Product Spaces and Nelson's Inequality

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Abstract. Certain inequalities for unbounded linear operators in L^2 extend to product spaces. This product space property is used to give a new proof that Gross's inequality for matrices implies Nelson's inequality for ordinary differential operators, and that this in turn implies Nelson's inequality for partial differential operators in infinitely many dimensions. A new proof of Gross's inequality is given.

There is also a discussion of the physical meaning of Nelson's inequality in quantum field theory. This article may serve as an introduction for non-specialists to some of the recent mathematical work in this subject.

1. Introduction

This article is devoted to a discussion and proof of Nelson's inequality. This is an inequality for partial differential operators which, unlike the Sobolev inequalities, does not involve the dimension n of space. Thus the inequality is valid with the same constants when $n = 1$, $n = 10^{25}$, or even $n = \infty$. This is why it has played an important role in constructive quantum field theory.

The method of proof follows an article by L. Gross [1]. He works directly with unbounded operators; however, in order to pass to a product space he uses the associated semigroups and a product space lemma of I. Segal [2]. In the present treatment the product space property is proved directly for the unbounded operators, and there is no need to carry out the reduction to semigroups.

Section 2 contains a statement of Nelson's inequality and a discussion of its physical meaning. The product space property is proved in sufficient generality for later applications. In Section 3 a new proof of Gross's inequality for matrices is given. The rest of the section contains Gross's proof of Nelson's inequality from Gross's inequality and the product space property.

Nelson's inequality for semigroups was proved in successively stronger forms by Nelson, Glimm (see [3]), and Nelson [4]. Federbush [5] showed that such inequalities can be written in a form involving the unbounded operators directly, and Gross [1] showed that the unbounded operator and semigroup formulations are equivalent.

Recently Brascamp and Lieb [6] and Beckner [7] have found new proofs in the semigroup context. The result is deduced from a generalization of Young's inequality. Their proofs of this inequality resemble Gross's proof of Nelson's inequality in that they are based on ideas related to the central limit theorem of probability.

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Generalizations and applications of the inequality have been given by Gross [8], Eckmann [9], Rosen [10], Wilde [11], and Simon [12]. The Orlicz spaces that occur in the unbounded operator formulation are discussed in standard works, such as that of Krasnosel'skii and Rotickii [13]. The notes by Faris [14] contain an elementary discussion of the implications of the inequality for quantum field Hamiltonians.

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2. The Product Space Property

If λ is a positive measure, then for each real number p with $1 \leqslant p < \infty$ the space L^p consists of the complex functions ψ such that $\|\psi\|_p = \left(\int |\psi(x)|^p \, d\lambda(x)\right)^{1/p} < \infty$. If λ is Lebesgue measure on $\mathbb{R}^n$ and if ψ and $\partial\psi/\partial x_j$ are in L^2 for $j = 1,\ldots, n$, then if $n > 2$ the classical Sobolev inequality implies that ψ is in L^r where $1/r = \frac{1}{2} - 1/n$, and that $\|\psi\|_r^2 \leqslant \text{const} \sum_{j=1}^n \|\partial\psi/\partial x_j\|_2^2$. Notice that this inequality depends on the dimension of space, and so is not very helpful in the limit $n \to \infty$. Nelson's inequality was designed to get around this difficulty. This inequality involves the number operator, so we begin with that.

Consider self-adjoint operators $q_1, q_2,\ldots, q_n$ and $p_1, p_2,\ldots, p_n$ satisfying $p_j q_k - q_k p_j = -i\delta_{jk}$. The number operator is defined by $N = \sum_{j=1}^n \frac{1}{2}(p_j^2 + q_j^2 - 1)$. (This operator is obviously closely related to the harmonic oscillator Hamiltonian.)

It is convenient to introduce operators $a_j = 2^{-1/2}(q_j + ip_j)$ and $a_k^* = 2^{-1/2}(q_k - ip_k)$. These satisfy $a_j a_k^* - a_k^* a_j = \delta_{kj}$ The operator N is given in terms of these operators by $N = \sum_j a_j^* a_j$.

