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Classical Electrodynamics with Extended Charges

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Abstract. In the first part, the problem of the existence, uniqueness, smoothness of the solution of the coupled Maxwell–Newton equations is investigated for a charge distribution with compact support. In the second part, a detailed study of this problem for the case of a spherical shell distribution shows that the limit of Lorentz–Dirac is not physical.

Introduction

In the discussion of radiation reaction in classical electrodynamics, one often takes comfort in observing that the difficulties of the theory appear in a region where the classical theory is no longer applicable, and that the ultimate solution of the problem must therefore be given by the quantum theory. We are not very happy with this answer, because in quantum electrodynamics not even the problem can be formulated in a mathematically precise way, at present. When one tries to understand things at least on the formal level, one comes to the famous divergencies which, furthermore, have some similarity with the difficulties of the classical theory. Then one takes comfort in presuming that the infinities appear because quantum electrodynamics is not a closed theory in itself, and that the ultimate solution must be given by a 'complete' theory including strong and weak interactions. In this way the problem of radiation reaction is disappeared in the shadowy contours of such a future 'super-theory'.

On the other hand, the problem of the motion of extended charges interacting with the electromagnetic field is classically well defined. This is a strong enough reason for us to take this point of view, being aware that nothing is learnt about the electron in this way, but something is learnt about classical electrodynamics. In order to avoid additional degrees of freedom, we will consider a rigid charge distribution. Then we must use non-relativistic mechanics to avoid difficulties with causality. This leads us to investigate the coupled Maxwell and Newton's equations.

Summary of Procedure and Results

In the first part of this work we consider the general problem of a charge distribution $\rho(\mathbf{x})$ of finite extension in interaction with an electromagnetic field. Starting from the Lagrange function for the coupled system particle-field, we formulate an initial value problem with Cauchy data given at $t = 0$. This goal is achieved by considering the canonical equations which arise from the hamiltonian formulation.

The canonical evolution equation contains a linear and a non-linear part. We solve the linear problem ($\rho = 0$) within a certain space X. Then we prove that the interaction is lipschitzian in the space X; this fact enables us to apply a theorem of Segal [1, page 343] which ensures the local existence and uniqueness of the solution if ρ is a C_0^1 function and if the potentials are smooth for $t = 0$ (Theorem 1). The hamilton function gives us a uniform bound for the velocity and the global existence and uniqueness follow from the theorem of Segal (Theorem 2).

In Section 2 we assume two supplementary conditions:

(a) ρ is C_0^∞,
(b) the potentials at $t = 0$ are in the domain of Δ^n for all n.

With (a) and (b) and a result of Segal [1, page 357] it follows that the solution is infinitely differentiable in $\mathbf{x}$ and t (Theorem 4).

In the second part of this paper we study the motion of an uniformly charged spherical shell (radius a) with a distribution $\rho(\mathbf{x}) = 1/4\pi a^2\ \delta(|\mathbf{x}| - a)$. This singular distribution, which is particular suitable for the explicit calculation of solutions, is not in the classes considered before. Therefore the Cauchy problem has to be investigated separately. In Section 3 we follow Bohm and Weinstein [3] in giving a derivation of the equation of motion for the shell by elimination of the fields. This equation is non-local in t and the complete determination of the solution requires initial data

$$\mathbf{g}(t) = \begin{pmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{pmatrix}$$

over a time interval $T_a = [0, 2a/c]$. In Section 4 we prove that the Cauchy data at $t = 0$ for the coupled problem fully determine an unique solution over T_a, giving the initial data for the Bohm–Weinstein equation.

In Section 5 we study the Bohm–Weinstein equation with a time dependent force $\mathbf{f}(t)$ for the special case where the homogeneous solution of the Maxwell equations is zero. The resulting equation is then linear. This problem is not very physical but is a good introduction to the general case. We investigate especially the limits $t \to \infty$ and $a \to 0$ for the kernel of the solution.

In Section 6 we study the general non-linear case. With some assumptions about the fields at $t = 0$, we obtain the existence and uniqueness of the solution. We show that the limit $t \to \infty$ for the velocity exists. The solutions are entirely well-behaved for finite a (bare mass $m > 0$).

The problem of the radiation damping is investigated in Section 7. We show that the shell radiates according to the law of classical electrodynamics. Expecially the non-relativistic hyperbolic motion gives a correct radiation. The results remain valid if the limit $a \to 0$ to the point charge is performed. However, the radiation rate then goes to zero because the particle becomes infinitely heavy due to the diverging electromagnetic mass $m_e = 2e^2/3ac^2$ and cannot be accelerated by finite force. If one compensates the increasing electromagnetic mass m_e by a decreasing bare mass m such that $m_T = m + m_e$ remains fixed (mass renormalisation), one gets the Lorentz–Dirac theory [8–10] as the rigorous limit. But as soon as the bare mass m becomes negative, i.e. for finite $a < 2e^2/3m_Tc^2 = r_e$ (r_e = the classical electron radius), the unphysical run-away solutions already appear. This shows that the Lorentz–Dirac limit is unphysical. The classical electron radius is the natural lower bound for the extension of a charge in classical electrodynamics.

At the end, we make some remarks about the possibilities of formulating a relativistic theory for a rigid charge of finite extension; in particular, the covariant theory of Nodvik [4] is shown to be not causal.

For a complete list of references concerning the subject of radiation damping in classical electrodynamics, see Refs. [11–12].

PART I

1. The Cauchy Problem for the Coupled Maxwell–Newton Equations

We shall deal with the problem of a finite rigid charge distribution moving in an electromagnetic field. Let us denote by $\mathbf{z}(t)$, $\mathbf{v}(t)$ the position and the velocity of the center of mass and by $\mathbf{A}(\mathbf{x}, t)$, $\phi(\mathbf{x}, t)$ the vector and scalar potentials respectively.

