Zeitschrift: Helvetica Physica Acta
Band: 47 (1974)
Heft: 4

Artikel: Photon correlations in the Dicke maser model
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DOI: https://doi.org/10.5169/seals-114577

Helvetica Physica Acta
Vol. 47, 1974.

Birkhäuser Verlag Basel

Photon Correlations in the Dicke Maser Model

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Abstract. The second-order photon correlation functions have been calculated for the Dicke maser model describing N spins ($s = \frac{1}{2}$) interacting with a single mode of the radiation field. Assuming an initial state with all N spins in the upper level and N_0 photons present, it is found that the correlation functions $g(t, t_1)$ factorize into products of the photon number $g(t, t_1) = n(t)\, n(t_1)\, (1 + 0(N_0^{-1}))$. This result gives some understanding as to why an inverted population emits coherent radiation.

1. Introduction

The Dicke maser model has attracted great interest in literature [1], because it is the only model describing the collective interaction of radiation and matter, which is to a large extent mathematically tractable. In its simplest version, the matter is represented by N two-level atoms interacting with a single mode of the radiation field. Simplifying the dipole coupling according to the rotating wave approximation, the model is characterized by the following Hamiltonian

$$H = \sum_{i=1}^{N} s_i^+ s_i^- + a^+ a + g \sum_i (a^+ s_i^- + s_i^+ a) \tag{1.1}$$

where $s_i^\pm$ are the usual spin flip operators ($s = \frac{1}{2}$) for the atom i and a^+, a are the creation and annihilation operators of the photon mode. For convenience, a resonant mode is considered.

With respect to the laser phenomenon, an interesting question to be answered for this model is the following: Assuming a fully excited initial state ψ_0 of all N atoms in the upper level, how does this inverted population radiate? A first step towards a solution to this problem was the calculation of the mean photon number

$$n(t) = (\psi_0, a^+(t)\, a(t)\, \psi_0) \tag{1.2}$$

and its mean square deviation

$$\sigma^2(t) = (\psi_0, [a^+(t)\, a(t) - n(t)]^2\, \psi_0) = \sigma_0^2(t) - n^2(t) \tag{1.3}$$

in a previous paper [2]. In the present paper, the methods developed in Ref. [2] will be used to calculate the second-order photon correlation functions

$$g_1(t, t_1) = (\psi_0, a^+(t_1)\, a(t_1)\, a^+(t)\, a(t)\, \psi_0)$$

$$g_2(t, t_1) = (\psi_0, a^+(t_1)\, a^+(t)\, a(t)\, a(t_1)\, \psi_0). \tag{1.4}$$

These quantities are of interest for two reasons: First, they give a measure for the coherence of the radiation field. If the radiation emitted by the inverted population is coherent in second order, then the correlation functions (1.4) must factorize

$$g_1(t, t_1) \approx n(t)\, n(t_1) \approx g_2(t, t_1). \tag{1.5}$$

That means experimentally, for instance, that no Hanbury–Brown–Twiss effect occurs, as it is typical for lasers. The factorization (1.5) holds, indeed, the relative error is of the order N_0^{-1} where N_0 is the number of photons at $t = 0$. The second reason for calculating the correlation functions lies in the fact that the knowledge of g_1 and g_2 enables one to analyse the influence of losses on the time evolution of the system for small times. This will be discussed elsewhere.

2. The Time Ordered Correlation Function g_1

As it has been discussed in detail in Ref. [2], the Schrödinger equation

$$i\frac{d\psi(t)}{dt} = H\psi(t) \tag{2.1}$$

for the Hamiltonian (1.1) can be reduced to a $N+1$-dimensional matrix equation where

$$\psi(t) = \{e_n(t)\} \in \mathbb{C}^{N+1}, \qquad n = 0, 1, \ldots, N \tag{2.2}$$

and H is a tridiagonal matrix

$$H_{n,n+1} = H_{n+1,n} = [(R-n)\,(N-n)\,(n+1)]^{1/2} \tag{2.3}$$

$$H_{nn} = R, \qquad \text{with } g = 1.$$

Here the integer $R = N + N_0$ is a constant of the motion, where $N_0 = 0, 1, \ldots$ is the number of excess photons, i.e. the number of photons being present if all N atoms are in the upper state.