We represent these operators on the Hilbert space consisting of all complex functions ψ on $\mathbb{R}^n$ such that $\|\psi\|_2^2 = \int |\psi(x)|^2 \, d\nu(x)$ is finite. Here ν is the Gaussian measure given by $d\nu(x) = \pi^{-n/2} \exp(-x^2) \, dx$. Notice that because of the Gaussian measure the adjoint of $\partial/\partial x_j$ is $-\partial/\partial x_j + 2x_j$. Thus $-i(\partial/\partial x_j - x_j)$ is self-adjoint. The operators p_j and q_k are represented by $-i(\partial/\partial x_j - x_j)$ and multiplication by x_k.

In this representation the operator

$$N = \sum_{j=1}^n \left(-\frac{1}{2}\frac{\partial}{\partial x_j^2} + x_j \frac{\partial}{\partial x_j}\right).$$

This may also be written as a sum of factors

$$N = \sum_{j=1}^n a_j^* a_j = \sum_{j=1}^n \frac{1}{2}\left(-\frac{\partial}{\partial x_j} + 2x_j\right)\frac{\partial}{\partial x_j}.$$

From this is is not difficult to see that its eigenfunctions are products of Hermite polynomials $H_n(x_j) = (-\partial/\partial x_j + 2x_j)^n 1$ and its eigenvalues are $n = 0, 1, 2, 3, \ldots$. For this reason N is called the number operator.

One point to emphasize is that this type of representation continues to work even when $n = \infty$. In this case the measure ν on $\mathbb{R}^\infty$ is the product of infinitely many Gaussian measures. (Since these all have total probability one the product measure is defined.) The operator N continues to make sense as a partial differential operator in infinitely many variables. This is because the constants have been chosen so that in each summand there is no constant term. The points of $\mathbb{R}^\infty$ may be thought of as (multiples of) the Fourier coefficients of a field.

In dealing with unbounded operators such as N there is often some question about their exact domain of definition. However we will always be dealing with numbers of the form $\langle\psi, N\psi\rangle$, where ψ is a vector in the Hilbert space. Since $N \geqslant 0$ we may interpret this as $\langle\psi, N\psi\rangle = \langle N^{1/2}\psi, N^{1/2}\psi\rangle = \|N^{1/2}\psi\|^2$. (The square root of N is defined by taking square roots of eigenvalues.) Thus $\langle\psi, N\psi\rangle$ is finite if $N^{1/2}\psi$ is in the Hilbert space, and $\langle\psi, N\psi\rangle = +\infty$ otherwise.

There is a different way to write $\langle\psi, N\psi\rangle$ which explicitly exhibits its positivity. In fact

$$\langle\psi, N\psi\rangle = \sum_{j=1}^{n} \|a_j\psi\|_2^2 = \frac{1}{2}\sum_{j=1}^{n}\left\|\frac{\partial\psi}{\partial x_j}\right\|_2^2 = \frac{1}{2}\sum_{j=1}^{n}\int\left|\frac{\partial\psi}{\partial x_j}\right|^2 d\nu(x).$$

In this form it looks quite similar to the expression occurring in Sobolev's inequality; the only difference is the Gaussian measure.

Nelson's inequality

Let ν be Gaussian measure on $\mathbb{R}^n$ (with variance $\frac{1}{2}$). Define the inner product $\langle\psi, \phi\rangle = \int \psi^*\phi \, d\nu$ and norm $\|\psi\|_2^2 = \int |\psi|^2 \, d\nu$. Let V be a real function on $\mathbb{R}^n$. Then

$$\langle\psi, V\psi\rangle \leqslant \langle\psi, N\psi\rangle + \log\|\exp(V)\|_2\|\psi\|_2^2.$$

The proof will be postponed to the next section. This section will be devoted to the implications of the inequality.