The coupled system particle-field can be described by a Lagrange function L. The complete description of the system involves rotations so that we obtain for L:

$$L = \frac{1}{2} m\mathbf{v}^2 + \frac{1}{c}\mathbf{v}\cdot \int \rho\mathbf{A} - \frac{1}{c}\int \rho\phi + \frac{1}{8\pi}\int \left(\frac{1}{c}\dot{\mathbf{A}} + \nabla\phi\right)^2 - \frac{1}{8\pi}\int (\nabla \wedge \mathbf{A})^2$$

$$+ \frac{1}{8\pi}\int \left(\frac{\dot{\phi}}{c} + \operatorname{div}\mathbf{A}\right)^2 + \frac{1}{2}\Theta\boldsymbol{\omega}^2 - \frac{1}{c}\int \omega_{\mu\nu}(x_\nu - z_\nu)A_\mu\rho, \tag{1.1}$$

where Θ is the moment of inertia and

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix};$$

further $\int \rho\mathbf{A}$ means $\int d^3x\mathbf{A}(\mathbf{x}, t)\rho(\mathbf{x} - \mathbf{z}(t))$ and so on. The conjugate momenta have the form

$$\mathbf{\Pi}_T(t) = m\mathbf{v} + \frac{1}{c}\int \rho\mathbf{A} \tag{1.2}$$

for the center of mass, and

$$\mathbf{\Pi}_\Omega(t) = \Theta\boldsymbol{\omega} - \frac{1}{c}\int \mathbf{A} \wedge (\mathbf{x} - \mathbf{z})\rho \tag{1.3}$$

for the angular coordinate $\boldsymbol{\Omega}$. Further

$$\mathbf{\Pi}_A(\mathbf{x}, t) = \frac{1}{4\pi c}\left(\frac{1}{c}\dot{\mathbf{A}}(\mathbf{x}, t) + \nabla\phi(\mathbf{x}, t)\right) \tag{1.4}$$

for the vector potential, and

$$\mathbf{\Pi}_\phi(\mathbf{x}, t) = \frac{1}{4\pi c}\left(\frac{1}{c}\dot{\phi}(\mathbf{x}, t) + \operatorname{div}\mathbf{A}(\mathbf{x}, t)\right) \tag{1.5}$$

for the scalar potential. Now we apply the canonical formalism and obtain easily the following Hamiltonian H:

$$H = \frac{1}{2m}\left(\mathbf{\Pi}_T - \frac{1}{c}\int \rho\mathbf{A}\right)^2 + 2\pi c^2 \int \mathbf{\Pi}_A^2 + \frac{1}{8\pi}\int (\nabla \wedge \mathbf{A})^2 + \int (\rho\phi - c\mathbf{\Pi}_A \cdot \nabla\phi)$$
$$+ \frac{1}{2\Theta}\left(\mathbf{\Pi}_\Omega + \frac{1}{c}\int \mathbf{A} \wedge (\mathbf{x} - \mathbf{z})\rho\right)^2; \tag{1.6}$$

from this Hamilton function we deduce the canonical equations of motion for the coupled system:

$$\dot{\mathbf{z}}(t) = \frac{\mathbf{\Pi}_T}{m} - \frac{1}{mc}\int \rho\mathbf{A}; \tag{1.7}$$

$$\dot{\mathbf{\Pi}}_T(t) = \frac{1}{mc}\left(\mathbf{\Pi}_T - \frac{1}{c}\int \rho\mathbf{A}\right)\cdot\left(\int \rho_{\mathbf{z}}\mathbf{A}\right) - \int \phi\rho_{\mathbf{z}}$$
$$+ \frac{1}{\Theta c}\left(\mathbf{\Pi}_\Omega + \frac{1}{c}\int \rho(\mathbf{A} \wedge (\mathbf{x} - \mathbf{z}))\right)\cdot\int \mathbf{A} \wedge (\mathbf{x} - \mathbf{z})\rho_{\mathbf{z}}$$
$$- \frac{1}{c}\int \rho\boldsymbol{\omega} \wedge \mathbf{A}, \tag{1.8}$$

where

$$\rho_{\mathbf{z}} = \nabla_{\mathbf{z}}\rho$$
$$\dot{\mathbf{\Omega}}(t) = \frac{1}{\Theta c}\int \rho(\mathbf{x} - \mathbf{z}) \wedge \mathbf{A}$$
$$\dot{\mathbf{\Pi}}_\Omega(t) = \mathbf{0}. \tag{1.9}$$

We integrate (1.9) with the initial condition $\boldsymbol{\omega}(0) = \mathbf{0}$ and obtain

$$\boldsymbol{\omega}(t) = \frac{1}{\Theta c}\int \rho(\mathbf{x} - \mathbf{z}) \wedge \mathbf{A} = \frac{d\mathbf{\Omega}}{dt}$$
$$\mathbf{\Pi}_\Omega(t) = \mathbf{0}; \tag{1.10}$$

the equations for the fields are

$$\dot{\mathbf{A}}(\mathbf{x}, t) = 4\pi c^2\mathbf{\Pi}_A - c\nabla\phi,$$
$$\dot{\phi}(\mathbf{x}, t) = 4\pi c^2\Pi_\phi - c \operatorname{div} \mathbf{A}, \tag{1.11}$$

and

$$\mathbf{\Pi}_A(\mathbf{x}, t) = -\frac{1}{4\pi}\operatorname{rot}\operatorname{rot}\mathbf{A} + c\nabla\Pi_\phi$$
$$+\frac{1}{mc}\rho\left(\mathbf{\Pi}_T - \frac{1}{c}\int \rho\mathbf{A}\right) - \frac{1}{c}\rho(\boldsymbol{\omega} \wedge (\mathbf{x} - \mathbf{z});$$
$$\dot{\Pi}_\phi(\mathbf{x}, t) = c \operatorname{div} \mathbf{\Pi}_A + \rho. \tag{1.12}$$

We choose the Lorentz gauge. Then it follows from equation (1.5) that Π_ϕ, $\dot{\Pi}_\phi$ are identically zero; furthermore the term $\int (\rho\phi - c\mathbf{\Pi}_A \cdot \nabla\phi)$ in H does not contribute to the energy because of equation (1.12).

