω-expectation values. Those are defined recursively for $X^1, \dots X^s \in \mathscr{A}$ by

$$\omega^T(X^1 \dots X^s) = \omega(X^1 \dots X^s) - \sum{}' \omega^T(X^{i(1)} \dots X^{i(r)}) \dots \omega^T(X^{j(1)} \dots X^{j(t)}) \tag{2.10}$$

where $\sum'$ extends over all partitions of $\{X^1, \dots X^s\}$ into more than one set and $\omega^T(x^i) = \omega(X^i)$. The relevant operators are the $A^k_n \equiv \tau_n A^k$ for $1 \leqslant k \leqslant l$ and $n = 1, 2, \dots$. ω is pure, if and only if

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \omega(A^k_n) = \alpha^k, \quad 1 \leqslant k \leqslant l, \tag{2.11}$$

exists and for all $s \geqslant 2$

$$\lim_{N\to\infty} N^{-s} \sum_{n(1), \dots n(s)=1}^{N} \omega^T(A^{k(1)}_{n(1)} \dots A^{k(s)}_{n(s)}) = 0. \tag{2.12}$$

ω has normal fluctuations around $\underline{\alpha}$, if and only if

$$\lim_{N\to\infty} N^{-1/2} \sum_{n=1}^{N} \{\omega(A^k_n) - \alpha^k\} \tag{2.13}$$

exists and

$$\lim_{N\to\infty} N^{-s/2} \sum_{n(1), \dots n(s)=1}^{N} \omega^T(A^{k(1)}_{n(1)} \dots A^{k(s)}_{n(s)}) \tag{2.14}$$

exists ($s \geqslant 2$).

One sees that a translation invariant direct product state ω with $\omega^T(A^{k(1)}_{n(1)}, \dots A^{k(s)}_{n(s)}) \equiv 0$ for $s \geqslant 2$, has normal fluctuations.

In the following theorem the assumption of $\mathscr{L}$ being a finite dimensional Lie algebra is important for the first time:

Theorem 2.1: Let the Hamiltonian $H_{(N)} = H^*_{(N)} = N\beta(\alpha^1_{(N)}, \dots \alpha^l_{(N)})$ be an extensive polynomial (associated with $\mathscr{L}$). Define

$$\alpha^k_{(N)}(t) = \exp iH_{(N)}t)\, \alpha^k_{(N)} \exp(-iH_{(N)}t). \tag{2.15}$$

Then for all $1 \leqslant k \leqslant l$ and all $t \in \mathbf{R}$ the $\alpha^k_{(N)}(t)$ are intensive observables and they are uniformly approximated by

$$\alpha^k_{(N)}(t) = n - \lim_{n(0),\dots n(r)\to\infty} \alpha^k_{(N)n(0)\dots n(r)}(t). \tag{2.16}$$

The polynomials $\alpha^k_{(N)n(0)\dots n(r)}$ in $\alpha^1_{(N)}, \dots \alpha^l_{(N)}$ are recursively defined by (2.22) as iterative solutions of the integral equation

$$\alpha^k_{(N)}(t) = \alpha^k_{(N)}(s) + i\int_s^t dr[H_{(N)}(r), \alpha^k_{(N)}(r)]. \tag{2.17}$$

These solutions are entire analytic in t and uniformly bounded in N in a neighbourhood of the real t-axis.

Proof: For finite N, $H_{(N)} \in \mathscr{A}$, and $\alpha^k_{(N)}(t) \in \mathscr{A}$ is norm-continuous in t and satisfies $\|\alpha^k_{(N)}(t)\| = \|A^k\|$ for all t and all $1 \leqslant k \leqslant l$. Let $0 \leqslant r < s$ and let $X_{(r,s)}$ be the Banach space of all norm-continuous curves

$$x = \{\underline{x}(t) = (x^1(t), \dots x^l(t)) \in \mathscr{A}^l,\ r \leqslant t \leqslant s\},$$
$$\|x\| = \sup_{r\leqslant t\leqslant s}\ \max_{1\leqslant k\leqslant l} \|x^k(t)\|. \tag{2.18}$$

Let $\alpha = \max\|A^k\|$ and $X^{2\alpha}_{(r,s)} = \{x \in X_{(r,s)},\ \|x\| \leqslant 2\alpha\}$. For $\underline{z} \in \mathscr{A}^l$ and $x \in X_{(r,s)}$, we define $y = T_{(r,s)}(\underline{z}, x) \in X_{(r,s)}$ by

$$y^k(t) = z^k + \int_r^t du\beta^k(x^1(u), \dots x^l(u)), \tag{2.19}$$

where

$$\beta^k(\alpha^1_{(N)}(u) \dots \alpha^l_{(N)}(u)) \equiv i[H_{(N)}(u), \alpha^k_{(N)}(u)]$$

is a polynomial which can be computed using the Lie structure (2.9) of $\mathscr{L}$. It is easy to see that there exists some $d > 0$ such that, whenever $|r - s| \leqslant d$ and $\|\underline{z}\| = \max\|z^k\| \leqslant 3\alpha/2$, $T_{(r,s)}(\underline{z}, \cdot)$ is a contraction on $X^{2\alpha}_{(r,s)}$ with a Lipschitz constant $\frac{1}{2}$. Thus, for $0 \leqslant t \leqslant d$, we can use the contraction fixed-point theorem to approximate the solutions to (2.17), uniformly in N, by a sequence of intensive polynomials $\alpha^k_{(N)n}(t) = T^n_0(\underline{\alpha}_{(N)})^k(t)$, which are recursively defined (as elements of $X^{2\alpha}_{(r,s)}$) by

$$T^n_0(\underline{\alpha}_{(N)}) = T_{(0,d)}(\underline{\alpha}_{(N)}, T^{n-1}_0(\underline{\alpha}_{(N)})),$$
$$T^0_0(\underline{\alpha}_{(N)}) = \underline{\alpha}_{(N)}. \tag{2.20}$$

Since for all $m = 1, 2, \dots$, $\|\underline{\alpha}_{(N)}(md)\| \leqslant \alpha$, one can use $\underline{\alpha}_{(N)}(md)$ to compute $\underline{\alpha}_{(N)}(t)$ for $md \leqslant t \leqslant (m+1)\,d$, by applying the fixed-point theorem. Let $T^n_m(\underline{z})$ be defined as in

(2.20), with $(0,d)$ replaced by $(md, (m+1)d)$ and $\underline{\alpha}_{(N)}$ by $\underline{z}$. We claim that for every $m = 0, 1, 2, \ldots$ and every $\epsilon > 0$ there exists some $K(m,\epsilon) < \infty$ such that for all $n(0), \ldots n(m) \geqslant K(m,\epsilon)$ and for all $md \leqslant t \leqslant (m+1)d$ and all N

$$\|\alpha^k_{(N)n(0)\ldots n(m)}(t) - \alpha^k_{(N)}(t)\| < \epsilon. \tag{2.21}$$

The $\alpha_{(N)n(0)\ldots n(m)}$ are recursively defined as elements in $X_{(md,(m+1)d)}$ by

$$\alpha_{(N)n(0)\ldots n(m)} = T^{n(m)}_m(\underline{\alpha}_{(N)n(0)\ldots n(m-1)}(md)), \tag{2.22}$$

where the recursion starts at (2.20). Since T_m is a contraction, one has for all n and for all $\underline{x}, \underline{y}$ with $\|\underline{x}\|, \|\underline{y}\| \leqslant 3\alpha/2$:

$$\|T^n_m(x) - T^n_m(y)\| \leqslant \|x-y\| + \tfrac{1}{2}\|T^{n-1}_m(x) - T^{n-1}_m(y)\| \leqslant 2\|x-y\|. \tag{2.23}$$

Given $m = 0, 1, \ldots$ and $\epsilon > 0$, there exists some $K(m,\epsilon) < \infty$, such that for all $n(0), \ldots n(m) \geqslant K(m,\epsilon)$ and for all $\mu = 0, 1, \ldots m$

$$\|\underline{\alpha}_{(N)} - T^{n(\mu)}_\mu(\underline{\alpha}_{(N)}(\mu d))\| < \epsilon 2^{m-\mu}(m+1)^{-1} \tag{2.24}$$

as elements of $X^{2\alpha}_{(\mu d,(\mu+1)d)}$. Hence, by (2.23),

$$\begin{aligned}\|\alpha_{(N)} - \alpha_{(N)n(0)\ldots n(m)}\| &\leqslant \|\alpha_{(N)} - T^{n(m)}_m(\underline{\alpha}_{(N)}(md))\| \\ &+ \|T^{n(m)}_m(\underline{\alpha}_{(N)}(md)) - T^{n(m)}_m(\underline{\alpha}_{(N)n(0)\ldots n(m-1)}(md))\| < \epsilon(m+1)^{-1} \\ &+ 2\|\alpha_{(N)} - \alpha_{(N)n(0)\ldots n(m-1)}\| \leqslant \epsilon.\end{aligned} \tag{2.25}$$

QED

Remark: Obviously this theorem can be generalized to a large class of non-polynomial extensive interactions and even beyond finite dimensional Lie algebras.

Theorem 2.2: Under the assumptions of Theorem 2.1 one has

$$\sup_{i,j,N} N\|[\alpha^i_{(N)}(s), \alpha^j_{(N)}(t)]\| < \infty \tag{2.26}$$

uniformly on compacts in s, t.

Proof: Starting from (2.19) one obtains

$$[A^i_{(N)}, \alpha^j_{(N)}(t)] = \sum_{k=1}^{l} C^{ij}_k \alpha^k_{(N)} + \sum_{k=1}^{l} \int_0^t dr \beta^{jk}(\underline{\alpha}_{(N)}(r), [A^i_{(N)}, \alpha^k_{(N)}(r)]), \tag{2.27}$$

where β^{jk} is linear in $[A^i_{(N)}, \alpha^k_{(N)}(r)]$. This linear Volterra equation can be solved by iteration, uniformly on compacts in t. By repeating this argument for

$$[A^i_{(N)}(s), \alpha^j_{(N)}(t)] = [A^i_{(N)}. \alpha^j_{(N)}(t)] + \sum_{k=1}^{l} \int_0^s dr \beta^{ik}(\underline{\alpha}_{(N)}(r), [A^k_{(N)}(r), \alpha^j_{(N)}(t)]) \tag{2.28}$$

with (2.27) as input, one obtains Theorem 2.2.

QED

The following theorem gives a classical interpretation to the time-evolution under extensive interaction for expectation values of intensive observables in classical states:

Theorem 2.3: Under the assumptions of Theorem 2.1, let ω be classical with respect to $\mathscr{L}$ with probability measure μ_ω. Then ω is classical with respect to all intensive observables $\alpha^k_{(N)}(t)$, $1 \leqslant k \leqslant l$, $t \in \mathbf{R}$, and for all monomials

$$\lim_{N\to\infty} \omega(\alpha^i_{(N)}(r) \ldots \alpha^j_{(N)}(t)) = \int \mu_\omega(d\underline{\gamma})\, \alpha^i(\underline{\gamma}, r) \ldots \alpha^j(\underline{\gamma}, t) \tag{2.29}$$

uniformly on compacts in $r, \ldots t$.