The inequality may be thought of either in terms of partial differential equations or in terms of quantum mechanics. In the partial differential equation setting it is natural to try to write it in a form similar to Sobolev's inequality. Look at the special case $V = \log|\psi|$. The inequality becomes

$$\int |\psi|^2 \log|\psi| \, d\nu \leqslant \langle\psi, N\psi\rangle + \|\psi\|_2^2 \log\|\psi\|_2.$$

Thus it says that if ψ is in L^2 and $\langle\psi, N\psi\rangle$ is finite, then ψ is in $L^2 \log L$. In Sobolev's inequality the conclusion is that ψ is in L^r with $1/r = \frac{1}{2} - 1/n$. In some sense this logarithmic Sobolev inequality is what remains of the ordinary Sobolev inequality when $n \to \infty$. (Of course it also helps that Lebesgue measure is replaced by Gaussian measure.)

In quantum mechanical terms the inequality says that in any state $\langle V(q)\rangle \leqslant \langle N\rangle + \frac{1}{2}\log\langle\exp(2V(q))\rangle_0$, where $\langle\phi(q)\rangle_0$ is the expectation of $\phi(q)$ in the zero particle state. In order to give a physical interpretation of the inequality we consider a special case. Let E be a subset of $\mathbb{R}^n$ and let $\nu(E)$ be the probability of E with respect to the Gaussian measure ν, that is, the probability that q is in E when the system is in the no particle state. Then we have the following corollary.

Particle number principle

$$\text{Prob}\{q \text{ in } E\} \leqslant (1/\log(1/\nu(E)))(2\langle N\rangle + 1).$$

Proof. Let $\pi(E)$ be the probability that q is in E. Let t be a positive real number and set $V(x) = t$ for x in E, $V(x) = 0$ otherwise. Then the inequality gives

$$t\pi(E) \leqslant \langle N\rangle + \tfrac{1}{2}\log[e^{2t}\nu(E) + e^0(1 - \nu(E))].$$

Now choose $2t = -\log \nu(E)$. Then

$$\begin{aligned}-\tfrac{1}{2}(\log \nu(E))\pi(E) &\leqslant \langle N\rangle + \tfrac{1}{2}\log[1 + 1 - \nu(E)]\\ &\leqslant \langle N\rangle + \tfrac{1}{2}\log 2\\ &\leqslant \langle N\rangle + \tfrac{1}{2}.\end{aligned}$$

Hence $\pi(E) \leqslant (1/(-\log \nu(E)))(2\langle N\rangle + 1)$.

This particle number principle has the interpretation that if E is a set of configurations that is highly improbable in the no particle state, and if the expected number of particles in the state of interest is small, then E is somewhat improbable in the state of interest. In other words, when the expected number of particles is small, the field configurations resemble those of the no particle state.

In one version or another the inequality is the basis for the construction of relativistic boson quantum field models. The idea is to prove the consistency of relativistic quantum field theory by constructing a non-trivial mathematical model. This is simplest in a world of one space (and one time) dimension. A preliminary step is to construct the model for a finite interval of space (a sort of quantum mechanical vibrating string). Then methods from statistical mechanics are employed to take the limit.

The Hamiltonian for this vibrating string problem may be written as $H = H_0 + V(q)$, where H_0 is a partial differential operator (in infinitely many dimensions) and $V(q)$ is a multiplication operator. The free Hamiltonian H_0 measures the relativistic energies of particles of mass m. Hence $H_0 \geqslant mN$, where N is the number of particles. The interaction term $V(q)$ is a function of the fields q which is unbounded above and below, but which is almost bounded below in the sense that $\exp(-V)$ is in L^p with respect to Gaussian measure for every $p < \infty$.

If $m > 0$ we may apply Nelson's inequality to $-(1/m)V$ to obtain

$$\begin{aligned}-\langle\psi, V(q)\psi\rangle &\leqslant m\langle\psi, N\psi\rangle + m\log\left\|\exp\left(-\frac{1}{m}V\right)\right\|_2\\ &\leqslant \langle\psi, H_0\psi\rangle + \log\left\|\exp\left(-\frac{1}{m}V\right)\right\|_2^m.\end{aligned}$$

This gives

$$\langle\psi, H\psi\rangle \geqslant -\log\|\exp(-V)\|_{2/m}.$$

Thus the total Hamiltonian is bounded below.

In the rest of this lecture we examine a basic property of this type of inequality which will be fundamental to the proof. First we need a general definition.