Now let us consider the free motion ($\rho = 0$). We obtain for the mechanical quantities

$$\frac{d}{dt}\begin{pmatrix}\mathbf{z}\\ \mathbf{\Pi}_T\end{pmatrix} = \begin{pmatrix}0 & \mathbb{I}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{z}\\ \mathbf{\Pi}_T\end{pmatrix}, \quad \begin{pmatrix}\mathbf{z}\\ \mathbf{\Pi}_T\end{pmatrix} \in \mathbb{R}^6 \tag{1.13}$$

and

$$\frac{d}{dt}\begin{pmatrix}\mathbf{\Omega}\\ \mathbf{\Pi}_\Omega\end{pmatrix} = \begin{pmatrix}0 & \mathbb{I}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{\Omega}\\ \mathbf{\Pi}_\Omega\end{pmatrix}, \quad \begin{pmatrix}\mathbf{\Omega}\\ \mathbf{\Pi}_\Omega\end{pmatrix} \in \mathbb{R}^6 \tag{1.14}$$

here $\mathbb{I}$ denotes the 3×3 unit matrix. We write the Maxwell equations in vector form

$$\frac{1}{c^2}\frac{d}{dt}\begin{pmatrix}A_\mu\\ \dot{A}_\mu\end{pmatrix} = \begin{pmatrix}0 & I\\ -\Delta & 0\end{pmatrix}\begin{pmatrix}A_\mu\\ \dot{A}_\mu\end{pmatrix} \tag{1.15}$$

where $A_\mu = (-\phi, \mathbf{A})$ and $\dot{A}_\mu = (-\dot{\phi}, \dot{\mathbf{A}})$; we have $\partial^\mu A_\mu = 0$, $\mu = 1, 2, 3, 4$.

Let us define M to be the real Hilbert space of square integrable functions $\begin{pmatrix}A_\mu\\ \dot{A}_\mu\end{pmatrix}$ on $\mathbb{R}^3$ with values in $\mathbb{R}^4 \oplus \mathbb{R}^4$. Denote the 4×4 operator $(-\Delta)^{1/2}\cdot I$ (defined by real Fourier transform) by B. Then $M_{3/2} = M^1_{3/2} \oplus M^2_{3/2}$ is defined as the domain of $B(I + B) \oplus (I + B)$ endowed with the norm

$$\left\|\begin{matrix}A_\mu\\ \dot{A}_\mu\end{matrix}\right\|_{M_{3/2}} = \left\{\sum_{\mu=1}^{4} \|B(I + B)A_\mu\|_2^2 + \|(I + B)\dot{A}_\mu\|_2^2\right\}^{1/2}.$$

M is a Hilbert space and the time evolution operator $M(t)$ which gives the solution of (1.15) in $M_{3/2}$ takes the form

$$M(t) = \begin{pmatrix}\cos(ctB) & B^{-1}\sin(ctB)\\ -B\sin(ctB) & \cos(ctB)\end{pmatrix}.$$

$M(t)$ is a continuous one-parameter group of orthogonal transformations [2, page 598] with the skew-adjoint generator

$$\begin{pmatrix}0 & I\\ -\Delta & 0\end{pmatrix}.$$

Note that $\|f\|_{M^1_{3/2}} \leqslant \text{cst}\|f\|_{22}$ where

$$\|f\|_{22} = \left(\sum_{|\alpha|\leqslant 2}\int |D^\alpha f|^2\right)^{1/2}$$

is the norm in the Sobolev space $W^{22}(\mathbb{R}^4)$ and

$$\|g\|_{M^2_{3/2}} \leqslant \text{cst}'\|g\|_{12}$$

in $W^{12}(\mathbb{R}^4)$.

Now let us define X to be

$$\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus M^1_{3/2} \oplus M^2_{3/2}$$

with the norm

$$\|\cdot\|_X = \{\|\cdot\|^2_{\mathbb{R}^3} + \cdots + \|\cdot\|^2_{M^1_{3/2}} + \|\cdot\|^2_{M^2_{3/2}}\}^{1/2};$$

here $\|\cdot\|_{\mathbb{R}^3}$ means $\sup_{i=1,2,3}|\cdot i|$. We consider the coupled problem in the space X and so write the evolution equation as follows:

$$\frac{d}{dt}u = Lu + K(u), \qquad u \in X. \tag{1.16}$$

$K(u)$ is the non-linear part and does not depend explicitly on time on our case. Such equations have been extensively studied by I. Segal [1] for the case of relativistic equations. Using his general results, we obtain the following

Theorem 1 (local existence and uniqueness). There exists an interval $[0, \bar{t})$ on which the solution of (1.16) is unique and continuous provided that the charge distribution (with compact support) is C_0^1 and

$$\begin{pmatrix} A_\mu(0) \\ \dot{A}_\mu(0) \end{pmatrix} \in M_{3/2}.$$

Proof. This follows from a Theorem of Segal [1, page 343] if we can show that $K(u)$ is locally lipschitzian and is a continuous function of $u \in X$. Because of the form of $K(u)$ it only remains to prove that it is locally lipschitzian. The calculation is lengthy but not difficult so that we only give a representative example. We use the notation

$$\rho_i = \frac{\partial}{\partial z_i}\rho(\mathbf{x} - \mathbf{z}(t)) \quad \text{and} \quad \rho_i' = \frac{\partial}{\partial z_i'}\rho(\mathbf{x} - \mathbf{z}'(t))$$

and consider the term

$$R = \sup_{i=1,2,3}\left|\int (\mathbf{\Pi}_T\cdot\mathbf{A})\rho_i - \int (\mathbf{\Pi}_T'\cdot\mathbf{A}')\rho_i'\right|.$$