The main tool in the discussion of the matrix problem (2.1) is the mapping on the following quantum mechanical problem in $L^2(0, \infty)$ ([2], Section 3)

$$-i\frac{\partial y}{\partial t} = \mathscr{H}_R y$$

$$\mathscr{H}_R = -\tfrac{1}{2}\left(x\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}x\right) + \tfrac{1}{4}x^3 - Ex + \frac{L^2}{x} \tag{2.4}$$

where

$$E = \tfrac{1}{2}(R+N) + 1$$

$$2L = N_0 = R - N. \tag{2.5}$$

This mapping is achieved by the linear embedding operator T_R

$$\mathbb{C}^{N+1} \ni \psi = \{e_n\} \overset{T_R}{\rightsquigarrow} y = \sum_{n=0}^{N} \binom{N}{n}^{1/2} ((R-n)!)^{-1/2} e_n x^{N+L-n} \exp -\tfrac{1}{4}x^2 \in L^2(0, \infty). \tag{2.6}$$

We have

$$\mathscr{H}_R = -T_R H T_R^{-1} \tag{2.7}$$

on the $N+1$-dimensional range of T_R in L^2. It is very useful to apply the mapping (2.6) to other observables of interest and to perform all further calculations entirely in mechanical terms. For the photon number operator a^+a, we get

$$\begin{aligned} T_R a^+ a T_R^{-1} y &= \sum_n (R-n) \binom{N}{n}^{1/2} ((R-n)!)^{-1/2} e_n x^{N+L-n} \exp -\tfrac{1}{4}x^2 \\ &= \left(x \frac{\partial}{\partial x} + \tfrac{1}{2}x^2 + L \right) y. \end{aligned} \tag{2.8}$$

Regarding the creation and destruction operators a^+, a, we must note that they operate between different subspaces with energy quantum numbers $R \pm 1$. Therefore, we obtain the relations

$$\begin{aligned} T_R a^+ T_{R-1}^{-1} &= x^{1/2} \\ T_{R-1} a T_R^{-1} &= x^{1/2} \frac{\partial}{\partial x} + L x^{-1/2} + \tfrac{1}{2} x^{3/2}. \end{aligned} \tag{2.9}$$

Now we turn to the time-ordered correlation function

$$g_1(t, t_1) = (\psi_0, a^+(t_1)\, a(t_1)\, a^+(t)\, a(t)\, \psi_0). \tag{2.10}$$

As the initial state ψ_0 of the system, we choose the fully excited state of all N atoms in the upper state and $R-N$ photons present, which is given by

$$\psi_0 = (0, 0, \ldots, 1)$$

in $\mathbb{C}^{N+1}$ (2.2). Let us introduce the orthonormal states

$$\psi_j = (0, \ldots, \underset{N-j}{1}, 0, \ldots). \tag{2.11}$$

For g_1 (2.10), it is only necessary to calculate the matrix elements of the photon number operator

$$A_j \stackrel{\text{def}}{=} (\psi_j, e^{iHt} a^+ a\, e^{-iHt} \psi_0). \tag{2.12}$$

This can be done in exactly the same way as in Ref. [2] for the photon number itself. Using equation (2.8), we can express equation (2.12) in mechanical terms

$$\begin{aligned} A_j &= \left(\psi_j, T_R^{-1} e^{-i\mathscr{H}t} \left(x \frac{\partial}{\partial x} + \tfrac{1}{2}x^2 + L \right) e^{i\mathscr{H}t} y_0 \right) \\ &= T_R^{-1} e^{-i\mathscr{H}t} \left(x \frac{\partial}{\partial x} + \tfrac{1}{2}x^2 + L \right) e^{i\mathscr{H}t} y_0 \Big|_{N-j} \end{aligned} \tag{2.13}$$

where

$$y_0 = (2L)!^{-1/2} x^L \exp - \tfrac{1}{4}x^2 \tag{2.14}$$

is the mechanical initial state corresponding to ψ_0 and the index $N-j$ means the $(N-j)$-component of the vector in $\mathbb{C}^{N+1}$. The right side of equation (2.13) is equal to the coefficient of the term $\sim x^{L+j}\exp - x^2/4$ of the function

$$A(x) = \int_0^\infty dx_0\, K(x, x_0, t)\left(x_0 \frac{\partial}{\partial x_0} + \tfrac{1}{2}x_0^2 + L\right)\int_0^\infty dx_1\, K(x_0, x_1, -t)\, x_1^L \exp - \tfrac{1}{4}x_1^2 \tag{2.15}$$

where K is the propagator of the mechanical problem (2.4)

$$(e^{-i\mathscr{H}t} y)(x) = \int_0^\infty dx_0\, K(x, x_0, t)\, y(x_0). \tag{2.16}$$