Definition. Let ν be a finite positive measure in the space M. Consider the Hilbert space L^2 consisting of complex functions ψ on M with $\|\psi\|_2^2 = \int |\psi|^2\, d\nu < \infty$. Let N be a positive self-adjoint operator acting in this Hilbert space. Then N is said to be a Gross operator if for all ψ in L^2 with $\langle\psi, N\psi\rangle$ finite and for all real V with $\exp(V)$ in L^2

$$\langle\psi, V\psi\rangle \leqslant \langle\psi, N\psi\rangle + \log\|\exp(V)\|_2\|\psi\|_2^2.$$

Thus Nelson's inequality asserts that the number operator is a Gross operator. The following lemma is the basic fact about Gross operators.

Product space property

Let N_1 be a Gross operator in $L^2(M_1, \nu_1)$ and N_2 be a Gross operator in $L^2(M_2, \nu_2)$. Let $M = M_1 \times M_2$ and $\nu = \nu_1 \times \nu_2$. Interpret N_1 and N_2 as operators in $L^2(M, \nu)$. Then $N = N_1 + N_2$ is a Gross operator.

Proof. Since N_1 is a Gross operator

$$\int V(x_1, x_2)|\psi(x_1, x_2)|^2\, d\nu_1(x_1) \leqslant \int \psi(x_1, x_2)^* N_1 \psi(x_1, x_2)\, d\nu_1(x_1) + W(x_2) \int |\psi(x_1, x_2)|^2\, d\nu_1(x_1),$$

where $W(x_2) = \frac{1}{2} \log \int \exp(2V(s, x_2))\, d\nu_1(s)$.

If we do the second integration we obtain

$$\begin{aligned} \langle \psi, V\psi \rangle &\leqslant \langle \psi, N_1\psi \rangle + \int W(x_2) \int |\psi(x_1, x_2)|^2\, d\nu_1(x_1)\, d\nu_2(x_2) \\ &= \langle \psi, N_1\psi \rangle + \int \left[\int W(x_2)|\psi(x_1, x_2)|^2\, d\nu_2(x_2)\right] d\nu_1(x_1). \end{aligned}$$

Now use the fact that N_2 is a Gross operator. This gives

$$\langle \psi, V\psi \rangle \leqslant \langle \psi, N_1\psi \rangle + \int \left[\int \psi(x_1, x_2)^* N_2 \psi(x_1, x_2)\, d\nu_2(x_2) + c \int |\psi(x_1, x_2)|^2\, d\nu_2(x_2)\right] d\nu_1(x_1)$$

where

$$c = \tfrac{1}{2} \log \int \exp(2W(t))\, d\nu_2(t) = \tfrac{1}{2} \log \iint \exp(2V(s, t))\, d\nu_1(s)\, d\nu_2(t).$$

In other words,

$$\langle \psi, V\psi \rangle \leqslant \langle \psi, N_1\psi \rangle + \langle \psi, N_2\psi \rangle + \log\|\exp V\|_2 \|\psi\|_2^2.$$

A consequence of the product space property is that once we have Nelson's inequality for $n = 1$ we immediately have it for arbitrary finite n. (If we want it for $n = 10^{25}$, we just have to apply the product space property 10^{25} times.) Also the inequality for $n = \infty$ may be proved by approximating with large finite n. So everything reduces to the case $n = 1$.

We conclude this section with a remark on the relation between the logarithmic Sobolev inequality and Nelson's inequality.

There is an inequality relating $L \log L$ to e^L. In one version it states that

$$\int |f| g\, d\nu \leqslant \left[\int |f| \log|f|\, d\nu - \|f\|_1 \log\|f\|_1\right] + \|f\|_1 \log\|\exp(g)\|_1.$$

The proof follows from the observation that if $y \geqslant 0$, then $g(x) = \exp(x) - yx$ has its minimum at $x = \log y$. Hence $g(x) \geqslant g(\log y)$. In other words $yx \leqslant y \log y - y + \exp(x)$. (This is a special case of Young's inequality for convex functions.)