We get

$$R \leqslant \sup_{i=1,2,3}\sum_{k=1}^{3}\left[|(\Pi_T)_k - (\Pi_T')_k|\int|A_k\rho_i| \right.$$
$$\left. + |(\Pi_T')_k|\int|(A_k - A_k')\rho_i'| + |(\Pi_T')_k|\int|A_k(\rho_i - \rho_i')|\right];$$

where k denotes the cartesian components. By Schwartz inequality we obtain

$$R \leqslant \sup_{i=1,2,3}\sum_{k=1}^{3}[|(\Pi_T)_k - (\Pi_T')_k|\,\|A_k\|_2\cdot\|\rho_i\|_2$$
$$+ |(\Pi_T')_k|\,\|A_k - A_k'\|_2\cdot\|\rho_i'\|_2 + |(\Pi_T')_k|\,\|A_k\|_2\|\rho_i - \rho_i'\|_2].$$

Since

$$\|\rho_i - \rho_i'\|_2 \leqslant b\|\mathbf{z} - \mathbf{z}'\|_{\mathbb{R}^3}$$

and

$$\|\rho_i\|_2 = \|\rho_i'\|_2 = a,$$

we have

$$R \leqslant a\|\mathbf{A}\|_2\|\mathbf{\Pi}_T - \mathbf{\Pi}_T'\|_{\mathbb{R}^3} + a\|\mathbf{\Pi}_T'\|_{\mathbb{R}^3}\|\mathbf{A} - \mathbf{A}'\|_2$$
$$+ b\|\mathbf{\Pi}_T'\|_{\mathbb{R}^3}\,\|\mathbf{A}\|_2\|\mathbf{z} - \mathbf{z}'\|_{\mathbb{R}^3};$$

it follows that R is lipschitzian because $\|f\|_2 \leqslant \|f\|_{M^1_{3/2}}$ for $f \in M^1_{3/2}$. A similar proof for the other terms may be given without difficulties.

Actually the Thoerem of Segal gives the solution of the evolution equation as the solution of the corresponding integral equation

$$u(t) = L(t)u(0) + \int_0^t L(t - s)K(u(s))\, ds. \tag{1.17}$$

We will show later that this solution satisfies the differential equation in the classical sense.

In order to study the global existence and uniqueness of the solution, let us write down the integral equation (1.17) explicitly. We write the equations for $\dot{\phi}$, $\dot{\mathbf{A}}$ instead of $\mathbf{\Pi}_A$, Π_ϕ which is possible because the interaction does not involve $\mathbf{\Pi}_A$, Π_ϕ. We assume that $\mathbf{z}(0) = \mathbf{0}$ and so obtain

$$\mathbf{z}(t) = \frac{\mathbf{\Pi}_T(0)}{m}\cdot t + \int_0^t \left[\frac{1}{mc}\int \rho(s)\mathbf{A}(s) + (t - s)\dot{\mathbf{\Pi}}_T(s)\right] ds; \tag{1.18}$$

$$\begin{aligned}\mathbf{\Pi}_T(t) = \mathbf{\Pi}_T(0) + \int_0^t \Bigg[&\frac{1}{mc}\left(\mathbf{\Pi}_T - \frac{1}{c}\int \rho\mathbf{A}\right)\cdot\int \rho_{\mathbf{z}}\cdot\mathbf{A} \\ &- \int \phi\rho_{\mathbf{z}} - \frac{1}{c}\int \rho\boldsymbol{\omega}\wedge\mathbf{A} \\ &- \frac{1}{\Theta c^2}\int \mathbf{A}\wedge(\mathbf{x} - \mathbf{z})\rho\cdot\int \mathbf{A}\wedge(\mathbf{x} - \mathbf{z})\rho_{\mathbf{z}}\Bigg] ds;\end{aligned} \tag{1.19}$$

$$\mathbf{\Omega}(t) = \frac{1}{\Theta c}\int_0^t \rho(\mathbf{x} - \mathbf{z})\wedge\mathbf{A}\, ds; \tag{1.20}$$

$$\mathbf{\Pi}_\Omega(t) = \mathbf{0} \tag{1.21}$$

$$\begin{aligned}\mathbf{A}(\mathbf{x}, t) = {}&\cos(ctB)\mathbf{A}(\mathbf{x}, 0) + B^{-1}\sin(ctB)\dot{\mathbf{A}}(\mathbf{x}, 0) \\ &+ \int_0^t \{B^{-1}\sin[c(t - s)B]\}\Bigg\{\left(\frac{1}{m}\mathbf{\Pi}_T - \frac{1}{mc}\int \rho\mathbf{A}\right)\rho \\ &- \frac{1}{c}\rho(\boldsymbol{\omega}\wedge(\mathbf{x} - \mathbf{z}))\Bigg\} ds;\end{aligned} \tag{1.22}$$

$$\begin{aligned}\phi(\mathbf{x}, t) = {}&\cos(ctB)\phi(\mathbf{x}, 0) + B^{-1}\sin(ctB)\dot{\phi}(\mathbf{x}, 0) \\ &+ \int_0^t \{B^{-1}\sin[c(t - s)B]\}\rho\, ds;\end{aligned} \tag{1.23}$$

$$\begin{aligned}\dot{\mathbf{A}}(\mathbf{x}, t) = {}&-B\sin(ctB)\mathbf{A}(\mathbf{x}, 0) + \cos(ctB)\dot{\mathbf{A}}(\mathbf{x}, 0) \\ &+ \int_0^t \{\cos[c(t - s)B]\}\Bigg\{\left(\frac{1}{m}\mathbf{\Pi}_T - \frac{1}{mc}\int \rho\mathbf{A}\right)\rho \\ &- \frac{1}{c}\rho\boldsymbol{\omega}\wedge(\mathbf{x} - \mathbf{z})\Bigg\} ds;\end{aligned} \tag{1.24}$$

$$\dot{\phi}(\mathbf{x}, t) = -B\sin(ctB)\phi(\mathbf{x}, 0) + \cos(ctB)\dot{\phi}(\mathbf{x}, 0) + \int_0^t \{\cos[c(t - s)]\}\rho\, ds. \tag{1.25}$$

Theorem 2. Under the assumptions of Theorem 1, the solution of (1.16) exists, is unique and continuous on $T = [0, \infty)$. Moreover the velocity and the fields are uniformly bounded on T.