The function $A(x)$ was studied in Ref. [2] by means of the W.K.B. method which is asymptotically exact for $N \to \infty$. The W.K.B. expression for the propagator K reads

$$K = \Phi \exp iS \tag{2.17}$$

where

$$\Phi^2 = \text{const}\, \frac{\partial^2 S(x, x_0, t)}{\partial x\, \partial x_0} \tag{2.18}$$

and

$$S(x, x_0, t) = \epsilon t + \int_{x_0}^{x} dx' \left[E - \tfrac{1}{4}x'^2 - \frac{L^2}{x'^2} - \frac{\epsilon}{x'}\right]^{1/2} \tag{2.19}$$

is the classical action integral along the mechanical path from x_0 to x in time t; ϵ denotes the corresponding (negative) mechanical energy. Using this form of K in equation (2.15), we obtain the following result for $A(x)$ in the limit $x \to 0$ ([2], equation (5.11))

$$\begin{aligned} A(x) \to \int dx_0 &\left(2L\frac{P+E}{P+L} - i\frac{\epsilon P'}{2(P+L)^2} + \frac{\epsilon^2}{2(P+L)^2} - 1 - x_0 \frac{1}{\Phi}\frac{\partial \Phi}{\partial x_0}\right) \\ &\times K(x, x_0, t)\int dx_1\, K(x_0, x_1, -t)\, x_1^L \exp - \tfrac{1}{4}x_1^2 \end{aligned} \tag{2.20}$$

where $P(t)$ is essentially Weierstrass' $\wp$-function

$$P(t) = \wp(t;\, g_2, g_3) - \frac{E}{3}, \qquad P'(t) = \frac{\partial P}{\partial t} \tag{2.21}$$

with the invariants

$$\begin{aligned} g_2 &= \tfrac{4}{3}E^2 + 4L^2 \\ g_3 &= -\tfrac{8}{27}E^3 + \tfrac{8}{3}EL^2 + (2L+1)N. \end{aligned} \tag{2.22}$$

The constant terms independent of ϵ in equation (2.20) contribute only to the power x^L, that means $j=0$, which gives the photon number $n(t)$ [2]

$$A_0 = n(t) = (2L+1)\frac{P+E}{P+L+1} - 1. \tag{2.23}$$

In the second term proportional to ϵ we insert

$$\epsilon = \frac{\partial S}{\partial t} = -\frac{i}{K}\frac{\partial K}{\partial t} + \frac{i}{\Phi}\frac{\partial \Phi}{\partial t} = -\frac{1}{K}\mathscr{H}K + \frac{i}{2}\frac{P'}{P}. \tag{2.24}$$

Here, the second term again contributes to A_0 and is already included in the result (2.23). The first term gives

$$-i\frac{P'}{2(P+L)^2}\int dx_0(-i\mathscr{H})\,K(x,x_0,t)\int dx_1\,K(x_0,x_1,-t)\,x_1^L \exp -\tfrac{1}{4}x_1^2 \tag{2.25}$$

and therefore

$$A_1 = -i\frac{P'}{2(P+L)^2}(\psi_1, H\psi_0) = -i\frac{P'}{2(P+L)^2}\sqrt{(2L+1)\,N}. \tag{2.26}$$

The remaining terms in equation (2.20) are small [2]. This follows for $L \ll E$ from the estimates ([2], Appendix II)

$$P = 0(E^{1-\alpha})$$

$$\epsilon = 0(E^{1/2})$$

$$x_0 = 0(E^{-(1/2)+\alpha}) \tag{2.27}$$

with

$$t = \alpha\omega, \qquad L \ll E, \tag{2.28}$$

where ω is the real half-period of the p-function (2.21).