Here $\underline{\alpha}(\gamma, t)$ is the solution of the ordinary differential equation

$$\dot{\alpha}^k = \{\beta, \alpha^k\} = \beta^k(\underline{\alpha}),$$
$$\alpha^k(\underline{\gamma}, 0) = \gamma^k, \tag{2.30}$$

where β^k is obtained from $H_{(N)}$ by treating β as a polynomial in commuting variables $\alpha^1, \ldots \alpha^l$ and by using the Poisson brackets $\{\alpha^i, \alpha^j\} = i \sum C^{ij}_k \alpha^k$.

In particular, if ω is pure with respect to $\alpha^1_{(N)}, \ldots \alpha^l_{(N)}$, then ω is also pure with respect to all $\alpha^k_{(N)}(t)$.

Remark: In this framework, the classical limit (the limit $N \to \infty$) establishes the following correspondence:

Quantum-classical correspondence

Quantum mechanics	Classical mechanics
Classical state	Ensemble in phase space†
Pure classical state	Point in phase space
Intensive observable	Function on phase space
Heisenberg time evolution	Hamiltonian time evolution
$iN[\alpha^p_{(N)}, \alpha^q_{(N)}]$	$\{\alpha^p, \alpha^q\}$

Proof: If one can show that the classical equations

$$\alpha^k(t) = \alpha^k(s) + \int_s^t dr \beta^k(\underline{\alpha}(r)) \tag{2.31}$$

have iterative solutions $\alpha^k_{n(0)\ldots n(m)}(t) \to \alpha^k(t)$ for $md \leqslant t \leqslant (m+1)d$, as in Theorem 2.1, then clearly (2.29) will hold. This is so, since every pair of finite approximations $\underline{\alpha}_{(N)\,n(0)\ldots n(m)}(t)$ and $\underline{\alpha}_{n(0)\ldots n(m)}(\underline{\gamma}, t)$ to the quantum and classical equations (the latter with initial condition $\underline{\gamma}$ at $t = 0$) satisfy

$$\lim_{N\to\infty} \omega(\alpha^i_{(N)m(0)\ldots m(\rho)}(r) \ldots \alpha^j_{(N)n(0)\ldots n(\tau)}(t))$$
$$= \int \mu_\omega(d\underline{\gamma})\, \alpha^i_{m(0)\ldots m(\rho)}(\underline{\gamma}, r) \ldots \alpha^j_{n(0)\ldots n(\tau)}(\underline{\gamma}, t), \tag{2.32}$$

since ω is a classical state. The left-hand side of (2.32) approximates the left-hand side of (2.29) uniformly in N, by Theorem 2.1. By the iterative construction for (2.31), the integrand of the right-hand side of (2.32) is uniformly bounded by an integrable function and converges pointwise for $m(0), \ldots m(\rho), \ldots, n(0), \ldots n(\tau) \to \infty$ to $\alpha^i(\underline{\gamma}, r) \ldots$

† A phase space $\mathbf{R}^{2f}$ with the symplectic structure of classical Hamiltonian mechanics can be obtained by group contraction from $SU(2)^f$ (see the laser model in Section 5).

$\alpha^j(\underline{\gamma},t)$. Hence the right-hand side of (2.32) converges to the right-hand side of (2.29) by the dominated convergence theorem.

That the classical equations have global solutions follows from a group-theoretical argument: Complete $A^1, \dots A^l$ to a real basis $A^1, \dots A^L$ of the Lie algebra $\mathfrak{G}$ of $GL(K,\mathbf{C})$, $L = K^2$. $\mathfrak{G}$ has the Casimir operator

$$C = \sum_{i,j=1}^{K} E^{ij*} E^{ij} = \sum_{i,j=1}^{L} x_{ij} A^i A^j, \tag{2.33}$$

where (x_{ij}) is a positive definite $L \times L$ matrix. In the extended phase space $\mathbf{R}^L$ one can study the differential equation

$$\dot{\alpha}^i = \{\beta, \alpha^i\}, \quad 1 \leqslant i \leqslant L, \tag{2.34}$$

where the α^i correspond to the A^i and have well-defined Poisson brackets $\{\alpha^i, \alpha^j\}$ and where $\beta = \beta(\alpha^1, \dots \alpha^l)$ is independent of $\alpha^{l+1}, \dots \alpha^L$. For initial conditions $\alpha^i(\gamma, 0) = \gamma^i$, $1 \leqslant i \leqslant l$, and $\alpha^i(\underline{\gamma}, 0) = 0$, $l+1 \leqslant i \leqslant L$, the $\alpha^i(\underline{\gamma}, t)$, $1 \leqslant i \leqslant l$, are the solutions of (2.30). Since

$$\{\beta, \sum x_{ij} \alpha^i \alpha^j\} = 0,$$

one has for all t

$$\sum_{i,j=1}^{L} x_{ij} \alpha^i(\underline{\gamma}, t)\, \alpha^j(\underline{\gamma}, t) = \sum_{i,j=1}^{l} x_{ij} \gamma^i \gamma^j. \tag{2.35}$$

Since (x_{ij}) is positive definite, the $\alpha^i(\underline{\gamma},t)$ stay bounded uniformly in t. Hence the iteration scheme can be used for all times in the classical equations (2.30).

QED

Theorem 2.4: Under the conditions of Theorem 2.1 let $\alpha^1_{(N)}, \dots \alpha^l_{(N)}$ have normal fluctuations in ω around $\gamma^1, \dots \gamma^l$. Then, for all times t and all $1 \leqslant k \leqslant l$, the $\alpha^k_{(N)}(t)$ have normal fluctuations in ω around $\alpha^k(\underline{\gamma}, t)$, and for all monomials in

$$\begin{aligned} &a^k_{(N)}(t) \equiv N^{1/2}(\alpha^k_{(N)}(t) - \alpha_k(\underline{\gamma}, t)) \\ &\lim_{N\to\infty} \omega(a^i_{(N)}(r) \dots a^j_{(N)}(t)) = (\Omega_\omega, a^i_\omega(r) \dots a^j_\omega(t)\, \Omega_\omega). \end{aligned} \tag{2.36}$$

Here, $a^k_\omega(t)$ is the solution of the linear variational equation of (2.30) around $\underline{\alpha}(\underline{\gamma}, t)$ with initial condition $\alpha^k_\omega(0) = a^k_\omega$.

Proof: By taking the difference between (2.17) and (2.31) one obtains

$$a^k_{(N)}(t) = a^k_{(N)} + \int_0^t ds N^{1/2}(i[H_{(N)}(s), \alpha^k_{(N)}(s)] - \{\beta(x), \alpha^k(s)\}). \tag{2.37}$$

In the integrand of (2.37) we combine corresponding terms

$$N^{1/2}[\alpha^{k(1)}_{(N)}(s) \dots \alpha^{k(m)}_{(N)}(s) - \alpha^{k(1)}(s) \dots \alpha^{k(m)}(s)] = \sum_{\mu=1}^{m} a^{k(\mu)}(s) \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{m} \alpha^{k(\nu)}(s) + \sum_{k=1}^{l} \delta^k_{(N)}(s). \tag{2.38}$$

Each $\delta^k_{(N)}(s)$ is a sum of monomials (with coefficients independent of N) in $\alpha^i_{(N)}(s)$,

$\alpha^j(s)$, $1 \leqslant i, j \leqslant l$, and is linear in $a^k_{(N)}(s)$. $\delta^k_{(N)}(s)$ contains at least one factor of the type $(\alpha^j_{(N)}(s) - \alpha^j(s))$. We consider (2.37) as a linear Volterra equation for the $a^k_{(N)}(t)$. The iterative solution converges in the ω-expectation to (2.36), since, by Theorem 2.3, the $\delta^k_{(N)}$-terms do not contribute in the limit $N \to \infty$.

QED

It is clear that the above class of non-linear differential equations does not show self-oscillatory behaviour, since the equations of motion generated by the time independent Hamiltonian, β, have at least β and the Casimir functions of the Lie group as integrals of motion. In other words, a system with a globally attractive limit cycle would necessarily have $\beta =$ constant on that limit cycle, and this cannot be reached by an initial state with a different value of β, because β is a non-constant analytic function of the phase space coordinates. In the context of these models, 'coarse graining' does not give rise to irreversible behaviour. Hence, in order to build a system having laser-like behaviour, dissipation is an absolute necessity. We turn to that question in the next section.

3. Linear Dissipation in Fermi Systems

Having previously considered a system consisting of many small systems coupled together by a mean field like Hamiltonian (i.e. an interaction which does not distinguish the relative location of the small systems), we now introduce the concept of the reservoir. The small systems will be composed of fermions, and each small system will have its own reservoir which is taken to be a fermion field of infinitely many degrees of freedom. We shall begin by studying the one system–one reservoir pair. It is then easy to describe many such pairs simultaneously and to see how the law of large numbers determines the properties of the intensive and fluctuation observables.

The use of fermion reservoirs, instead of boson or two-level atom reservoirs, has a mathematical and a physical motivation. From the mathematical point of view, the simplicity of dealing with bounded operators is obvious, and it turns out that the equations of motion have a closed solution (this boundedness would not be true if one used the boson reservoir mentioned in the introduction [S1]). Physically, dissipation by fermion–fermion coupling can hold for superconductors and for atoms in a solid state laser. In both cases the small system fermions are really identifiable as electrons (cf. Section 5).

We first consider one fermion $a^{\#}$ coupled to two Fermi fields $A^{\#}_w$, $B^{\#}_w$, where the parameter w characterizes the degrees of freedom of the Fermi field, and w varies over $\mathbf{R}^1$ for simplicity. All operators anticommute except for

$$\{a, a^{\#}\} = 1, \quad \{A_w, A^*_{w'}\} = \{B_w, B^*_{w'}\} = \delta(w - w'). \tag{3.1}$$

Let $\mathscr{A}^F$ be the C^*-algebra generated by $a^{\#}$ and the smeared-out $A^{\#}_w$, $B^{\#}_w$. The Hamiltonian

$$H^0 = \int_{-\infty}^{+\infty} dww(A^*_w A_w + B^*_w B_w) \tag{3.2}$$

is the infinitesimal generator (in the Fock representation) of the continuous 1-parameter group α^0_t of automorphisms of $\mathscr{A}^F$, which acts (in distribution language) as $a \to a$,

$C_w \to C_w \exp(-iwt)$. For simplicity, we shall often use the symbol C to stand either for A or for B. For $\epsilon > 0$

$$V_\varepsilon = \int dw\, e^{-\varepsilon w^2}\{(g_A^* A_w^* + g_B^* B_w^*)\, a + \text{h.c.}\} \tag{3.3}$$

belongs to $\mathscr{A}^F$. Hence $H = H^0 + V$ also generates a 1-parameter group α_t^ε of automorphisms of $\mathscr{A}^F$, and the action of α_t^ε can be computed by a norm-convergent Dyson series. The operators $a^\varepsilon(t) = \alpha_t^\varepsilon(a)$ and $C_w^\varepsilon(t) = \alpha_t^\varepsilon(C_w)$ are solutions of the linear Heisenberg equations

$$\begin{aligned} \dot{a}^\varepsilon(t) &= -i \int dw\, e^{-\varepsilon w^2}\{g_A A_w^\varepsilon(t) + g_B B_w^\varepsilon(t)\}, \\ \dot{C}_w^\varepsilon(t) &= -iwC_w^\varepsilon(t) - ig_C^* \exp(-\epsilon w^2)\, a^\varepsilon(t). \end{aligned} \tag{3.4}$$