Proof. The statement follows from the above Theorem of Segal [1, page 343] if we can show that the norm remains bounded for all t. To illustrate the method of estimation, let us consider the translational motion. A similar proof is valid for the angular coordinates. We use the orthogonality of $M(t)$ on $M_{3/2}$ to obtain

$$\left\|\begin{matrix} A_\mu \\ \dot{A}_\mu \end{matrix}\right\|_2 \leqslant \mathrm{cst} \left\|\begin{matrix} A_\mu \\ \dot{A}_\mu \end{matrix}\right\|_{M_{3/2}} \leqslant \left\|\begin{matrix} A_\mu(0) \\ \dot{A}_\mu(0) \end{matrix}\right\|_{M_{3/2}} + n(a) \sup_{s\in[0,t)} \|\dot{\mathbf{z}}(s)\|_{\mathbb{R}^3}; \tag{1.26}$$

here $n(a)$ is a constant which depends on the radius a of the charge distribution. $n(a)$ is of the order $2a/c$; (1.26) follows from the fact that the charge distribution has compact support; in other words, for $\mathbf{x}$, t fixed, the integration over the time is restricted to a finite interval.

Since H (1.6) is a constant of the motion, it follows that the velocity and the integrals of the fields are bounded for arbitrary time t if

$$\left\|\begin{matrix} A_\mu(0) \\ \dot{A}_\mu(0) \end{matrix}\right\|_{M_{3/2}} < \infty.$$

Because the norm of the fields is bounded by the velocity according to (1.26) we immediately conclude that the norm of $u \in X$ remains bounded for all t. This completes the proof.

2. Smoothness of the Solution

In this section we assume that $\rho(\mathbf{x})$ is C_0^∞ as a function of $\mathbf{x}$ and that

$$\begin{pmatrix} A_\mu(0) \\ \dot{A}_\mu(0) \end{pmatrix} \text{ are in the domain of } \Delta^m \text{ for } m = 1, 2, \ldots \tag{2.1}$$

With the notation of (1.16) we prove the following.

Lemma 1. If $u \in X$ then

$$\begin{aligned}
&K(u) \in D(L^{n-1}) && \text{if } u \in D(L^{n-1}) \\
&\frac{d}{dt} K(u) \in D(L^{n-2}) && \text{if } u, \dot{u} \in D(L^{n-2}) \\
&\vdots \\
&\frac{d^{(n-2)}}{dt^{(n-2)}} K(u) \in D(L) && \text{if } u, \dot{u}, \ldots, u^{(n-2)} \in D(L)
\end{aligned}$$

where $D(L^m)$ denotes the domain of L^m.

Proof. We note that the integrations occurring in (1.18), (1.19), and (1.20) are restricted to a compact domain so that the quantities $\mathbf{z}, \dot{\mathbf{z}}, \ldots, \mathbf{\Pi}_T, \dot{\mathbf{\Pi}}_T, \ldots,$ are bounded as soon as $\mathbf{A}, \dot{\mathbf{A}}, \ldots, \phi, \dot{\phi}, \ldots,$ are continuous. Further $j_\mu^{(n-2)} = (\rho, \dot{\mathbf{z}}\rho)^{(n-2)}$ is in $D(L)$ if $u, \dot{u}, \ldots, u^{n-2}$ are in $D(L)$ because $\rho(\mathbf{x} - \mathbf{z})$ is C_0^∞ by assumption. In fact we have

$$\left(\frac{d}{dt}\right)^n K(u) \in D(L^r)$$

as soon as $u, \dot{u}, \ldots, u^n \in D(L^r)$; $r, n = 0, 1, 2, \ldots$

Theorem 3. $K(u)$ is a C^∞-Fréchet differentiable function of $X \to X$.

Proof. We consider as an example the term

$$F = \int \rho(\mathbf{x} - \mathbf{z}) \cdot A_i(\mathbf{x}, t)\, d^3x$$

where i denotes the cartesian components. Assuming $y \in \mathbb{R}^3$ and $a_i \in M_{3/2}^1$ the Fréchet derivative takes the form

$$F_u = \lim_{\epsilon \to 0} \epsilon^{-1} \int \{\rho(\mathbf{z} + \epsilon\mathbf{y} - \mathbf{x}) \cdot (A_i + \epsilon a_i) - \rho(\mathbf{z} - \mathbf{x}) \cdot A_i\};$$

in the limit $\epsilon \to 0$ we obtain

$$F_u = \int a_i \rho + \sum_{k=1}^{3} \int \frac{\partial \rho}{\partial z_k} A_i.$$

It is possible to repeat the operation because ρ is C_0^∞. We conclude that $K(u)$ is C^∞ as a function of $X \to X$. It should be pointed out that, apart from ρ, the interaction is a polynomial.

Theorem 4. The function $u(t)$ is infinitely differentiable as a function of the time and (after a suitable modification on a Lebesque set of measure zero) of the space variables. This solution then satisfies the differential equation in the classical sense if the conditions (2.1) are fulfilled.

Proof. The statement follows from a result of Segal [1, corollary 3.2].