Insert $y = |f|$ and $x = g$ and integrate. This gives

$$\int |f| g \, d\nu \leqslant \int |f| \log|f| \, d\nu - \|f\|_1 + \|\exp(g)\|_1.$$

Now multiply f by a constant to arrange that $\|f\|_1 = \|\exp(g)\|_1$. This gives $\int |f| g \, d\nu \leqslant \int |f| \log|f| \, d\nu$, which is the inequality in this special case. But since both sides of the inequality are homogeneous in f, it must hold in general.

This inequality may be applied to show the logarithmic Sobolev inequality implies Nelson's inequality in the form we are using it. In fact, if we set $f = \psi^2$ and $g = 2V$ and divide by 2, we obtain

$$\langle \psi, V\psi \rangle \leqslant \left[\int |\psi|^2 \log|\psi| \, d\nu - \|\psi\|_2^2 \log\|\psi\|_2 \right] + \|\psi\|_2^2 \log\|\exp(V)\|_2,$$

and the logarithmic Sobolev inequality says that the term in brackets is bounded by $\langle \psi, N\psi \rangle$.

3. Gross's Inequality

Recall that a Gross operator is an operator for which an inequality like Nelson's inequality is satisfied. There would not be much point in giving a general definition of Gross operator if the harmonic oscillator Hamiltonian were the only example. However there is at least one other interesting example.

Consider a space M consisting of two points. Let σ be the measure which assigns measure $\frac{1}{2}$ to each point. The Hilbert space of the example will be $\mathscr{H} = L^2(M, \sigma)$. This is a two dimensional space. It will be convenient to use the basis consisting of the vectors $1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. These will play a role analogous to Hermite polynomials.

There will be operators acting in $\mathscr{H}$ which are analogous to the operators q and p of the representation of the commutation relations. Let Q be multiplication by x, so that $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $P = \frac{1}{i}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then P and Q satisfy the Clifford algebra relations $P^2 = 1$, $Q^2 = 1$, and $PQ + QP = 0$.

We write $A = \frac{1}{2}(Q + iP)$ and $A^* = \frac{1}{2}(Q - iP)$. Then $A^2 = 0$, $(A^*)^2 = 0$, and $AA^* + A^*A = 1$. The operator we are interested in is $B = A^*A = \frac{1}{2}(1 + iQP)$. The only thing we really need to know about B is its explicit form: $B = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Gross's inequality

The operator B is a Gross operator.

Proof. We must show that for any multiplication operator $V = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$ we have $\langle \phi, V\phi \rangle \leqslant \langle \phi, B\phi \rangle + \log\|\exp V\|_2 \|\phi\|_2^2$ for all ϕ in the Hilbert space. In other words we must show that $\langle \phi, (V - B)\phi \rangle \leqslant \log\|\exp V\|_2$ for all ϕ with $\|\phi\|_2 = 1$. But since $V - B$ is Hermitian, this is the same as showing that the eigenvalues of $V - B$ are less than $\log\|\exp V\|_2$.

The largest eigenvalue of $V - B$ is $\frac{1}{2}(v_1 + v_2 - 1 + ((v_1 - v_2)^2 + 1)^{1/2})$. On the other hand, $\log\|\exp V\|_2 = \log(\frac{1}{2}e^{2v_1} + \frac{1}{2}e^{2v_2})^{1/2}$. If we set $x = v_1 + v_2$ and $y = v_1 - v_2$ the inequality we have to prove is thus

$$\tfrac{1}{2}(x - 1 + (y^2 + 1)^{1/2}) \leqslant \tfrac{1}{2}(x + \log\cosh y).$$

The proof is completed by the following lemma.

Lemma. $(1 + y^2)^{1/2} \leqslant 1 + \log\cosh y$.

Proof. Let $f(y) = (1 + \log\cosh y)^2 - (1 + y^2)$. Check that $f(0) = 0, f'(0) = 0$, and $f''(y) \geqslant 0$. It follows that $f(y) \geqslant 0$.