Hence, one can express $C_w^\varepsilon(t)$ in terms of the initial condition C_w at $t = 0$ and by $a^\varepsilon(s)$ for $|s| \leqslant t$:

$$C_w^\varepsilon(t) = e^{-iwt} C_w - ig_C^* \int_0^t ds \exp -\,(\epsilon w^2 + iw(t-s))\, a^\varepsilon(s). \tag{3.5}$$

This leads to the linear Volterra equation

$$\begin{aligned} a^\varepsilon(t) = a &- (|g_A|^2 + |g_B|^2) \int_0^t ds a^\varepsilon(s)\, \delta^\varepsilon(t-s) \\ &- i \int_0^t ds \int dw \{g_A A_w + g_B B_w\}\cdot \exp -(\epsilon w^2 + iws). \end{aligned} \tag{3.6}$$

As $\epsilon \downarrow 0$, $\delta^\varepsilon(s) \to 2\pi\delta(s)$. One easily deduces from the iterative solution of (3.6):

Theorem 3.1: Let $\gamma = \pi(|g_A|^2 + |g_B|^2)$. Then for all $t \in \mathbf{R}^1$ and $h \in \mathscr{S}(\mathbf{R}^1)$

$$n - \lim_{\varepsilon \downarrow 0} a^\varepsilon(t) = a(t) = e^{-\gamma|t|}\, a - i \int_0^t ds\, e^{-\gamma|t-s|}\{g_A A(s) + g_B B(s)\}, \tag{3.7}$$

$$n - \lim_{\varepsilon \downarrow 0} \int dw h(w)\, C_w^\varepsilon(t) = \int dw h(w)\, C_w(t), \tag{3.8}$$

where

$$C(s) = \int dw\, C_w \exp(-iws), \tag{3.9}$$

$$C_w(t) = e^{-iwt} C_w - ig_C^* \int_0^t ds a(s) \exp - iw(t-s). \tag{3.10}$$

$(a, C_w) \leftrightarrow (a(t), C_w(t))$ can be extended to a strongly continuous 1-parameter group of automorphisms of $\mathscr{A}^F$, $\alpha_t(A) = n - \lim \alpha_t^\varepsilon(A)$.

Comment: When $\epsilon > 0$, the reservoir, as defined by $H^0 + V_\varepsilon$, is called regular. By modifying H^0 (e.g. $w \to (w^2+1)^{1/2}$), one could even make the energy bounded from below. The equation (3.6) can be solved by Laplace transform. We shall return to the study of regular reservoirs in a later paper.

In the sequel we shall always take $t \geqslant 0$. Then $a(t)$ is the solution of the differential equation

$$\dot{a}(t) = -\gamma a(t) - ig_A A(t) - ig_B B(t) \tag{3.11}$$

with 'damping constant' $\gamma \geqslant 0$ and 'fluctuation force' $F(t) = -ig_A A(t) - ig_B B(t)$. The Hilbert space of the system $a^\#$ and the reservoir $A_w^\#$, $B_w^\#$ will be taken as $\mathscr{H} = \mathscr{H}_S \otimes \mathscr{H}_R$. The initial state ω at $t=0$ of the combined system has to describe the coupling of the system in any state ω_S (an arbitrary density matrix in $\mathscr{H}_S \simeq \mathbf{C}^2$) to the reservoir in the state ω_R, which is 'full' of B-quanta and 'empty' of A-quanta. Hence we take $\omega = \omega_S \otimes \omega_R$, with $\omega_R = (\Omega_R, \cdot\Omega_R)$, i.e. the vector state Ω_R in the Fock space $\mathscr{H}_R$ which satisfies

$$A_w \Omega_R = B_w^* \Omega_R = 0, \quad \forall\, w \in \mathbf{R}. \tag{3.12}$$

The $A^\#(t)$ and $B^\#(t)$ act like 'white noise': they are 'Gaussian' in the sense that all truncated ω_R-expectations of order >2 vanish. They are also 'Markovian' and stationary with zero mean, since the only non-vanishing ω_R-expectations of order $\leqslant 2$ are

$$\omega_R(A(s)\, A^*(t)) = \omega_R(B^*(s)\, B(t)) = 2\pi\delta(s-t). \tag{3.13}$$

One deduces from (3.7) and (3.13) that the following limits are attained with errors that vanish exponentially in time:

$$\begin{aligned} &\lim_{t\to+\infty} \omega(a^\#(t)) = 0 \\ &\lim_{t\to+\infty} \omega(a^*(t)\, a(t)) = |g_B|^2(|g_A|^2 + |g_B|^2)^{-1}. \end{aligned} \tag{3.14}$$

Let $\omega_S(t)$ be the partial expectation over ω_R of $\omega = \omega_S \otimes \omega_R$ evolved under α_t. By (3.14), $\omega_S(t)$ converges to the Gibbs state for $a^* a$ with a temperature $(kT)^{-1} = \ln|g_A|^2 - \ln|g_B|^2 \in (-\infty, \infty)$.

From Theorem 3.1, the anticommutation relations between $a^\#(r)$, $C_w^\#(s)$ and $C^\#(t)$ follow for all times. For instance,

$$\begin{aligned} &\{a(s), a^*(t)\} = \exp - \gamma|t-s|, \\ &\{A(t), a^*(t)\} = i\pi g_A^*, \\ &\{B(t), a^*(t)\} = i\pi g_B^*, \\ &\{C(t), a(t)\} = 0. \end{aligned} \tag{3.15}$$

(3.15) is not in contradiction with the distribution identity $\{A(t), a^*(0)\} = 0$ for all t. One has to be careful in computations in the singular reservoir. Ambiguities are resolved by taking limits after a regularization: $a(t) = \lim a^\varepsilon(t)$.

The above formalism of linear dissipation will now be generalized to a 'small' system of infinitely many fermions, each coupled to its private big reservoir. In the spirit of Section 2, we have a countable number of C^*-algebras $\mathscr{A}_n^F$, $n \in \mathbf{Z}$, each of which is generated by M Fermi operators $a_n^{m\#}$, $1 \leqslant m \leqslant M$, with their own reservoir operators $A_{nw}^{m\#}$, $B_{nw}^{m\#}$. For all $1 \leqslant m \leqslant M$, $n \in \mathbf{Z}$, we assume

$$\{a_n^m, a_n^{m*}\} = 1,$$
$$\{A_{nw}^m, A_{nw'}^{m*}\} = \{B_{nw}^m, B_{nw'}^{m*}\} = \delta(w - w') \tag{3.16}$$

and that all other anticommutators between first degree fermion operators vanish. $\mathscr{A}_n^F$ contains the quasilocal C^*-algebra $\mathscr{A}_n$, generated by all particle-conserving fermion polynomials at site n. $\mathscr{A}_n$ contains the operators

$$S_n^{kl} = a_m^{k*} a_n^l, \quad 1 \leqslant k, l \leqslant M, \tag{3.17}$$

which are a basis of the Lie algebra $\mathscr{L} = \mathrm{Lie}\, U(M)$ of $U(M)$. By taking instead of $\mathscr{A}_n$ the *even* fermion algebra $\mathscr{A}_n^e \subset \mathscr{A}_n^F$, we could have included operators of the type $(a_n^k a_n^l)^\#$ and obtain the Lie algebra of $O(2M)$. Since all our examples are inbedded in $\mathrm{Lie}\, U(M)$, we refrain from this notational complication.

For finitely many sites, $1 \leqslant n \leqslant N$, we shall consider the linear dissipation described by the Hamiltonian

$$H_{(N)}^R = \sum_{n=1}^N H_n^R,$$

$$H_n^R = \sum_{m=1}^M \left\{ \int dw (A_{nw}^{m*} A_{nw}^m + B_{nw}^{m*} B_{nw}^m)\, w + \int dw (g_A^{m*} A_{nw}^{m*} a_n^m + g_B^{m*} B_{nw}^{m*} a_n^m + \text{h.c.}) \right\}, \tag{3.18}$$

where we assume

$$\gamma^m \equiv \pi(|g_A^m|^2 + |g_B^m|^2) > 0 \quad \text{for all } 1 \leqslant m \leqslant M.$$

With

$$C_n^m(t) = \int dw C_{nw}^m \exp(-iwt) \tag{3.19}$$

the time evolution of the local fermion operators, $a_n^m(t)$, is again of the type (3.11). Using (3.15), the resulting differential equations for the

$$S_n^{kl}(t) = \exp(iH_n^R t)\, S_n^{kl} \exp(-iH_n^R t) \quad \text{for } t \geqslant 0$$

become

$$\dot{S}_n^{kl}(t) = \delta^{kl} \eta^k - \gamma^{kl} S_n^{kl}(t) + F_{n+}^{kl}(t) + F_{n-}^{kl}(t), \tag{3.20}$$

where δ^{kl} is the Kronecker delta and

$$\eta^k = 2\pi |g_B^k|^2, \quad \gamma^{kl} = \gamma^k + \gamma^l,$$
$$F_{n+}^{kl}(t) = i g_A^{k*} A_n^k(t)^* a_n^l(t) + i g_B^l B_n^l(t)\, a_n^k(t)^*,$$
$$F_{n-}^{kl}(t) = -i g_B^{k*} a_n^l(t)\, B_n^k(t)^* - i g_A^l a_n^k(t)^* A_n^l(t). \tag{3.21}$$

The separation (3.20) into damping constants γ^{kl}, pump parameters η^k, and fluctuation forces $F_{n\pm}^{kl}(t)$ is natural, if one considers states $\omega = \omega_S \otimes \omega_R$ in which the reservoir $\omega_R = (\Omega_R, \cdot\, \Omega_R)$ is full of B-quanta and empty of A-quanta:

$$A_{nw}^m \Omega_R = B_{nw}^{m*} \Omega_R = 0 \tag{3.22}$$

for all m, n, w. It follows from (3.21) that, however complicated the $a_n^m(t)^{\#}$ may be (and they will be complicated in the next section), the partial trace

$$\omega_R(F_{n\pm}^{kl}(t)) = 0,$$

since

$$F_{n-}^{kl}(t)\,\Omega_R = F_{n+}^{kl}(t)^*\,\Omega_R = 0. \tag{3.23}$$

The fact that every local fermion operator $a_n^{m\#}$ is coupled to its own reservoir implies a strong law of large numbers for the M^2 intensive observables (which we collectively write in vector notation as $\underline{\sigma}_{(N)}$),

$$\sigma_{(N)}^{kl}(t) = N^{-1} \sum_{n=1}^{N} S_n^{kl}(t), \tag{3.24}$$

under the time evolution $H_{(N)}^R$:

Theorem 3.2: Assume that ω has the form $\omega_S \otimes \omega_R$, where ω_R satisfies (3.22) and ω_S is classical with respect to $\underline{\sigma}_{(N)}$ with probability measure μ_ω. Then ω is classical with respect to $\underline{\sigma}_{(N)}(t)$ and for all monomials

$$\lim_{N\to\infty} \omega(\sigma_{(N)}^{ij}(t)\ldots\sigma_{(N)}^{pq}(s)) = \int \mu_\omega(d\underline{\alpha})\,\sigma^{ij}(\underline{\alpha},t)\ldots\sigma^{pq}(\underline{\alpha},s) \tag{3.25}$$

uniformly on compacts in $t, \ldots s$. For all $t \in \mathbf{R}$, $\underline{\sigma}(\underline{\alpha}, t)$ is the solution of the linear dissipative equation

$$\dot{\sigma}^{kl}(\underline{\alpha},t) = \delta^{kl}\,\eta^k - \operatorname{sgn}(t)\,\gamma^{kl}\,\sigma^{kl}(\underline{\alpha},t),$$
$$\underline{\sigma}(\underline{\alpha},0) = \underline{\alpha}, \tag{3.26}$$

i.e.

$$\sigma^{kl}(\underline{\alpha},t) = \exp(-\gamma^{kl}|t|)\,\alpha^{kl} + \delta^{kl}[\eta^k - \exp(-\gamma^{kl}|t|)]/\gamma^{kl}. \tag{3.27}$$

Proof: From (3.20) one obtains for $t \geqslant 0$:

$$\dot{\sigma}_{(N)}^{kl}(t) = \delta^{kl}\,\eta^k - \gamma^{kl}\,\sigma_{(N)}^{kl}(t) + \varphi_{(N)}^{kl}(t), \tag{3.28}$$

where

$$\varphi_{(N)}^{kl}(t) = \varphi_{(N)+}^{kl}(t) + \varphi_{(N)-}^{kl}(t),$$

and

$$\varphi_{(N)\pm}^{kl}(t) = N^{-1} \sum_{n=1}^{N} F_{n\pm}^{kl}(t). \tag{3.29}$$

Hence,

$$\sigma_{(N)}^{kl}(t) = \sigma_{(N)}^{kl} \exp(-\gamma^{kl}\,t) + \delta^{kl}[\eta^k - \exp(-\gamma^{kl}\,t)]/\gamma^{kl} + \hat{\sigma}_{(N)}^{kl}(t). \tag{3.30}$$

Here,

$$\hat{\sigma}_{(N)}^{kl}(t) = \int_0^t ds \exp - \gamma^{kl}(t-s)\,\varphi_{(N)}^{kl}(s) \tag{3.31}$$

is explicitly known, using the solution (3.7) for the $a_n^m(s)^{\#}$ occurring in $\varphi_{(N)}^{kl}(s)$. One obtains (3.27), if one can show that, for all $r, \ldots t$ and all monomials (with $i = (i_1, i_2)$, $1 \leqslant i\rho \leqslant M$),

$$\lim_{N\to\infty} \omega(\sigma_{(N)}^i \ldots \sigma_{(N)}^j \,\hat{\sigma}_{(N)}^k(r) \ldots \hat{\sigma}_{(N)}^l(t)\, \sigma_{(N)}^m \ldots \sigma_{(N)}^n) = 0. \tag{3.32}$$

Since

$$\hat{\sigma}_{(N)}^k(r) = \hat{\sigma}_{(N)+}^k(r) + \hat{\sigma}_{(N)-}^k(r),$$

where the first term annihilates ω on the left- and the second on the right-hand side, it suffices to show that

$$[\sigma_{(N)}^j, \hat{\sigma}_{(N)\pm}^k(t)] = O(N^{-1}), \tag{3.33}$$

$$[\hat{\sigma}_{(N)\pm}^j(s), \hat{\sigma}_{(N)\pm}^k(t)] = O(N^{-1}) \tag{3.34}$$

for $N \to \infty$. To prove this we use (3.15) and (3.21):

$$[\sigma_{(N)}^{kl}, \hat{\sigma}_{(N)+}^{pq}(t)] = -\frac{i}{N^2} \sum_{n=1}^{N} \int_0^t ds \exp -\gamma^{pq}(t-s)$$
$$\times \{g_A^{p*} A_n^p(s)^* a_n^l \delta^{kq} \exp(-\gamma^q s) - g_B^q B_n^q(s) a_n^{k*} \delta^{lp} \exp(-\gamma^p s)\}. \tag{3.35}$$

Since the smeared-out A- and B-fields are bounded operators, (3.33) is manifestly $O(N^{-1})$. In the $(-,+)$-case, which is the most interesting, (3.34) leads to the majorization of

$$[\varphi_{(N)-}^{kl}(s), \varphi_{(N)+}^{pq}(t)]$$
$$= N^{-2} \sum_{n=1}^{N} [(g_B^{k*} a_n^l(s) B_n^k(s)^* + g_A^l a_n^k(s)^* A_n^l(s)), (g_A^{p*} A_n^p(t)^* a_n^q(t) + g_B^q B_n^q(t) a_n^p(t)^*]. \tag{3.36}$$

(3.36) can be evaluated in terms of anticommutators, by using the identity

$$[UV, XY] = U\{V,X\}Y - \{U,X\}\{V,Y\} + XU\{V,Y\} + \{U,X\}YV - X\{U,Y\}V, \tag{3.37}$$

and one obtains

$$N^{-2} \sum_{n=1}^{N} [2\pi |g_B^k|^2 \delta^{kq} \delta(s-t)\, a_n^l(s)\, a_n^p(s)^* + 2\pi |g_A^l|^2 g^{lp} \delta(s-t)\, a_n^k(s)^* a_n^q(s)$$
$$- g_B^{k*} g_A^{p*} \{a_n^l(s), A_n^p(t)^*\}\{B_n^k(s)^*, a_n^q(t)\} \tag{3.38}$$

and similar terms of the type $\{U,X\}\{V,Y\}$

$$+ g_B^{k*} g_B^q B_n^q(t)\, a_n^l(s) \{B_n^k(s)^*, a_n^p(t)^*\}$$

and similar terms with either B or A^* to the left or B^* or A to the right.

The first two terms are $O(N^{-1})$, when integrated over s and t. From (3.7), one obtains

$$\{a_n^l(s), A_n^p(t)^*\} = -2\pi i g_A^l \delta^{lp} \theta(s-t) \exp -\gamma^l(s-t). \tag{3.39}$$

Hence the third term is $O(N^{-1})$ and has support in $\{s=t\}$, and can therefore be omitted under the s–t-integration. All other terms are $O(N^{-1})$ and annihilate ω either to the left- or to the right-hand side. The commutators of the type $[\varphi^{kl}_{(N)-}(s), \varphi^{pq}_{(N)-}(t)]$ are of similar structure, except that the $\delta(s-t)$-terms are missing.

QED

(3.35) and (3.38) are useful in the following investigation up to $o(1)$ of the commutators of fluctuation observables, which are larger by a factor N:

Theorem 3.3: Under the assumption of Theorem 3.2, let $\underline{\sigma}_{(N)}$ have normal fluctuations in ω_S around $\underline{\alpha}$. Then the $\underline{\sigma}_{(N)}(t)$ have normal fluctuations in ω around $\underline{\sigma}(\underline{\alpha},t)$ (cf. (3.26)), and the limits $s(t)$ of the fluctuation observables

$$\sqrt{N}(\underline{\sigma}_{(N)}(t)-\underline{\sigma}(\underline{\alpha},t)) = \underline{s}_{(N)}(t),$$
$$\lim_{N\to\infty} \omega(s^{ij}_{(N)}(r)\ldots s^{pq}_{(N)}(t)) = (\Omega, s^{ij}(r)\ldots s^{pq}(t)\,\Omega),$$

are solutions of the linear inhomogeneous equations ($t \geqslant 0$):

$$\begin{aligned}\dot{s}^{kl}(t) &= -\gamma^{kl}s^{kl}(t) + f^{kl}_{\omega}(t),\\ s^{kl}(0) &= s^{kl}_{\omega},\end{aligned} \tag{3.40}$$

where the fluctuation forces $f^{kl}_{\omega}(t)$ will be defined by (3.45).

Proof: For $t \geqslant 0$, $\underline{s}_{(N)}(t)$ can be evaluated from (3.30) and (3.27):

$$s^{kl}_{(N)}(t) = s^{kl}_{(N)}\exp[-\gamma^{kl}t] + \int_0^t ds\, f^{kl}_{(N)}(s)\exp[-\gamma^{kl}(t-s)], \tag{3.41}$$

where $\underline{f}_{(N)}(s) = \sqrt{N}\,\underline{\varphi}_{(N)}(s)$. Instead of (3.32) we have now to investigate the limit $N\to\infty$ of

$$\omega(s^{i}_{(N)}\ldots s^{j}_{(N)} f^{k}_{(N)}(r)\ldots f^{l}_{(N)}(t)\, s^{m}_{(N)}\ldots s^{n}_{(N)}). \tag{3.42}$$

Now, $[s^{kl}_{(N)}, f^{pq}_{(N)+}(t)]$ leads to N times the integrand of (3.35) and, when integrated over t, is $O(1)$ for $N\to\infty$. However, this commutator is a sum of two terms which annihilate ω either to the left- or to the right-hand side. The commutator of this commutator with $s^{xy}_{(N)}$ or $f^{xy}_{(N)}(r)$ is $O(N^{-1/2})$, by the same argument. Hence $[s^{kl}_{(N)}, f^{pq}_{(N)}(t)]$ can be dropped in (3.42) for $N\to\infty$.

Similarly, $[f^{kl}_{(N)-}(s), f^{pq}_{(N)+}(t)]$ is $O(1)$, when integrated over s and t. Here N times the first two terms in (3.38) converge in (3.42) to the c-numbers

$$\Delta^{kl,pq}(s,t) \equiv 2\pi\delta(s-t)\{\delta^{kq}\delta^{lp}|g^{k}_{B}|^2 - \delta^{kq}\sigma^{pl}(\alpha,t)|g^{k}_{B}|^2 + \delta^{lp}\sigma^{kq}(\underline{\alpha},t)|g^{l}_{A}|^2\}. \tag{3.43}$$

The $\{U,X\}\{V,Y\}$-terms vanish when integrated over s and t. The others have $O(N^{-1/2})$ commutators with $s^{xy}_{(N)}$ and $f^{xy}_{(N)}(r)$ and annihilate ω either to the left- or to the right-hand side. Hence they can be dropped for $N\to\infty$.