PART II

3. The Equation of Bohm–Weinstein

For the sake of completeness we first include the derivation of the Bohm–Weinstein equation of motion. Let us consider a spherically symmetric charge distribution with the Fourier transform $\tilde{\rho}(k)$ $(k = [k_1^2 + k_2^2 + k_3^2]^{1/2})$. The moving charge distribution leads to the current density

$$\mathbf{j}(\mathbf{k}, t) = \frac{e}{c}\, \dot{\mathbf{z}}(t) \cdot \exp(-i\mathbf{k} \cdot \mathbf{z}(t)) \cdot \tilde{\rho}(k), \tag{3.1}$$

where $\mathbf{z}(t)$, $\dot{\mathbf{z}}(t)$ denotes the position and the velocity of the center of mass respectively. It is convenient to choose a gauge in which the transverse part of the vector potential is divergentless. In this case we have

$$\Delta\phi(\mathbf{x}, t) = -4\pi\rho(\mathbf{x}, t)$$

for the scalar potential and it follows that $\phi(\mathbf{x}, t)$ will produce no net force on the center of mass. The transverse part $\tilde{\mathbf{A}}_\perp$ will satisfy

$$\left(c^2k^2 + \frac{\partial^2}{\partial t^2}\right)\tilde{\mathbf{A}}_\perp = 4\pi c^2 \tilde{\mathbf{j}}_\perp \tag{3.2}$$

with

$$\tilde{\mathbf{j}}_\perp = \frac{\mathbf{k} \wedge (\tilde{\mathbf{j}} \wedge \mathbf{k})}{k^2}. \tag{3.3}$$

The retarded solution of equation (3.2) is

$$\tilde{\mathbf{A}}^r_\perp(\mathbf{k}, t) = \frac{4\pi c}{k}\int_{-\infty}^{t} dt' (\tilde{\mathbf{j}}(\mathbf{k}, t'))_\perp \sin[ck(t - t')], \tag{3.4}$$

up to a solution $\tilde{\mathbf{A}}^h$ of the homogeneous equation. The retarded fields are given by

$$\mathbf{E}^r_\perp(\mathbf{x}, t) = \frac{1}{c}\frac{\partial \mathbf{A}^r_\perp(\mathbf{x}, t)}{\partial t}.$$

$$\mathbf{B}^r(\mathbf{x}, t) = \nabla \wedge \mathbf{A}^r_\perp(\mathbf{x}, t), \tag{3.5}$$

and the total force acting on the center of mass is then

$$\mathbf{F}_T = \mathbf{F}_{\mathrm{self}} + \mathbf{F}_{\mathrm{ext}}$$

with

$$\mathbf{F}_{\mathrm{self}} = \int [\rho(\mathbf{x}, t)\cdot \mathbf{E}^r_\perp(\mathbf{x}, t) + \mathbf{j}(\mathbf{x}, t) \wedge \mathbf{B}^r(\mathbf{x}, t)]\, d^3x \tag{3.6}$$

and

$$\mathbf{F}_{\mathrm{ext}} = \mathbf{f}(\mathbf{z}, \dot{\mathbf{z}}, t) + \mathbf{F}_h, \tag{3.7}$$

where

$$\mathbf{F}_h = \int d^3x \rho(\mathbf{x} - \mathbf{z}(t))[\mathbf{E}^h(\mathbf{x}, t) + \dot{\mathbf{z}}(t) \wedge \mathbf{B}^h(\mathbf{x}, t)]; \tag{3.8}$$

here h denotes the solution of the homogeneous Maxwell equations and $\mathbf{f}$ stands for other non-electromagnetic forces like gravitational ones.

Using Fourier transformation and putting $t - t' = \tau$ it follows that

$$\mathbf{F}_{\mathrm{self}} = -4\pi e^2 \int_0^\infty d\tau \int \frac{d^3k}{k} |\tilde{\rho}|^2 e^{i\mathbf{k}\cdot\mathbf{s}} \cdot \left[k\cdot\cos(ck)\frac{\mathbf{k} \wedge [\dot{\mathbf{z}}(t - \tau) \wedge \mathbf{k}]}{k^2} - i\sin(ck\tau)\frac{\dot{\mathbf{z}}(t)}{c} \wedge \{\mathbf{k} \wedge \dot{\mathbf{z}}(t - \tau)\}\right], \tag{3.9}$$

where

$$\mathbf{s} = \mathbf{z}(t) - \mathbf{z}(t - \tau). \tag{3.10}$$

Further the computation of (3.9) with

$$\tilde{\rho}(k) = \frac{1}{(2\pi)^{3/2}}\cdot\frac{\sin ka}{ka} \qquad \text{(spherical shell)}$$

gives the simple result

$$\mathbf{F}_{\text{self}} = \frac{e^2}{3a^2c}\left[\dot{\mathbf{z}}\left(t - \frac{2a}{c}\right) - \dot{\mathbf{z}}(t)\right] + 0(\dot{\mathbf{z}}/c)^2, \qquad t \geqslant \frac{2a}{c}. \tag{3.11}$$

The equation of motion now takes the form

$$m\ddot{\mathbf{z}}(t) = \mathbf{F}_{\text{self}} + \mathbf{F}_{\text{ext}}. \tag{3.12}$$

The solution $\mathbf{z}(t)$ can be calculated if

$$\mathbf{g}(t) = \begin{pmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{pmatrix} \quad \text{is known for} \quad t \in \left[0, \frac{2a}{c}\right].$$

These values are uniquely determined by the Cauchy data at $t = 0$ for the whole problem as will be shown in the next section. Since the actual calculation of $\mathbf{g}(t)$ requires the solution of the original coupled problem in the time interval $[0, 2a/c]$, the equation (3.12) of Bohm–Weinstein solves this problem only partially, i.e. for $t > 2a/c$. But we can consider an initial field with compact support which is space-like separated from the charge such that we have a free motion for $t \in [0, 2a/c]$. Note that this motion is a solution of (3.12) if $\mathbf{F}_{\text{ext}} = \mathbf{0}$. This scattering formulation of the problem is quite physical and we will show that the solution is unique and continuous if $\mathbf{F}_{\text{ext}}(t) \in C^1[0, \infty)$.