Consequently, only the matrix elements A_0 and A_1 contribute to the correlation function

$$g_1(t,t_1) = n(t)\,n(t_1) + A_1(t)\,\bar{A}_1(t_1). \tag{2.29}$$

By comparison with the exact power series expansion around $t=t_1=0$, we find that the denominator $(P+L)^2$ in equation (2.26) should be replaced by $(P+L+1)^2$. Then the final result is

$$g_1(t,t_1) = n(t)\,n(t_1) + \tfrac{1}{4}(2L+1)\,N\frac{P'}{(P+L+1)^2}\frac{P_1'}{(P_1+L+1)^2}$$

$$n(t) = (2L+1)\frac{P+E}{P+L+1} - 1, \qquad P_1 = P(t_1). \tag{2.30}$$

3. The Normal Ordered Correlation Function g_2

The determination of the function

$$g_2(t, t_1) = (\psi_0, a^+(t_1)\, a^+(t)\, a(t)\, a(t_1)\, \psi_0) \tag{3.1}$$

presents a much more difficult problem. Let us start from the expression in mechanical terms

$$\begin{aligned} g_2(t, t_1) = \lim_{x\to 0} x^{-L} \int dx_0\, K_R(x, x_0, t_1)\, x^{1/2} \int dx_1\, K_{R-1}(x_0, x_1, t_2) \\ \times \left(x_1 \frac{\partial}{\partial x_1} + \tfrac{1}{2}x_1^2 + L - \tfrac{1}{2}\right) \int dx_2\, K_{R-1}(x_1, x_2, -t_2) \\ \times \left(x_2^{1/2} \frac{\partial}{\partial x_2} + L x_2^{-1/2} + \tfrac{1}{2}x_2^{3/2}\right) \int dx_3\, K_R(x_2, x_3, -t_1)\, x_3^L \exp - \tfrac{1}{4}x_3^2, \\ t_2 = t - t_1 \end{aligned} \tag{3.2}$$

which follows from equations (2.8) and (2.9). First, we consider the second integral $\int dx_1$. After partial integration, using [2]

$$-x_1 \frac{\partial}{\partial x_1} K_{R-1}(x_0, x_1, t_2) = -x_1\, i \frac{\partial}{\partial x_1} S_-(x_0, x_1, \epsilon_-)\, K_{R-1} = \tfrac{1}{2}\sqrt{-R_-(x_1)}\, K_{R-1} \tag{3.3}$$

where

$$R_-(x_1) = -x_1^4 + 4E_- x_1^2 - 4\epsilon_- x_1 - 4L_-^2 \tag{3.4}$$

$$E_- = \tfrac{1}{2}(R - 1 + N) + 1 \qquad 2L_- = R - 1 - N, \tag{3.5}$$

we are faced with the problem of treating a factor in front of $K_{R-1}(x_0, x_1, t - t_1)$, which is a function of x_1 and $\sqrt{-R_-(x_1)}$. It is an important consequence of Weierstrass' transformation of elliptic integrals [3], that both quantities can be expressed algebraically by x_0 and $\sqrt{R_-(x_0)}$:

$$x_1 = x_0 + \frac{(4Ex_0 - 2x_0^3 - 2\epsilon)\,(P_2 + \tfrac{1}{2}x_0^2) + x_0 R(x_0) - P_2' \sqrt{R(x_0)}}{2(P_2 + \tfrac{1}{2}x_0^2)^2 + \tfrac{1}{2}R(x_0)} \tag{3.6}$$

$$\begin{aligned} \sqrt{R(x_1)} = \{-P_2'[(P_2 + \tfrac{1}{2}x_0^2)^2\,(-4x_0^3 + 8Ex_0 - 4\epsilon) - R(x_0)\,(-x_0^3 + 2Ex_0 \\ - \epsilon + 4x_0(P_2 + \tfrac{1}{2}x_0^2))] + \sqrt{R(x_0)}\,[-(12p^2 - \tfrac{4}{3}E^2 - 4L^2) \\ \times ((P_2 + \tfrac{1}{2}x_0^2)^2 + \tfrac{1}{4}R(x_0)) + 4P_2'^2(P_2 + \tfrac{1}{2}x_0^2)]\}/\{2(P_2 + \tfrac{1}{2}x_0^2)^2 + \tfrac{1}{2}R(x_0)\}^2 \end{aligned} \tag{3.7}$$

where

$$P_2 = P_-(t - t_1), \qquad P_2' = \frac{d}{dt} P_2, \qquad t_2 = t - t_1. \tag{3.8}$$

The subscripts "$-$" have been omitted for convenience.