Thus the limit $N\to\infty$ of (3.42) has the form

$$(\Omega_S, s^{i}_{\omega}\ldots s^{n}_{\omega}\,\Omega_S)(\Omega_R, f^{k}_{\omega}(r)\ldots f^{l}_{\omega}(t)\,\Omega_R), \tag{3.44}$$

where the operators s^{i}_{ω} and $f^{k}_{\omega}(r)$ act in different Hilbert spaces $\mathscr{H}_S$ and $\mathscr{H}_R . \mathscr{H}_S$, with vacuum Ω_S, is the Hilbert space of the GNS construction for the fluctuation operators s_ω^i at $t=0$ (cf. (2.7)). The $f_\omega^{kl}(s) = f_{\omega+}^{kl}(s) + f_{\omega-}^{kl}(s)$ are generalized Bose fields in a Fock space $\mathscr{H}_R$ with vacuum Ω_R satisfying for all s, t and all k, l, p, q:

$$\begin{aligned} &f_{\omega-}^{kl}(s)\,\Omega_R = 0 = f_{\omega+}^{kl}(s)^*\,\Omega_R, \\ &[f_{\omega-}^{kl}(s), f_{\omega-}^{pq}(t)] = 0, \\ &[f_{\omega-}^{kl}(s), f_{\omega+}^{pq}(t)] = \Delta^{kl,pq}(s,t). \end{aligned} \tag{3.45}$$

Hence we obtain (3.40). To the damping of the fluctuations, γ^{kl}, the dissipation adds a fluctuation force $f_{(N)}^{kl}(t)$, which does not vanish for $N \to \infty$. However, from the messy expression for finite N, $f_{(N)}^{kl}(t)$ is 'purified' by the law of large numbers to a Gaussian and Markovian external force, $f_\omega^{kl}(t)$. For $t \to \infty$, the $f_\omega^{kl}(t)$ become stationary and independent of ω, since

$$\Delta^{kl,pq}(s,t) \to 2\pi\delta(s-t)\,\delta^{kp}\,\delta^{lp}|g_B^k|^2. \tag{3.46}$$

QED

Remark: (1) If the model is changed so that all fermions $a_n^{m\#}$ are coupled to one and the same reservoir, then the macroscopic equations are drastically changed even in $O(N)$, and they no longer lead to the phenomenologically accepted exponential decay law (3.27).

(2) The limits $\epsilon \downarrow 0$ (from a regular to a singular reservoir) and $N \to \infty$ can be interchanged in Theorems 3.2 and 3.3.

4. Fermi Systems with Mean Field Interaction and Linear Dissipation

Now we shall combine the structures of Section 2 and 3. The main result will be that for $N \to \infty$ the intensive observables obey ordinary differential equations which are the sum of a Hamiltonian and a linear dissipative part. In a normally fluctuating state, the fluctuations around the expectation values of the intensive observables obey linearized equations with a Gaussian and Markovian external force. These fluctuation forces are Bose fields in a Fock representation with commutators determined by the instantaneous values of the intensive observables.

As in Section 3, the building blocks of the dynamics are fermions $a_n^{m\#}$ with reservoirs $A_{nw}^{m\#}$, $B_{nw}^{m\#}$ and the Lie algebra is Lie $U(M)$. As Hamiltonian for the finite system ($N < \infty$) we take

$$H_{(N)} = H_{(N)}^S + H_{(N)}^R. \tag{4.1}$$

$H_{(N)}^S = N\beta(\underline{\sigma}_{(N)})$ is a self-adjoint extensive polynomial, and $H_{(N)}^R$ has the form (3.18). $H_{(N)}^S \in \mathscr{A}$ can be considered to be a perturbation of the $H_{(N)}^R$-automorphism group of Theorem 3.1. Therefore $H_{(N)}$, too, determines a strongly continuous 1-parameter group of automorphisms $\alpha_{(N)}^t$ of $\mathscr{A}$.

For $t \geqslant 0$, the equations of motion for the local fermion operators $a_{n(N)}^m(t) = \alpha_{(N)}^t(a_n^m)$ are

$$\dot{a}_{n(N)}^m(t) = -\gamma^m a_{n(N)}^m(t) - ig_A^m A_n^m(t) - ig_B^m B_n^m(t) + i[H_{(N)}^S(t), a_{n(N)}^m(t)]. \tag{4.2}$$

The intensive observables satisfy

$$\dot{\sigma}_{(N)}^{kl}(t) = \delta^{kl}\eta^k - \gamma^{kl}\sigma_{(N)}^{kl}(t) + \varphi_{(N)}^{kl}(t) + i[H_{(N)}^S(t), \sigma_{(N)}^{kl}(t)]. \tag{4.3}$$

By the Lie structure, $i[H^S_{(N)}(t), \sigma^{kl}_{(N)}(t)]$ is a polynomial in $\underline{\sigma}_{(N)}(t)$. The fluctuation forces $\varphi^{kl}_{(N)}(t)$ have the form (3.21), (3.29) with $a^m_n(t)$ replaced by the very complicated solution $a^m_{n(N)}(t)$ of the local equations (4.2). However, if we consider initial states at $t=0$ of the type $\omega = \omega_S \otimes \omega_R$ with $\omega_R = (\Omega_R, \cdot \Omega_R)$ satisfying (3.22), one still has

$$\begin{aligned} &\varphi^{kl}_{(N)}(t) = \varphi^{kl}_{(N)+}(t) + \varphi^{kl}_{(N)-}(t), \\ &\varphi^{kl}_{(N)-}(t)\,\Omega_R = \varphi^{kl}_{(N)+}(t)^*\,\Omega_R = 0. \end{aligned} \tag{4.4}$$

$\sum_{m=1}^M \sigma^{mm}_{(N)}(t)$ commutes with $H^S_{(N)}$. For suitable dissipations and special $H^S_{(N)}$, the equations of motion might fortuitously close on a subgroup of $U(M)$ (e.g. on $SU(2) \times SU(2)$). In this section we shall disregard this possibility, which will be discussed in the examples of Section 5.

As in Section 3, the direct integration of the equations of motion of the intensive observables,

$$\begin{aligned} \sigma^{kl}_{(N)}(t) = {} & \sigma^{kl}_{(N)}(s) \exp[-\gamma^{kl}(t-s)] + \delta^{kl}(\eta^k - \exp[-\gamma^{kl}(t-s)]/\gamma^{kl} \\ & + \int_s^t dr \exp[-\gamma^{kl}(t-r)]\{i[H^S_{(N)}(r), \sigma^{kl}_{(N)}(r)] + \varphi^{kl}_{(N)}(r)\} \end{aligned} \tag{4.5}$$

can be effected with only a qualitative knowledge of the $\varphi^{kl}_{(N)}(r)$, which depend on the local operators $a^m_{n(N)}(r)$:

Theorem 4.1: Let $\omega = \omega_S \otimes \omega_R$, where ω_S is classical with respect to $\underline{\sigma}_{(N)}$ with probability measure μ_ω, and ω_R satisfies (3.22). Then for all monomials

$$\lim_{N\to\infty} \omega\,(\sigma^{ik}_{(N)}(r) \dots \sigma^{pq}_{(N)}(t)) = \int \mu_\omega(d\underline{\alpha})\, \sigma^{ik}(\underline{\alpha}, r) \dots \sigma^{pq}(\underline{\alpha}, t) \tag{4.6}$$

uniformly on compacts in $r, \dots t$. Here $\underline{\sigma}(\underline{\alpha}, t)$ is the solution of the classical equation

$$\dot{\sigma}^{kl} = \delta^{kl}\,\eta^k - \gamma^{kl}\,\sigma^{kl} + \{\beta, \sigma^{kl}\} \tag{4.7}$$

with $\underline{\sigma}(\underline{\alpha}, 0) = \underline{\alpha}$. In particular, if ω_S is pure with respect to $\underline{\sigma}_{(N)}$, then ω is a pure classical state with respect to all $\underline{\sigma}_{(N)}(t)$.

Proof: We first remark that the classical equations (4.7) have global solutions for every initial condition $\underline{\alpha} \in \mathbf{R}^{M^2}$: The Casimir function $\tau = \sum_{i,j=1}^M |\sigma^{ij}|^2$ of $U(M)$ satisfies $\{\beta, \tau\} = 0$, and therefore along every integral curve of (4.7)

$$\dot{\tau} = -\sum_{i,j=1}^M 2\gamma^{ij}|\sigma^{ij}|^2 + \sum_{k=1}^M 2\eta^k\,\sigma^{kk}. \tag{4.8}$$

Hence $\dot{\tau} = 0$ on the compact surface $\sum$ given by

$$\sum_{i \neq j} \gamma^{ij}|\sigma^{ij}|^2 + \sum_k \gamma^{kk}(\sigma^{kk} - \eta^k/2\gamma^{kk})^2 = \sum_k (\eta^k)^2(4\gamma^{kk})^{-1}. \tag{4.9}$$

Here we use the assumption on (3.18) that $\gamma^m > 0$ for all $1 \leqslant m \leqslant M$. Hence for all ρ sufficiently large, the regions $D_\rho = \{\sigma | \sum |\sigma^{ij}|^2 \leqslant \rho\}$ are invariant under the flow of (4.7), since $\dot{\tau} \leqslant 0$ on the boundary of D_ρ.

As in the proof of Theorem 2.2 we shall show that the quantum equations (4.5) again have the fixed point property, that one can approximate $\underline{\sigma}_{(N)}(t)$ by $\underline{\sigma}_{(N)n(0)\ldots n(r)}(t)$ uniformly in N and that the ω-expectation values of the iterative solutions $\underline{\sigma}_{(N)n(0)\ldots n(r)}(t)$ converge to the μ_ω-mean of the corresponding classical approximations $\underline{\sigma}_{n(0)\ldots n(r)}(\underline{\alpha},t)$.

By solving (4.5) for $\varphi^{kl}_{(N)}(r)$ and using the fact that $\|\sigma^{kl}_{(N)}(r)\| = 1$ uniformly in N, one proves that

$$\left\| \int_s^t dr \varphi^{kl}_{(N)}(r) \exp - \gamma^{kl}(t-r) \right\| \leqslant \text{const}\,(1 + |t-s|) \tag{4.10}$$

uniformly in N. Hence (4.5) again has the fixed point property, and the solution can be approximated uniformly in N by polynomials in the initial conditions $\underline{\sigma}_{(N)}$ (having the same structure as the approximating polynomials for the classical equations) and by integrals over the fluctuation forces $\varphi_{(N)}(s)$. We have to show that in every fixed order in the iteration scheme the fluctuation forces give a vanishing contribution in the ω-expectation

$$\omega(\sigma^i_{(N)} \ldots \sigma^j_{(N)}\, \varphi^k_{(N)}(r) \ldots \varphi^l_{(N)}(t)\, \sigma^m_{(N)} \ldots \sigma^n_{(N)}) \tag{4.11}$$

for $N \to \infty$. Using (4.4) this follows if one can commute the $\varphi^k_{(N)\pm}(r)$ freely among themselves and with the $\sigma^j_{(N)}$, with $O(N^{-1})$ errors for $N \to \infty$ when integrated over r.