Now we come to the analogy between B and a second order differential operator. Since $Bx = x$ and $B1 = 0$, B is the orthogonal projection onto x. Let ψ be a complex function on the real line. Then $\psi(x)$ is a function on M. We compute $B(\psi(x))$. This is

$$\begin{aligned} B(\psi(x)) &= \langle x, \psi(x)\rangle\, x \\ &= \tfrac{1}{2}[\psi(1) - \psi(-1)]x \\ &= \tfrac{1}{4}[\psi(x + 2)(x - 1) + 2\psi(x) - \psi(x - 2)(x + 1)] \\ &= -\tfrac{1}{4}[\psi(x + 2) - 2\psi(x) + \psi(x - 2)] + \tfrac{1}{4}x[\psi(x + 2) - \psi(x - 2)]. \end{aligned}$$

Notice that in order to check that this holds, it is sufficient to check the cases $x = 1$ and $x = -1$. To be sure, the expressions $\psi(3)$ and $\psi(-3)$ will occur, but always with coefficient zero.

Finally, we come to the proof of Nelson's inequality. The following elegant proof is due to Leonard Gross.

Proof. Let M^n be the space consisting of all sequences of n points selected from M. Let σ^n be the measure which assigns weight $(1/2^n)$ to each such sequence. Let $x_1, x_2, \ldots, x_n$ be the coordinate functions on M^n (so that x_j is $+1$ or -1 depending on whether x is $+1$ or -1 on the jth term of the sequence). Finally, let $B_1, B_2, \ldots, B_n$ be operators acting in $L^2(M^n, \sigma^n)$ such that B_j acts on functions of the x_j coordinate in the same way that B acts on functions of x.

Let $C_n = B_1 + B_2 + \cdots + B_n$. Let $y_n = (x_1 + \cdots + x_n)/(2n)^{1/2}$. Let ψ be a complex function on the reals. Then $\psi(y_n)$ is a complex function on M^n. We compute $C_n(\psi(y_n))$.

Notice that $(x_1 + x_2 + \cdots + x_n + 2)/(2n)^{1/2} = y_n + h$, where $h = (2/n)^{1/2}$. It follows that

$$\begin{aligned} C_n(\psi(y_n)) &= \sum_{j=1}^{n} B_j(\psi(y_n)) \\ &= \sum_{j=1}^{n} -\tfrac{1}{4}[\psi(y_n + h) - 2\psi(y_n) + \psi(y_n - h)] \\ &\quad + \tfrac{1}{4}x_j[\psi(y_n + h) - \psi(y_n - h)] \\ &= -\frac{n}{4}[\psi(y_n + h) - 2\psi(y_n) + \psi(y_n - h)] \\ &\quad + y_n\frac{(2n)^{1/2}}{4}[\psi(y_n + h) - \psi(y_n - h)]. \end{aligned}$$

In other words, $C_n(\psi(y_n)) = (D_n\psi)(y_n)$, where D_n is the difference operator defined by

$$(D_n\psi)(t) = -\frac{1}{2h^2}[\psi(t+h) - 2\psi(t) + \psi(t-h)] + t\frac{1}{2h}[\psi(t+h) - \psi(t-h)].$$

Notice that as $n \to \infty$ ($h \to 0$) the operator D_n approaches $-\frac{1}{2}(d^2/dt^2) + t(d/dt)$.

Let ν_n be the measure defined on subsets E of the reals by $\nu_n(E) = \sigma^n(y_n \text{ in } E)$. This measure may be evaluated as follows. The number of sequences in M^n such that exactly k of the numbers $x_1, x_2, \ldots, x_n$ are $+1$ is the binomial coefficient $\binom{n}{k}$. Therefore the σ^n measure of this subset of M^n is $\frac{1}{2^n}\binom{n}{k}$. The value of $x_1 + x_2 + \cdots + x_n$ on this subset is $2k - n$, and so the value of y_n is $(2k - n)/(2n)^{1/2}$. Hence ν_n assigns the weight $\frac{1}{2^n}\binom{n}{k}$ to the points $(2k - n)/(2n)^{1/2}$.

Thus ν_n is given by a binomial distribution. It is known from experience with coin tossing that the sequence of measures ν_n approaches the Gaussian measure $d\nu(t) = \pi^{-1/2}\exp(-t^2)\,dt$. (In fact $\int u(t)\,d\nu_n(t) \to \int u(t)\,d\nu(t)$ for every bounded continuous function u. We will save the proof of this fact for later.)