In these formulas the x_1-dependence is concentrated in ϵ_- and, weakly, in the invariant g_3 of the p-functions [2]. The latter can be neglected, if only not too large times t_2 are considered. Instead of the energy ϵ_- the Hamiltonian $\mathscr{H}_-$ can be introduced as above (equation 2.24). Since $\mathscr{H}$ commutes with the time evolution (2.16), the integral $\int dx_1$ is reduced to some function of $\mathscr{H}$. Proceeding in the same way with the other integrals in equation (3.2), the whole expression can in principle be reduced to expectation values of functions of $\mathscr{H}$ in the initial state ψ_0. In practice, this general method is restricted by our ability of handling the rapidly growing complicated terms. A considerable simplification occurs, if we consider the quantum region $L \ll E$ [2], where the estimates (2.27) can be used. Then we have

$$x_0 = 0(E^{-(1/2)+\alpha_1}), \qquad R(x_0) = 0(E^{2\alpha_1}) \qquad \text{for } t_1 = \alpha_1 \omega < \omega \tag{3.9}$$

and

$$x_1 = 0(E^{-(1/2)+\alpha_1+\alpha_2}), \qquad P_2 = 0(E^{1-\alpha_2}), \qquad t_1 = \alpha_2 \omega,$$

and therefore

$$P_2 \gg R(x_0)^{1/2} \gg x_0^2 \qquad \text{if } \alpha_1 + \alpha_2 < 1. \tag{3.10}$$

In addition, we find

$$R(x_1)^{1/2} = 0(E^{\alpha_1+\alpha_2}) \gg x_1^2.$$

Using these estimates in equation (3.7), the leading order in E of equation (3.2) is

$$g_2(t,t_1) = \lim_{x\to 0} x^{-L} \int dx_0\, K_R(x,x_0,t_1)\, x_0^{1/2} \int dx_1 \left[\frac{P_2+2E}{2P_2}\sqrt{-R_-(x_0)} + L - \tfrac{3}{2} - i\frac{P_2'}{2P_2^2}(\epsilon_- - 2Ex_0)\right] K_{R-1}(x_0,x_1,t_2)\ldots. \tag{3.11}$$

Let us denote the first term $\sim\sqrt{-R_-(x_0)}$ by B and the rest by C. With (3.3)

$$\sqrt{-R_-(x_0)}\, K_{R-1}(x_0,x_1,t_2) = 2x_0 \frac{\partial}{\partial x_0} K_{R-1},$$

B can be simplified to

$$B = \frac{P_2+2E}{2P_2} \lim x^{-L} \int dx_0\, K_R(x,x_0,t_1)\, x_0^{1/2}\, 2x \frac{\partial}{\partial x_0}\left(x_0^{1/2}\frac{\partial}{\partial x_0} + Lx_0^{-1/2} + \tfrac{1}{2}x_0^{3/2}\right)$$
$$\times \int dx_3\, K_R(x_0,x_3,-t_1)\, x_3^L \exp - \tfrac{1}{4}x_3^2$$
$$= \frac{P_2+2E}{P_2}\left\langle x_0^2\frac{\partial^2}{\partial x_0^2} + (-L+\tfrac{7}{2})\, x_0\frac{\partial}{\partial x_0} - \tfrac{3}{2}L + \tfrac{3}{2}\right\rangle \tag{3.12}$$

where

$$\langle D\rangle \stackrel{\text{def}}{=} \lim_{x\to 0} x^{-L} \int dx_0\, D_{x_0}\, K_R(x,x_0,t_1) \int dx_3 \ldots. \tag{3.13}$$

Expressions of this type (3.12) have been calculated in Ref. [2]. The final result is

$$B = \frac{P_2 + 2E}{P_2}\left[(4L^2 + 6L + 2)\frac{(P_1 + E)^2}{P_1^2} - (2L^2 + 8L + \tfrac{7}{2})\frac{P_1 + E}{P_1} + L + \tfrac{3}{2}\right]. \tag{3.14}$$