4. The Cauchy Problem for the Spherical Shell Distribution

We first consider the time interval $T_a = [0, 2a/c]$ and start from the equations (1.18)–(1.25) with Cauchy data at $t = 0$. After elimination of the fields by the method of Bohm and Weinstein we obtain the following differential equation

$$\frac{d}{dt}\begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{F}_h(\mathbf{z}, \dot{\mathbf{z}}, t) - \dfrac{e^2}{3a^2c}\dot{\mathbf{z}} \end{pmatrix} \tag{4.1}$$

for $t \in [0, 2a/c)$. Here $\mathbf{F}_h(\mathbf{z}, \dot{\mathbf{z}}, t)$ denotes

$$\frac{1}{(2\pi)^{3/2}}\int d^3k\, \frac{\sin(ka)}{ka}\, e^{i\mathbf{k}\cdot\mathbf{z}}\left[-\mathbf{k}\tilde{\phi}(\mathbf{k}, t) - \frac{1}{c}\tilde{\dot{\mathbf{A}}}(\mathbf{k}, t) + \dot{\mathbf{z}}(t) \wedge (\mathbf{k} \wedge \tilde{\mathbf{A}}(\mathbf{k}, t))\right], \tag{4.2}$$

where

$$\begin{pmatrix} A_\mu(\mathbf{k}, t) \\ \tilde{\dot{A}}_\mu(\mathbf{k}, t) \end{pmatrix} = \begin{pmatrix} \cos(ctk) & \dfrac{\sin(ctk)}{k} \\ -k\sin(ctk) & \cos(ctk) \end{pmatrix}\begin{pmatrix} \tilde{A}_\mu(\mathbf{k}, 0) \\ \tilde{\dot{A}}_\mu(\mathbf{k}, 0) \end{pmatrix}, \qquad \mu = 1, 2, 3, 4. \tag{4.3}$$

Lemma 4.1. The interaction terms in (4.1) are Lipschitzian in the space $\mathbb{R}^3 \oplus \mathbb{R}^3$ provided that the potentials $\begin{pmatrix} \tilde{A}_\mu(\mathbf{k}, 0) \\ \tilde{\dot{A}}_\mu(\mathbf{k}, 0) \end{pmatrix}$ satisfy

$$\sum_{\mu=1}^{4} [\|k(1 + k^2)\tilde{A}_\mu(\mathbf{k}, 0)\|_2 + \|(1 + k^2)\tilde{\dot{A}}_\mu(\mathbf{k}, 0)\|_2] < \infty. \tag{4.4}$$

Proof. The statement follows from the Schwartz inequality.

Theorem 4.1. With the assumptions of Lemma 4.1 the solution of equation (4.1) is unique and continuous over T_a.

Proof. The local existence and uniqueness is a direct consequence of the lipschitzian character of the interaction. It remains to prove that the norm remains bounded over T_a. The latter statement follows from

$$\begin{pmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{pmatrix} = \begin{pmatrix} \mathbb{1} & t\cdot\mathbb{1} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbf{z}(0) \\ \dot{\mathbf{z}}(0) \end{pmatrix} + \int_0^t \begin{pmatrix} \mathbb{1} & (t-s)\cdot\mathbb{1} \\ 0 & \mathbb{1} \end{pmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_h(s) - \dfrac{e^2}{3a^2c}\dot{\mathbf{z}}(s) \end{bmatrix} ds$$

by means of the Cronwall inequality. This completes the proof.

The study of this equation for $t \geqslant 2a/c$ will be the purpose of Section 6.

5. Existence and Uniqueness of the Solution of the Bohm–Weinstein Equation

We first investigate the linear problem ($\mathbf{F}_h = \mathbf{0}$) with an external mechanical force $\mathbf{f}(t)$. This formulation is not very physical but is an adequate starting point for the study of the non-linear problem. The equation of motion takes the form

$$m\ddot{\mathbf{z}}(t) = \frac{e^2}{3a^2c}\left[\dot{\mathbf{z}}\left(t - \frac{2a}{c}\right) - \dot{\mathbf{z}}(t)\right] + \mathbf{f}(t), \qquad t > \frac{2a}{c}. \tag{5.1}$$

Let us define

$$b = \frac{e^2}{3a^2c}, \quad \omega = \frac{2a}{c}, \quad \mathbf{u}(t) = \dot{\mathbf{z}}(t). \tag{5.2}$$

Then (5.1) takes the form

$$m\mathbf{u}(t) + b\mathbf{u}(t) - b\mathbf{u}(t-\omega) = \mathbf{f}(t), \qquad t > \frac{2a}{c} \tag{5.3}$$

$$\mathbf{u}(t) = \mathbf{g}(t), \qquad t \in [0, \omega]. \tag{5.4}$$

In view of the limit $\omega \to 0$ we assume that

$$\mathbf{f}(t) = \mathbf{0} \qquad \text{for } t \in [0, \alpha] \tag{5.5}$$

where α does not depend on ω. The existence and uniqueness of the solution of (5.3) follow from results of Bellmann and Cooke [5].

Theorem 5.1 [5, page 50]. The solution of (5.3), (5.4) is unique and continuous for $0 \leqslant t < \infty$ if $\mathbf{g}(t) \in C^0[0, \omega]$ and $\mathbf{f}(t) \in C^1[0, \infty)$. Further, if $\mathbf{g}(t) \in C^1[0, \omega]$, then the acceleration $\dot{\mathbf{u}}(t)$ is continuous for $t = \omega$ if and only if

$$m\dot{\mathbf{g}}(\omega - 0) + b\mathbf{g}(\omega) - b\mathbf{g}(0) = \mathbf{f}(\omega). \tag{5.6}$$

This means that in the case

$$\mathbf{g}(t) = \mathbf{v}_0, \quad \dot{\mathbf{u}}(t) \text{ is continuous at } t = \omega \text{ only if } \mathbf{f}(\omega) = \mathbf{0}. \tag{5.7}$$

Now let us represent the solution more explicitly in a similar way as the solutions of linear differential equations may be expressed by means of Green's functions. Let $k(t)$ be a real-valued function with the following properties:

$$k(t) = 0, \qquad t < 0, \tag{5.8a}$$

$$k(0) = \frac{1}{m}, \tag{5.8b}$$

$$k(t) \in C^0[0, \infty), \tag{5.8c}$$

$$k(t) \text{ satisfies } m\dot{k}(t) + bk(t) - bk(t - \omega) = 0 \qquad \text{for } t > 0. \tag{5.8d}$$

It is easy to see that this function exists and is unique. We formulate the following.