The crucial observation now is that by the product space property, the operator C_n is a Gross operator. Hence if V is a real function on the reals

$$\langle \psi(y_n), V(y_n)\psi(y_n)\rangle \leqslant \langle \psi(y_n), C_n(\psi(y_n))\rangle + \log\|\exp(V(y_n))\|_2\|\psi(y_n)\|_2^2.$$

The integrals may be written over the reals instead of over M^n. This gives an inequality involving ν_n and D_n:

$$\int V(t)|\psi(t)|^2\,d\nu_n(t) \leqslant \int \psi(t)^* D_n\psi(t)\,d\nu_n(t)$$
$$+ \log\left[\int \exp(2V(t))\,d\nu_n(t)\right]^{1/2}\int |\psi(t)|^2\,d\nu_n(t).$$

We wish to show that we may take the limit $n \to \infty$ to get an inequality involving ν and N.

It is not necessary to prove Nelson's inequality directly for all ψ such that $\langle\psi, N\psi\rangle$ is finite. It is sufficient that these can be approximated by ψ for which the inequality holds. In particular, it is sufficient to prove it for ψ which are linear combinations of Hermite polynomials, that is, for ψ which are polynomials.

It is even sufficient (in the case $n < \infty$, at least) to prove it for ψ which are smooth and have compact support. In fact polynomials may be approximated by such functions. (To see this, let u be a smooth function with compact support which is 1 near the origin. Let $u_n(x) = u(x/n)$. If ψ is a polynomial, then $u_n\psi$ is smooth and has compact support. But

$$\frac{\partial}{\partial x_j}u_n\psi = \frac{1}{n}\frac{\partial u}{\partial x_j}\psi + u_n\frac{\partial\psi}{\partial x_j} \to \frac{\partial\psi}{\partial x_j}$$

in L^2 as $n \to \infty$.)

Similarly, it is enough to prove the inequality for V which are, say, continuous and bounded. Thus we take ψ smooth with compact support and V continuous and bounded and let $n \to \infty$ in the inequality involving ν_n and D_n. Since all the integrands are bounded continuous functions, the terms in the inequality converge to the corresponding terms in Nelson's inequality for the case $n = 1$. This proves the inequality for $n = 1$, and hence for the general case.

For completeness we give a proof of the central limit theorem for coin tossing. This says that the sequence of measures ν_n approaches the Gaussian measure ν.

The trick is to compute Fourier transforms. The Fourier transform of the Gaussian measure is

$$\hat{\nu}(k) = \int \exp(ikt)\, d\nu(t) = \exp(-k^2/4).$$

On the other hand, the Fourier transform of ν_n is

$$\hat{\nu}_n(k) = \int \exp(ikt)\, d\nu_n(t) = \int \exp(iky_n)\, d\sigma_n.$$

Since $y_n = (x_1 + \cdots + x_n)/(2n)^{1/2}$, this factors to give

$$\begin{aligned}\hat{\nu}_n(k) &= \int \prod_{j=1}^{n} \exp(i(k/(2n)^{1/2})x_j)\, d\sigma_n \\ &= \prod_{j=1}^{n} \int \exp(i(k/(2n)^{1/2})x)\, d\sigma \\ &= (\cos(k/(2n)^{1/2}))^n.\end{aligned}$$

Hence $\hat{\nu}_n(k) = (1 - k^2/4n + \cdots)^n \to \exp(-k^2/4) = \hat{\nu}(k)$ as $n \to \infty$.

Now let u be a smooth rapidly decreasing function. Then its Fourier transform $\hat{u}$ is also such a function and $u(t) = \int \exp(itk)\hat{u}(k)\, dk$. Hence

$$\int u(t)\, d\nu_n(t) = \int \hat{u}(k)\hat{\nu}_n(k)\, dk \to \int \hat{u}(k)\hat{\nu}(k)\, dk = \int u(k)\, d\nu(k).$$

The result $\int u(t)\, d\nu_n(t) \to \int u(t)\, d\nu(t)$ may be extended to all u which are continuous and bounded by an approximation argument.

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