The term $C_1 \sim x_0$ in equation (3.11) gives

$$C_1 = iE\frac{P_2'}{P_2^2}\left\langle -x_0^2\frac{\partial}{\partial x_0} + (L - 2)\,x_0\right\rangle$$

$$= E\frac{P_2' P_1'}{P_2^2 P_1^2}\left[2L^2 + L + (2L^2 + 3L + 1)\frac{P_1 + E}{P_1}\right]. \tag{3.15}$$

In the remaining term $C_2 \sim \epsilon_-$, we again insert equation (2.24). The term with the Hamiltonian $\mathscr{H}_-$ vanishes, and from the second term in equation (2.24) we obtain

$$C_2 = \frac{1}{4}\frac{P_2' P_2'}{P_2^2 P_2}\,n(t_1) = \frac{P_2 + E}{P_2}\,n(t_1) \tag{3.16}$$

to leading order, because

$$P'^2 = 4(P + E)\,(P^2 - L^2) - \epsilon^2. \tag{3.17}$$

Summing up, we arrive at

$$g_2(t, t_1) = B + C_1 + C_2 + (L - \tfrac{3}{2})\,n(t_1)$$

$$= 4L^2 - 2L + (8L^2 + 4L)\frac{E}{P_1} + (4L^2 + 6L + 2)\frac{E^2}{P_1^2}$$

$$+ \frac{E}{P_2}\left[4L^2 + (12L^2 + 10L + 2)\frac{E}{P_1} + (8L^2 + 12L + 4)\frac{E^2}{P_1^2}\right]$$

$$+ E\frac{P_2' P_1'}{P_2^2 P_1^2}\left[2L^2 + L + (2L^2 + 3L + 1)\,E\frac{P_1'}{P_1^3}\right]. \tag{3.18}$$

Since we expect approximate factorization (1.5) if the result (3.18) is expressed in the variables t, t_1, we substitute $P_2 = P(t - t_1)$ and P_2' by means of the addition formulas for the $\wp$-function [4]. In leading order in E we have

$$\frac{E}{P_2} = \frac{E}{P_1} + \frac{E}{P} + 2\frac{E^2}{PP_1} - \frac{E}{2}\frac{P'}{P^2}\frac{P_1'}{P_1^2}$$

$$\frac{P_2'}{P_2^2} = -\frac{P_1'}{P_1^2} + \frac{P'}{P^2} - 2E\frac{P_1'}{PP_1^2} + 2E\frac{P'}{P^2 P_1}. \tag{3.19}$$

After substitution into equation (3.18), the result greatly simplifies:

$$g_2(t,t_1) = \left[2L - 1 + 2L\frac{E}{P}\right]\left[2L + (2L+1)\frac{E}{P_1}\right]$$

$$+ \frac{E}{P_1}\left(1 + 2\frac{E}{P}\right) + LE\frac{P'}{P^2}\frac{P_1'}{P_1^2}. \tag{3.20}$$

Here indeed, the first term is a product of two photon numbers. It is important to note that in the first factor the quantity $L_- = L - \frac{1}{2}$ (3.5) appears. This shows that, with respect to t, the photon number has to be calculated with the quantities E_-, L_- (3.5), which is also to be expected from the general structure of the correlation function g_2 (3.1) and from the special value

$$g_2(t,0) = (a\psi_0, a^+(t)\,a(t)\,a\psi_0) = 2Ln_-(t). \tag{3.21}$$

Our result

$$g_2(t,t_1) = n_-(t)\,n(t_1) + \frac{E}{P_1}\left(1 + 2\frac{E}{P}\right) + LE\frac{P'}{P^2}\frac{P_1'}{P_1^2} \tag{3.22}$$

agrees very well with the power series expansion

$$g_2(t,t_1) - n_-(t)\,n(t_1) = Nt_1^2 + Nt_1^4(\tfrac{1}{3}N - \tfrac{4}{3}L - \tfrac{2}{3})$$

$$+ Nt^2t_1^2(2N - 4L - 2) + 4LNtt_1[1 + t^2(\tfrac{2}{3}N - \tfrac{4}{3}L - \tfrac{2}{3})]$$

$$\times [1 + t_1^2(\tfrac{2}{3}N - \tfrac{2}{3}L - \tfrac{1}{3})] + \dots. \tag{3.23}$$