For this purpose we investigate the behaviour of the anticommutators $\{a^m_{n(N)}(t)^{\#}, C^p_q(s)^{\#}\}$ for $N \to \infty$. For definiteness we consider

$$\{a^m_{n(N)}(t)^{\#}, A^1_1(s)^*\} \exp \gamma^m t \equiv b^m_{n(N)}(s,t)^{\#}. \tag{4.12}$$

We define $b^1 \equiv b^1_1$, $b^2 \equiv b^{1*}_1$, $\ldots$, $b^{2M} \equiv b^{M*}_1$, $\ldots$ $b^{2MN} \equiv b^{M*}_N$. Then the local equations (4.2) lead to the following linear Volterra equation for the anticommutators (4.12):

$$b^1_{(N)}(s,t) = -2\pi i g^1_A\, \theta(t-s) \exp \gamma^1 s + \int_0^t dr \left\{ \sum_{k=1}^{2M} P^{1k}_{(N)}(s,r) + N^{-1} \sum_{k=2M+1}^{2NM} P^{1k}_{(N)}(s,r) \right\},$$

$$b^l_{(N)}(s,t) = \int_0^t dr \left\{ \sum_{k=2M(L-1)+1}^{2ML} P^{lk}_{(N)}(s,r) + N^{-1} \sum_{\substack{k=1 \\ k \notin [2M(L-1),\,2ML]}}^{2MN} P^{lk}_{(N)}(s,r) \right\}, \tag{4.13}$$

$(l > 1)$

where L satisfies $2M(L-1) < l \leqslant 2ML$ and where the $P^{lk}_{(N)}(s,r)$ are fixed polynomials in the local and intensive observables and linear in $b^k_{(N)}(s,r)$ with coefficients which are uniformly bounded in N. Hence one can 'compute' the $b^k_{(N)}(s,t)$ by iterating (4.13), and this series converges uniformly in N, since the $a^m_{n(N)}(r)^{\#}$ and $\underline{\sigma}_{(N)}(r)$ are uniformly bounded. Since the input vanishes for $s > t$, $b^k_{(N)}(s,t) \equiv 0$ for $s > t$. For $0 \leqslant s \leqslant t$, $b^k_{(N)}(s,t)$ is norm continuous in s, t and satisfies

$$\sup_N \|b^k_{(N)}(s,t)\| < \infty, \quad 1 \leqslant k \leqslant 2M,$$

$$\sup_N N \|b^k_{(N)}(s,t)\| < \infty, \quad 2M+1 \leqslant k \leqslant 2MN, \tag{4.14}$$

uniformly in N and uniformly on compacts in t. By a similar system of linear Volterra equations one obtains for $N \to \infty$

$$\sup_{n,N} \|\{a^m_{n(N)}(t)^{\#}, a^p_{n(N)}(s)^{\#}\}\| < \infty$$

$$\sup_{n \neq q, N} N\|\{a^m_{n(N)}(t)^{\#}, a^p_{q(N)}(s)^{\#}\}\| < \infty, \tag{4.15}$$

uniformly on compacts in s, t, and hence

$$[\sigma^{kl}_{(N)}(s), \sigma^{pq}_{(N)}(t)] = O(N^{-1}). \tag{4.16}$$

Now the proof of Theorem 3.2 can be repeated, with the only difference being that the mean field interaction contributes $O(N^{-1})$ to the non-diagonal part in the anti-commutators, as shown in the second line of (4.15). Formerly this part vanished.

QED

Similarly, the limit of the fluctuation observables in a normally fluctuating state can be discussed by generalizing Theorems 2.4 and 3.3. Let $\mathscr{H}_S$ be the GNS Hilbert space of the fluctuation operators s^{kl} at time zero with cyclic vector Ω_S. Let $\mathscr{H}_R$ with vacuum Ω_R be the GNS Hilbert space of field operators $f^{ki}(s) = f^{kl}_{+}(s) + f^{kl}_{-}(s)$ satisfying (3.45) with $\Delta^{kl,pq}(s,t)$ of the form (3.43), where $\sigma^{kl}(\underline{\alpha},t)$ are the solutions of the classical equations (4.7).

Theorem 4.2: Under the assumptions of Theorem 4.1 let $\underline{\sigma}_{(N)}$ have normal fluctuations in ω_S around $\underline{\alpha}$. Then all $s^{ik}_{(N)}(t) = N^{1/2}(\sigma^{ik}_{(N)}(t) - \sigma^{ik}(\underline{\alpha},t))$ have normal fluctuations in $\omega = \omega_S \otimes \omega_R$. For every monomial

$$\lim_{N\to\infty} \omega(s^{ik}_{(N)}(r) \dots s^{pq}_{(N)}(t)) = (\Omega, s^{ik}(r) \dots s^{pq}(t)\, \Omega), \tag{4.17}$$

where $s^{ik}(t)$ is the solution (in $\mathscr{H} = \mathscr{H}_S \otimes \mathscr{H}_R$ with vacuum $\Omega = \Omega_S \otimes \Omega_R$) of the linearized equations of (4.7) around $\sigma^{ik}(\underline{\alpha},t)$ with inhomogeneity $f^{ik}(t)$.

Remark: The $f^{ik}(t)$ are Gaussian and Markovian, but not stationary, unless $\underline{\sigma}(\underline{\alpha},t)$ is stationary.

Proof: By subtracting the quantum mechanical and the classical equations one obtains

$$s^{ik}_{(N)}(t) = s^{ik}_{(N)} \exp(-\gamma^{ik} t) + \int_0^t ds\, f^{ik}_{(N)}(s) \exp(-\gamma^{ik}(t-s))$$

$$+ \int_0^t ds\, N^{1/2} (i[H^S_{(N)}(s), \sigma^{ik}_{(N)}(s)] - \{\beta, \sigma^{ik}\}(s)) \exp(-\gamma^{ik}(t-s)). \tag{4.18}$$

As in Theorem 2.4, $N^{1/2}(i[H^S_{(N)}, \sigma^{ik}_{(N)}] - \{\beta, \sigma^{ik}\})(s)$ can be expressed in terms of polynomials in intensive observables $\underline{\sigma}_{(N)}(s)$ and classical solutions $\underline{\sigma}(s)$. Moreover, it contains the $\underline{s}_{(N)}(s)$ linearly. Hence one can 'solve' (4.18) by iteration in terms of the initial conditions $\underline{s}_{(N)}$ and the fluctuation forces $\underline{f}_{(N)}(s) = N^{1/2}\varphi_{(N)}(s)$. This iterative solution has coefficients which are uniformly bounded, and their ω-expectation values converge for $N \to \infty$. The computation of the Wightman functions starts from expressions of the type (3.42) in the manner given in Theorems **3.3** and **4.1**.

QED

We conclude this section with some qualitative statements about the classical equations (4.7), which describe the approach to equilibrium of our class of systems to $O(N)$. We have already remarked that all solutions of (4.7) are global and uniformly bounded in t in terms of the initial conditions $\underline{\alpha}$. In the purely lossy case (i.e. $\eta^m = 0$ for $1 \leqslant m \leqslant M$, $\sigma^{ik}(\underline{\alpha}, t) \to 0$ for $t \to \infty$ and all $\underline{\alpha}$. For small pumping, η, the stationary solution (for certain β)

$$\sigma^{ij} = 0, \quad i \neq j,$$

$$\sigma^{kk} = \eta^k / 2\gamma^{kk} \tag{4.19}$$

persists as a stable attractor until it becomes unstable. For some critical value of the coefficients of (4.7) (e.g. of the pump parameters η^k, which are sometimes under the control of the experimenters), another attracting critical point or a closed orbit [H8] or even more complicated attractors [J1], [R4] can bifurcate from (4.19).

In many cases the set of attractors $A(\eta)$ will be piecewise analytic in η. At every η_c, where the structure of the attractors changes, we speak of a *reservoir-induced phase transition of an open system* which is in general far away from thermal equilibrium. Experimentally and theoretically, the best understood example is the laser threshold (see Section 5), but other phenomena in solid state physics, in chemistry [G1] and in biology [P1] might have a similar idealized microscopic explanation.

It is to be remembered that, for small pumping, every classical state converges to the pure classical state (4.19) as $t \to \infty$. Consequently the probability measure μ_ω^t loses its classical entropy.

If the system is in a normal classical state, then the linearized equations around the classical observables can have stability, asymptotic stability, or instability with respect to fluctuations. All three cases can occur: For instance the stable but not asymptotically stable case holds for attracting periodic solutions, e.g. for the laser above threshold. In that case, there is one characteristic exponent zero, and the fluctuations are not all damped in directions parallel to the periodic orbit. The fluctuation forces succeed, in a sense, to restore the broken symmetry in this case. It is tempting to identify the unstable attractors with metastable phases [W2]. In normal classical states, Onsager's hypothesis [O1] about the 'regression of fluctuations' is valid: The equations of motion for the fluctuations, when averaged over ω_R, are equal to the linearized equations of the intensive observables. This conjectured profound connection between transport theory and fluctuation theory can easily be derived in our mean field type models.

We have always taken the limit $N \to \infty$ first and then the limit $t \to \infty$. Physically one wants to take the limit the other way around. We can show in our models that such an interchange is justified if the damping is very large with respect to the coupling constants of the non-linearities in H^S. We do not expect this to hold for all values of the parameters, at least not at critical points. This accounts for the difference between our results and those of Risken [R1] and of Hempstead and Lax [H2] at the laser threshold.

5. Examples of Reservoir-Induced Phase Transitions

The mean field Josephson oscillator

As a first illustration we shall study a model of two strongly coupled superconductors. Let $\underline{k}_1, \dots \underline{k}_N$ be the momenta characteristic of 1-electron states in the neighbourhood of the Fermi surface of a many electron system in a box of volume V. N and V are assumed to be proportional. The 1-electron states are denoted by

$$+n = (\underline{k}_n, \uparrow), \quad -n = (-\underline{k}_n, \downarrow), \tag{5.1}$$

where the arrow gives the spin state. Now imagine two such boxes, V, of electrons, and let $a^{\#}_{\pm n}$ and $b^{\#}_{\pm n}$ be their electron creation and annihilation operators with the usual anticommutation relations. Let us introduce quasi-spin operators for the two electron systems:

$$R^+_n = a^*_n a_{-n} = (R^-_n)^*, \quad R^3_n = (a^*_n a_n - a^*_{-n} a_{-n})/2$$

$$S^+_n = b^*_n b_{-n} = (S^-_n)^*, \quad S^3_n = (b^*_n b_n - b^*_{-n} b_{-n})/2$$

$$R^i_{(N)} = \sum_{n=1}^{N} R^i_n, \quad S^i_{(N)} = \sum_{n=1}^{N} S^i_n. \tag{5.2}$$

They are the generators of Lie$(SU(2) \times SU(2)) = \mathscr{L}$. The total Hamiltonian of the system is

$$H^S_{(N)} = \epsilon(R^3_{(N)} + S^3_{(N)}) + \mu N^{-1}(R^+_{(N)} R^-_{(N)} + S^+_{(N)} S^-_{(N)}) + \lambda N^{-1}(R^+_{(N)} S^-_{(N)} + S^+_{(N)} R^-_{(N)}). \tag{5.3}$$

The first two terms are the usual BCS strong coupling Hamiltonians for the two boxes [T2], which we henceforth refer to as superconductors. The last term constitutes a tunnelling Hamiltonian which one might think of as a mean field Josephson junction.

The closed system shows a second-order phase transition [H5]. As a caricature of a driven Josephson junction from a constant current source, we shall assume that electrons have the R-superconductor by linear electron dissipation into a large empty normal electron reservoir and enter the S-superconductor from a full reservoir. The reservoir fermion field operators are denoted by $A^{\#}_{\pm nw}$ and $B^{\#}_{\pm nw}$ with $w \in \mathbf{R}$ and the usual anticommutation relations. According to Section 3, we consider states $\omega = \omega_S \otimes \omega_R$, where $\omega_R = (\Omega_R, \cdot \Omega_R)$, which satisfy

$$A_{\pm nw} \Omega_R = 0 = B^*_{\pm nw} \Omega_R \tag{5.4}$$

for all $\pm n$ and all w.