Therefore, one might hope that one could extrapolate to larger t, t_1, beyond (3.10), in the usual manner by completing the rational functions of P, P_1 in equation (3.22) and adjusting a few parameters. This is possible with respect to t. However, it does not work for t_1. As a point of illustration, let us consider the odd term g_2^0 in equation (3.22)

$$g_2^0(t,t_1) = LN\frac{P'}{P^2 + aP + b}\,\frac{P_1'}{P_1^2 + a_1P + b_1},$$

which is the most important correction term. By expansion in powers of t, t_1 and comparison with equation (3.23), we find a reasonable value $a = 2L + 1$, but a strange value $a_1 = \frac{4}{3}L + 1$, causing initial doubts towards the validity of the procedure. We have therefore calculated the difference $h_2 = g_2 - n_-(t)\,n(t_1)$ numerically in the superradiant case [2] for $N = 50$, $R = 101$. As a function of t with t_1 fixed, $h_2(t,t_1)$ is nicely periodic (Fig. 1), but as a function of t_1 with t fixed, it is not (Fig. 2)! This fact is very important, because it shows that $g_2(t,t_1)$ is not an elliptic function in t_1, while in t it seems to be elliptic.

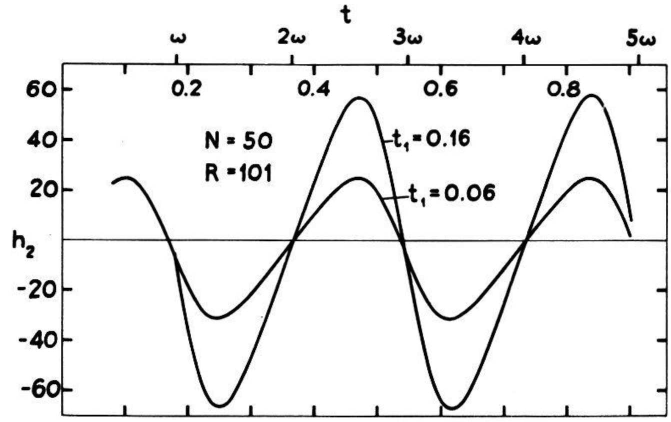

Figure 1
$h_2 = g_2 - n_-(t)\, n(t_1)$ as a function of t for fixed t_1. ω is the half-period of the $\wp$-function.

Consequently, the extrapolation of the result (3.22) to larger t_1 is a highly non-trivial problem. This can also be seen from the special case $t = t_1$, in which the correlation function is known as well (equation 1.3),

$$g_2(t, t) = (\psi_0, a^+(t)\, a(t)\, a^+(t)\, a(t)\, \psi_0) - (\psi_0, a^+(t)\, a(t)\, \psi_0)$$

$$= \sigma_0^2(t) - n(t)$$

$$= 2 - 4(2L+1)\frac{P+E}{P+L+1} + (2L+1)\,(2L+2)\frac{(P+E)\,(P+E+1)}{(P+L+1)\,(P+L+2)}. \tag{3.24}$$

Since here all $\wp$-functions have to be taken with quantities E, L in the invariants (2.22), the structure of equation (3.24) is quite different from (3.21) and (3.22). Apparently, a result uniformly valid in t_1 cannot be obtained with the present technique. Thus, a different method must be found. That a different approach must actually exist, is suggested by the following reasons: The present method is based on the estimates (3.9), which hold only in the quantum region $L \ll E$. There are no corresponding estimates in the classical region $L \approx E$; there, in fact, almost all the terms contribute.

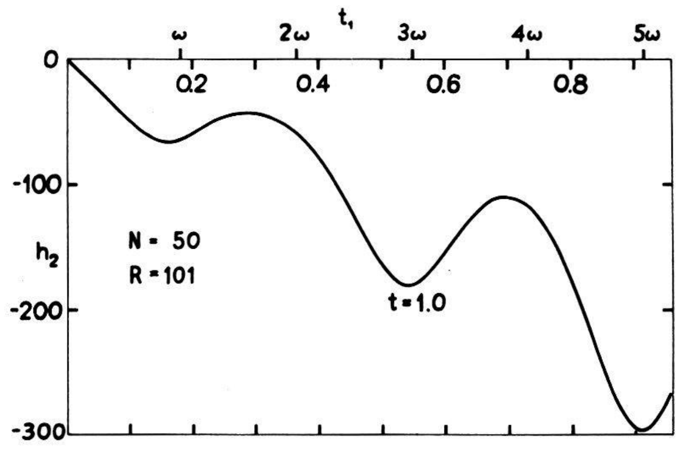

Figure 2
$h_2 = g_2 - n_-(t)\, n(t_1)$ as a function of t_1 for fixed t.

On the other hand, the W.K.B. theory is most accurate in the classical region, but our method of proof applies only to the quantum region. This leaves much to be desired.

For small t_1, let us say for the first few periods, the result (3.22) is quite accurate. For example, the second minimum in Fig. 2 is only 3% of the total value of g_2, *uniformly in t*. The fixed times $t_1 = 0.16$ or $t = 1.0$ in Figs. 1 or 2, respectively, have been chosen in such a way that the function $h_2(t, t_1)$ becomes maximal.

4. Discussion

For the properties of the elliptic functions, the fundamental cubic equation

$$4e^3 - g_2 e - g_3 = 0 \tag{4.1}$$

is of central importance. It is a curious fact, that equation (4.1), with invariants g_2, g_3 given by equation (2.22), has very simple solutions if $L \gg 1$, namely

$$e_1 = \frac{E}{3} + L + \tfrac{1}{4} + 0\left(\frac{L}{E}\right)$$

$$e_2 = \frac{E}{3} - L - \tfrac{1}{4} + 0\left(\frac{1}{L}\right)$$

$$e_3 = -e_1 - e_2 = -\tfrac{2}{3}E + 0\left(\frac{1}{L}, \frac{L}{E}\right), \tag{4.2}$$

which can be verified by inspection. This allows us to express the Weierstrass elliptic functions by Jacobian ones, which is sometimes convenient for numerical reasons. For instance, the photon number (2.23) becomes for $L \gg 1$ [5]

$$n(t) = (2L+1)\frac{\wp + \tfrac{2}{3}E}{\wp - \left(\dfrac{E}{3} - L\right)} = (2L+1)\frac{\wp - e_3}{\wp - e_2}$$

$$= (2L+1)\, nd^2(u, k) \tag{4.3}$$

where

$$u = (e_1 - e_3)^{1/2}\, t = R^{1/2}\, t$$

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{N}{R}. \tag{4.4}$$

This formula (4.3) was essentially given by Bonifacio and Preparata [1].

Now, let us consider the correlation function g_1. The striking feature of our result (2.30) is the appearance of the odd term, proportional to $P' P_1'$, which destroys the factorization of g_1. In order to estimate the magnitude of this term, let us define the quantity

$$\alpha(t,t_1) = \frac{g_1(t,t_1) - n(t)\,n(t_1)}{n(t)\,n(t_1)}, \tag{4.5}$$

which measures the degree of factorization or the degree of "coherence" of the emitted radiation. We have for $L \gg 1$

$$\alpha(t,t_1) = \frac{1}{4}\frac{N}{2L}\frac{P'}{(P+L)(P+E)}\frac{P_1}{(P_1+L)(P_1+E)} \tag{4.6}$$

First, let us consider

$$\beta(t) = \frac{P'}{(P+L)(P+E)} = \frac{p'}{(p-e_2)(p-e_3)}. \tag{4.7}$$

Since

$$p'(t) = -2[(p-e_1)(p-e_2)(p-e_3)]^{1/2}$$

we get

$$|\beta(t)| = 2\left[\frac{p-e_1}{(p-e_2)(p-e_3)}\right]^{1/2} < \frac{2}{(p-e_3)^{1/2}} = 2\frac{sn(u,k)}{R^{1/2}} < \frac{2}{R^{1/2}} \tag{4.8}$$

uniformly in t, and therefore

$$|\alpha(t,t_1)| < \frac{N}{2LR} < \frac{1}{2L} = \frac{1}{N_0}. \tag{4.9}$$

The relative deviation from the factorization is uniformly bounded by the reciprocal number of excess photons. For $N_0 \gg 1$, the emitted radiation is coherent. The situation is less clear for the normal ordered correlation function g_2, because the result (3.22) could not be extrapolated to larger t_1. For small times t_1, however, the same argument as for g_1 applies. It is an interesting question as to whether there is some decay of coherence in g_2 for larger t_1 in contrast to g_1.

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