The linear dissipation is described by

$$H^R_{(N)} = \sum_{\pm n} H^R_{\pm n},$$

$$H^R_{\pm n} = \int dw (A^*_{\pm nw} A_{\pm nw} + B^*_{\pm nw} B_{\pm nw})\, w + \int dw (g^* A^*_{\pm nw} a_{\pm n} + g^* B^*_{\pm nw} b_{\pm n} + \text{h.c.}), \tag{5.5}$$

where $\gamma = 2\pi |g|^2 > 0$.

The results of Section 4 apply to this model. We shall discuss the behaviour of the intensive observables $\underline{\rho}_{(N)} = N^{-1}\underline{R}_{(N)}$ and $\underline{\sigma}_{(N)} = N^{-1}\underline{S}_{(N)}$ in classical states ω_S. The classical equations of motion are

$$\begin{aligned}
\dot{\rho}^+ &= (i\epsilon - \gamma)\,\rho^+ - 2i\mu\rho^+\rho^3 - 2i\lambda\sigma^+\rho^3,\\
\dot{\rho}^3 &= -\gamma(\rho^3 + \tfrac{1}{2}) + i\lambda(\rho^-\sigma^+ - \rho^+\sigma^-),\\
\dot{\sigma}^+ &= (i\epsilon - \gamma)\,\sigma^+ - 2i\mu\sigma^+\sigma^3 - 2i\lambda\rho^+\sigma^3,\\
\dot{\sigma}^3 &= -\gamma(\sigma^3 - \tfrac{1}{2}) + i\lambda(\sigma^-\rho^+ - \sigma^+\rho^-).
\end{aligned} \tag{5.6}$$

We remark that the dissipation does not couple operators from $\mathscr{L}$ to other operators in Lie $U(4)$. From (5.7) we obtain the time dependence of the Casimir functions $\underline{\rho}^2 = |\rho^+|^2 + (\rho^3)^2$ and $\underline{\sigma}^2$:

$$\begin{aligned}
\frac{d}{dt}\underline{\rho}^2 &= -2\gamma(|\rho^+|^2 + (\rho^3 + \tfrac{1}{4})^2 - \tfrac{1}{16}),\\
\frac{d}{dt}\underline{\sigma}^2 &= -2\gamma(|\sigma^+|^2 + (\sigma^3 - \tfrac{1}{4})^2 - \tfrac{1}{16}).
\end{aligned} \tag{5.7}$$

The physical region

$$D = \{|\underline{\rho}| \leqslant \tfrac{1}{2},\ |\underline{\sigma}| \leqslant \tfrac{1}{2}\} \tag{5.8}$$

is invariant under time evolution, and even if one starts outside of D, one would eventually come into D, as $t \to \infty$.

For the simplest typical case, let us set $\mu = 0$. Then the global flow can be discussed further, if one uses the following consequences of (5.6):

$$\frac{d}{dt}(\rho^3 + \sigma^3) = -\gamma(\rho^3 + \sigma^3), \tag{5.9}$$

$$\frac{d}{dt}(\rho^+ + i\sigma^+) = (i\epsilon - \gamma + 2\lambda\sigma^3)(\rho^+ + i\sigma^+) - 2i\lambda\sigma^+(\rho^3 + \sigma^3), \tag{5.10}$$

$$\frac{d}{dt}(\rho^+ - i\sigma^+) = (i\epsilon - \gamma - 2\lambda\sigma^3)(\rho^+ - i\sigma^+) - 2i\lambda\sigma^+(\rho^3 + \sigma^3). \tag{5.11}$$

One sees from (5.9) that $\rho^3(t) + \sigma^3(t) = (\rho^3(0) + \sigma^3(0))\exp{-\gamma t}$. For $\lambda < \gamma$ one deduces from (5.10) and (5.11), using $\sigma^3 \leqslant \frac{1}{2}$ for $(\underline{\rho}, \underline{\sigma}) \in D$, that both $\rho^+ + i\sigma^+$ and $\rho^+ - i\sigma^+$ converge to zero as $t \to \infty$. By going back to (5.6) one proves that the stationary solution

$$\begin{aligned}
\rho^+ &= \sigma^+ = 0,\\
\rho^3 &= -\sigma^3 = -\tfrac{1}{2}
\end{aligned} \tag{5.12}$$

is a global attractor for $\lambda < \gamma$.

For $\lambda \geqslant \gamma$, (5.12) becomes unstable, as the discussion of the linearized equations around (5.12) shows. If one writes (5.6) as a real system, then a pair of complex conjugate eigenvalues for the linearized system passes the imaginary axis from the left, away from the origin. By the Hopf bifurcation theorem [H8], a stable attracting periodic solution should bifurcate from (5.12). The harmonic Ansatz $\rho^+(t) = \rho^+\exp i\omega t$, $\sigma^+(t) = \sigma^+\exp i\omega t$ with time-independent ρ^+, σ^+, ρ^3, σ^3 and ω leads to the Hopf solution

$$\omega = \epsilon, \quad \sigma^3 = -\rho^3 = \gamma/2\lambda,$$

$$\rho^+ = i\sigma^+, \quad |\sigma^+|^2 = \gamma(1-\gamma/\lambda)/4\lambda. \tag{5.13}$$

Of course, the common phase of ρ^+ and σ^+ is not uniquely determined by (5.13), because of time translation invariance.

We see that this system of strongly coupled superconductors interacting with reservoirs of different 'voltages' can exist in two regimes: For $\lambda \leqslant \gamma$, the friction dominates, and $\underline{\rho}$ is drained and $\underline{\sigma}$ drowned by the reservoirs. For λ slightly larger than γ, the system becomes self-oscillatory. The asymptotic state of the system does not depend on the initial conditions (except on a set of codimension two and except for a phase factor). The mean field Josephson oscillator produces a periodic process at the expense of a non-periodic source of energy, just as a clock or a steam engine does.

The Dicke–Haken–Lax laser model

This model has already been physically characterized in the introduction. We take $H^S_{(N)}$ as in (1.1) and H^P as in (1.2) with $k(w) = k > 0$ and $\kappa = \pi k^2$ and $E(w) = w$. In this paper we want to avoid difficulties associated with unbounded boson operators. Therefore we shall consider a 'solid state laser', where the dissipation (or pumping) proceeds not by the coupling of two-level atoms with a reservoir consisting of two-level atoms but by an exchange of electrons with an electron reservoir.

The atomic 'spin' operators are represented, as in the Josephson model, in terms of two Fermi operators per atom n

$$S^+_n = b^*_n b_{-n} = (S^-_n)^*, \quad S^3_n = (b^*_n b_n - b^*_{-n} b_{-n})/2. \tag{5.14}$$

Each $b^\#_{\pm n}$ is coupled to its own Fermi bath with creation and annihilation operators $B^\#_{\pm nw}$ and $C^\#_{\pm nw}$ by

$$H^A_{(N)} = \sum_{\pm n} H^A_{\pm n}$$

$$H^A_{\pm n} = \int dw (B^*_{\pm nw} B_{\pm nw} + C^*_{\pm nw} C_{\pm nw})\, w + \int dw \{(f^*_\pm B^*_{\pm nw} + g^*_\pm C^*_{\pm nw})\, b_{\pm n} + \text{h.c.}\}. \tag{5.15}$$

The state Ω_R of the reservoir is again characterized by

$$A_w \Omega_R = B_{\pm nw} \Omega_R = C^*_{\pm nw} \Omega_R = 0, \tag{5.16}$$

and we assume customary commutation and anticommutation relations.

We choose the coupling constants in (5.15) to satisfy

$$|f_+|^2 + |g_+|^2 = |f_-|^2 + |g_-|^2,$$

$$|g_+|^2 = |f_-|^2, \tag{5.17}$$

which insures that the mean electron number per atom

$$\nu_{(N)} = N^{-1} \sum_{n=1}^{N} (b^*_n b_n + b^*_{-n} b_{-n}) \tag{5.18}$$

does not couple in $O(1)$ through the dissipation to the other operators $\underline{\sigma}_{(N)}$ and $\alpha_{(N)}$, and has the stationary value 1 (macroscopic charge neutrality). There remain two free parameters describing the dissipation:

$$\gamma = 2\pi(|g_+|^2 + |g_-|^2),$$
$$\eta = (|g_+|^2 - |g_-|^2)/2(|g_+|^2 + |g_-|^2). \tag{5.19}$$

Then the macroscopic equations of the laser take the form (1.14), (1.15) and (1.16), except that in (1.16) the factor 2γ is changed into γ. This will neither change the laser frequency (5.39) nor the threshold condition (5.37), but only the slope of $|\alpha|^2_{\text{stat}}$ as a function of $\eta - \eta_c$ (5.40). It is not surprising that different models for the reservoirs can lead to different physical predictions in $O(N)$. The 'quantum mechanical consistency' [H1] allows for one more free parameter ($\delta > 0$ instead of 2γ in (1.16)) in the Dicke laser model.

The time dependence of the Casimir function becomes

$$\frac{d}{dt}\sigma^2 = -2\gamma(|\sigma^+|^2 + (\sigma^3 - \eta/2)^2 - \eta^2/4), \tag{5.20}$$

and for $\mu = 0$ one has, in addition,

$$\frac{d}{dt}(|\alpha|^2 + \sigma^3) = -2\kappa|\alpha|^2 - \gamma(\sigma^3 - \eta). \tag{5.21}$$

One can prove [H6] and expects from the equilibrium thermodynamics of the system [H4] that the mean photon number per atom, $|\alpha|^2$, should approach a stationary value $|\alpha|^2_{\text{stat}} < \infty$ for every reasonable initial condition. In order to avoid having to deal with unbounded boson operators, we therefore introduce a high photon number cut-off at $a^*a = N\zeta$, where $\zeta \gg |\alpha|^2_{\text{stat}}$ is an arbitrarily large integer. Technically, we replace (see e.g. [A1])

$$a \to (\zeta N)^{-1/2} R^-_{(\zeta N)}. \tag{5.22}$$

Here the $R^i_{(\zeta N)}$ are spin operators in the irreducible representation of $SU(2)$ with spin $\zeta N/2$. If one replaces the n-photon states Ω_n by eigenstates Ω'_n of $R^3_{(\zeta N)}$ with eigenvalue $n - \zeta N/2$, then one has

$$\Omega'_n = (n!)^{-1}\binom{\zeta N}{n}^{-1/2}(R^+_{(\zeta N)})^n\,\Omega'_0, \tag{5.23}$$

$$R^+_{(\zeta N)}\,\Omega'_n(\zeta N)^{-1/2} = (n+1)^{1/2}\,\Omega'_{n+1}(1 - n/\zeta N)^{1/2},$$

$$[R^-_{(\zeta N)}, R^+_{(\zeta N)}]\,\Omega'_n(\zeta N)^{-1} = \Omega'_n(1 - 2n/\zeta N).$$

In the sense of (5.23), the $R^\pm_{(\zeta N)}(\zeta N)^{-1/2}$ act as photon creation and annihilation operators uniformly in N for mean photon numbers $n/N \ll \zeta$.

The final form of our Dicke Hamiltonian is

$$H^S_{(N)} = \nu(R^3_{(\zeta N)} + \zeta N/2) + \epsilon S^3_{(N)} + \zeta^{-1/2}N^{-1}(\lambda R^+_{(\zeta N)}S^-_{(N)} + \mu R^+_{(\zeta N)}S^+_{(N)} + \text{h.c.}). \tag{5.24}$$

There exists a model of a photon reservoir, which leads to the same phenomenological dissipative part as (1.2) (up to $O(\zeta^{-1})$): One only has to introduce $2\zeta N$ Fermi operators $a^{\#}_{\pm m}$, $1 \leqslant m \leqslant \zeta N$ and

$$R^{+}_{m} = a^{*}_{m} a^{*}_{-m}, \quad R^{3}_{m} = (a^{*}_{m} a_{m} - a_{-m}\, a^{*}_{-m})/2,$$

$$R^{i}_{(\zeta N)} = \sum_{m=1}^{\zeta N} R^{i}_{n}, \tag{5.25}$$

$$a_{\pm m} \Omega'_{0} = 0, \quad 1 \leqslant m \leqslant \zeta N.$$

Then one couples the $a^{\#}_{\pm m}$ to Fermi reservoirs $A^{\#}_{\pm mw}$ by $H^{P}_{(N)} = \sum H^{P}_{\pm m}$, where

$$H^{P}_{\pm m} = \int dw \left(w A^{*}_{\pm mw} A_{\pm mw} + \frac{k}{\sqrt{2}} (A^{*}_{\pm mw}\, a_{\pm m} + \text{h.c.}) \right) \tag{5.26}$$

and $A_{\pm mw} \Omega_{R} = 0$.

For ζ fixed, the results of Section 4 apply for the intensive observables

$$\sigma^{i}_{(N)} = N^{-1} S^{i}_{(N)}, \quad \rho^{i}_{(\zeta N)} = (\zeta N)^{-1} R^{i}_{(\zeta N)}. \tag{5.27}$$

More useful for the limit $\zeta \to \infty$ will be

$$\alpha_{(\zeta, N)} = \zeta^{1/2} \rho^{-}_{(\zeta N)}. \tag{5.28}$$

The resulting equations of motion are

$$\dot{\alpha}_{(\zeta N)} = -(i\nu + \kappa)\, \alpha_{(\zeta, N)} + 2i\lambda\, \rho^{3}_{(\zeta N)}\, \sigma^{-}_{(N)} + 2i\mu \rho^{3}_{(\zeta N)}\, \sigma^{+}_{(N)} + \varphi_{(\zeta, N)}, \tag{5.29}$$

$$\dot{\sigma}^{-}_{(N)} = -(i\epsilon + \gamma)\, \sigma^{-}_{(N)} + 2i\lambda\, \sigma^{3}_{(N)}\, \alpha_{(\zeta, N)} + 2i\mu \sigma^{3}_{(N)}\, \alpha^{*}_{(\zeta, N)} + \chi^{+}_{(\zeta, N)}, \tag{5.30}$$

$$\dot{\sigma}^{3}_{(N)} = -\gamma(\sigma^{3}_{(N)} - \eta) + (i\lambda \sigma^{-}_{(N)}\, \alpha^{*}_{(\zeta, N)} + i\mu \sigma^{-}_{(N)}\, \alpha_{(\zeta, N)} + \text{h.c.}) + \chi^{3}_{(\zeta, N)}. \tag{5.31}$$

These equations are the same as in Section 1, except for different fluctuation forces and for the fact that there is $\rho^{3}_{(\zeta N)}$ instead of $\frac{1}{2}$ in (5.29). $\rho^{3}_{(\zeta N)}$ is not a c-number but satisfies

$$\dot{\rho}^{3}_{(\zeta N)} = -\kappa(\rho^{3}_{(\zeta N)} + \tfrac{1}{2}) + \varphi^{3}_{(\zeta, N)} + \zeta^{-1}(i\lambda \sigma^{+}_{(N)}\, \alpha_{(\zeta, N)} + i\mu \sigma^{+}_{(N)}\, \alpha^{*}_{(\zeta, N)} + \text{h.c.}). \tag{5.32}$$

We note, however, that if ζ is so large that the last term, which is $O(\zeta^{-1})$, can be neglected then, provided we start at time zero with $\rho^{3}_{(\zeta N)} = -\frac{1}{2}$, (5.29) reduces to (1.14).

We shall consider states $\omega = \omega_{S} \otimes \omega_{R}$ with ω_{S} classical with respect to $\underline{\rho}_{(\zeta N)}$ and $\underline{\sigma}_{(N)}$. Then the ω-expectation values of monomials in $\underline{\sigma}_{(N)}(t)$ and $\alpha_{(\zeta, N)}(t)$ can be expressed, in the limit *first* $N \to \infty$ *then* $\zeta \to \infty$ by the solutions of the classical equations

$$\begin{aligned} \dot{\alpha} &= -(i\nu + \kappa)\, \alpha - i\lambda \sigma^{-} - i\mu \sigma^{+}, \\ \dot{\sigma}^{-} &= -(i\epsilon + \gamma)\, \sigma^{-} + 2i\lambda \sigma^{3}\, \alpha + 2i\mu \sigma^{3}\, \alpha^{*}, \\ \dot{\sigma}^{3} &= -\gamma(\sigma^{3} - \eta) + (i\lambda \sigma^{-}\, \alpha^{*} + i\mu \sigma^{-}\, \alpha + \text{h.c.}), \end{aligned} \tag{5.33}$$

where the parameters satisfy

$$\gamma, \kappa, \epsilon, \nu > 0, \quad \lambda, \mu \in \mathbf{R}, \quad \lambda \neq 0. \tag{5.34}$$

For all μ there exists one time-independent stationary solution of (5.33):

$$\sigma^{3} = \eta, \quad \sigma^{-} = \alpha = 0. \tag{5.35}$$

For $\mu = 0$ (and hence for small $|\mu|$) this stationary solution is unique, since for $(\sigma^-, \alpha) \neq (0,0)$ the coefficient determinant of the equation $\dot{\sigma}^- = \dot{\alpha} = 0$ has to vanish. The latter is analytic in μ and, for $\mu = 0$, it takes the value $(i\nu + \kappa)(i\epsilon + \gamma) - 2\lambda^2 \sigma^3 \neq 0$.

For the stability of (5.35) one has to discuss the linearized equations around (5.35):

$$\begin{aligned} \dot{\alpha} &= -(i\nu + \kappa)\,\alpha - i\lambda\sigma^- - i\mu\sigma^+, \\ \dot{\sigma}^- &= -(i\epsilon + \gamma)\,\sigma^+ + 2i\lambda\eta\alpha + 2i\mu\eta\alpha^*, \\ \dot{\sigma}^3 &= -\gamma\sigma^3. \end{aligned} \tag{5.36}$$

For $\mu = 0$, one root of the characteristic polynomial

$$z^2 + \{\kappa + \gamma + i(\nu + \epsilon)\}\,z - 2\lambda\eta + (i\nu + \kappa)(i\epsilon + \gamma) = 0$$

crosses the imaginary axis away from the origin, if

$$\eta = \frac{\kappa\gamma}{2\lambda^2}\left(1 + \frac{(\epsilon - \nu)^2}{(\kappa + \gamma)^2}\right) \equiv \eta_c, \tag{5.37}$$

Hence, for strong damping or far away from resonance (i.e. for $\eta_c > \frac{1}{2}$) the non-radiating solution is stable for all physical values of the pump parameter, namely $|\eta| \leqslant \frac{1}{2}$.

By the Hopf bifurcation theorem, a stable periodic solution will branch from (5.35), if $\eta > \eta_c$. This picture will persist for small μ, with small changes in η_c.

For $\mu = 0$ and $\eta_c < \eta$ there exists a one-parameter family of harmonic solutions of (5.33) of the type $\alpha(t) = \alpha \exp -i\omega t$, $\sigma^-(t) = \sigma^- \exp -i\omega t$ with time-independent α, σ^-, σ^3 and ω given by

$$\sigma^3 = \eta_c, \quad \sigma^- = (i\kappa + \omega - \gamma)\,\alpha/\lambda, \tag{5.38}$$

$$\omega = (\nu\gamma + \epsilon\kappa)(\gamma + \kappa)^{-1}, \tag{5.39}$$

$$|\alpha|^2 = \gamma(\eta - \eta_c)/2\kappa. \tag{5.40}$$

This solution lies in the physical region $|\underline{\sigma}| \leqslant \frac{1}{2}$. Note that σ^3 is independent of the pump parameter η and that the mean photon number increases linearly with η above threshold. These formulae are well-known from semi-classical laser theory [H1] and are in fair agreement with experiments on finite lasers [D2]. In the gas laser, $\gamma/2\kappa$ has to be replaced by γ/κ in (5.40).

This harmonic solution is stable near threshold. We have verified stability for all $\eta - \eta^c$, if either $2\kappa = \gamma$ or $\nu = \epsilon$. In the former case one can even show, that (5.38, 5.39, 5.40) is the unique stable attractor of the system: (5.36) with $\mu = 0$ is equivalent to (5.21) and

$$\ddot{\beta} + [i(\epsilon + \nu - 2\omega) + \kappa + \gamma]\,\dot{\beta} + 2\lambda^2(\eta_c - \sigma^3)\,\beta = 0, \tag{5.41}$$

where

$$\beta(t) \equiv \alpha(t) \exp - i\omega t.$$

For

$$2\kappa = \gamma, \quad \sigma^3(t) \to \eta - |\beta(t)|^2$$

exponentially. Hence the attractors can be determined from (5.41) with σ^3 replaced by $\eta - |\beta|^2$. This equation has a Ljapounov function

$$H(\beta, \dot{\beta}) = |\dot{\beta}|^2 + \lambda^2|\beta|^4 + 2\lambda^2(\eta_c - \eta)|\beta|^2 \tag{5.42}$$

with $\dot{H} = -6\kappa|\dot{\beta}|^2$, and $\dot{H} = 0$ only for $\dot{\beta} = 0$. Hence for every integral curve $\dot{\beta}(t) \to 0$, which entails $\sigma^3(t) \to \eta_c$ and hence $|\beta(t)|^2 \to \eta - \eta_c$.

We see that the laser, too, shows a non-equilibrium phase transition at η_c (provided that $\eta_c < \frac{1}{2}$) between a unique non-radiating time-independent macrostate and a 1-parameter family of stable 'self-oscillatory' attracting macrostates. In both regimes, the fluctuation theory can be worked out and we hope to return to this interesting question in the future.

Conclusion

The main point of this paper is to exhibit a many-body system showing irreversible behaviour, where one can really separate classical and quantum effects. It has always been the faith of physicists that, for reservoirs which are very large with respect to the system of interest, the effect of the reservoir can be described to $O(N)$ by phenomenological classical dissipative equations and by a Langevin-type linear fluctuation theory in $O(\sqrt{N})$ around the classical macrostate. We have solved a non-linear quantum mechanical model (without using uncontrolled approximations such as master equations, quasilinearization and 'adiabatic' elimination procedures), where the conventional wisdom of irreversible statistical mechanics far from thermal equilibrium can be deduced from a microscopic theory.

It incidentally happens to be the case that our theory leads to a model of the laser and that it gives qualitative predictions which have some resemblance to experimental facts.

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