Theorem 5.2 [5, page 75]. If $\mathbf{u}(t)$ denotes the continuous solution of (5.3) and if

$\mathbf{g}(t) \in C^0[0, \omega]$,
$\mathbf{f}(t) \in C^1[0, \infty)$,

then

$$\mathbf{u}(t) = m\mathbf{g}(\omega)k(t - \omega) + b\int_0^\omega \mathbf{g}(t_1)k(t - t_1 - \omega)\,dt_1 + \int_\omega^t \mathbf{f}(t_1)k(t - t_1)\,dt_1 \qquad \text{for } t > \omega; \tag{5.9}$$

further, if $\mathbf{g}(t) \in C^1[0, \omega]$ then for $t > 0$

$$\mathbf{u}(t) = m\mathbf{g}(0)k(t) + \int_0^\omega [m\dot{\mathbf{g}}(t_1) + b\mathbf{g}(t_1)]k(t - t_1)\,dt_1 + \int_\omega^t \mathbf{f}(t_1)k(t - t_1)\,dt_1; \tag{5.10}$$

here $\dot{}$ denotes d/dt. As in the case of differential equations the solutions (5.9), (5.10) have the form

$$\mathbf{u}(t) = \mathbf{u}_0(t) + \int_\omega^t k(t - t_1)\mathbf{f}(t_1)\,dt_1 \tag{5.11}$$

where $\mathbf{u}_0(t)$ is the solution of the homogeneous equation ($\mathbf{f} = \mathbf{0}$).

Remark 5.1. If we put $\mathbf{g}(t) = \mathbf{v}_0$ for $t \in [0, \omega]$ we obtain $\mathbf{u}_0(t) = \mathbf{v}_0$ for $t \in [0, \infty)$; this is the free motion if $\mathbf{f} = \mathbf{0}$.

$k(t)$ can be determined by the following simple method: we solve equation (5.8d) for $[0, \omega]$, $[0, 2\omega], \ldots, [0, n\omega]$. We can easily find the recurrence formula

$$k(t) = \frac{1}{m}\sum_{l=0}^{N}\frac{1}{l!}\left[\frac{b}{m}(t - l\omega)\right]^l e^{-(b/m)(t - l\omega)} \tag{5.12}$$

for $N\omega \leqslant t \leqslant (N + 1)\omega$. $k(t)$ is $C^n(n\omega, \infty)$ as we immediately see from (5.12).

Let us verify the formula (5.12). We substitute $k(t)$ into (5.8d) and obtain

$$\sum_{l=1}^{N} \frac{1}{(l-1)!} \frac{b}{m} \left[\frac{b}{m}(t - l\omega)\right]^{l-1} e^{-(b/m)(t-l\omega)} - \sum_{l=0}^{N} \frac{1}{l!} \frac{b}{m} \left[\frac{b}{m}(t - l\omega)\right]^{l} e^{-(b/m)(t-l\omega)}$$

$$+ \frac{b}{m} \sum_{l=0}^{N} \frac{1}{l!} \left[\frac{b}{m}(t - l\omega)\right]^{l} e^{-(b/m)(t-l\omega)}$$

$$- \frac{b}{m} \sum_{l=0}^{N-1} \frac{1}{l!} \left[\frac{b}{m}(t - (l+1)\omega)\right]^{l} e^{-(b/m)(t-(l+1)\omega)};$$

(5.8d) follows if we put $(l - 1) = s$ in the first term. We can easily see that (5.8a, b, c) are also fulfilled.

Remark 5.2. If we put $t = \lambda\omega$ with $\lambda \in \mathbb{R}_+$, it results

$$k(t) = \frac{1}{m} \sum_{l=0}^{[\lambda]_-} \frac{1}{l!} \left[\frac{m_e}{m}(\lambda - l)\right]^{l} e^{-(m_e/m)\cdot(\lambda - l)}, \tag{5.13}$$

where $m = b\omega$ denotes the electromagnetic mass and $[\lambda]_-$ is the smallest integer $\leqslant (t/\omega)$.

We now study the asymptotic behaviour $t \to \infty$ (a fixed) of $k(t)$. We try to solve the problem with

$$k(t) = \sum_{\{\lambda_i\}} P_{\lambda_i}(t) e^{\lambda_i t}. \tag{5.14}$$

The characteristic equation is

$$h(s) = ms + b - be^{-\omega s}. \tag{5.15a}$$

We show that $s = 0$ is a simple zero and has the greatest real part. Let

$$s = \gamma + i\delta, \qquad \gamma, \delta \in \mathbb{R}.$$

Then $h(s) = 0$ implies

$$\gamma + i\delta = \frac{b}{m}(e^{-\omega(\gamma + i\delta)} - 1)$$

and

$$\frac{b}{m}(e^{-\omega\gamma} \cos \delta - 1) = \gamma,$$

$$-\frac{b}{m} e^{-\omega\gamma} \sin(\omega\delta) = \delta; \tag{5.15b}$$

since $e^{-\omega\gamma} \cos \delta - 1 \leqslant 0$ for $\gamma > 0$ the statement follows.

The determination of the P_{λ_i} is very difficult. In view of $\lim_{t\to\infty} k(t)$ we only need P_0. It results by Laplace transformation from equation (5.8) that

$$\int_0^\infty k(t) e^{-st}\, dt = h^{-1}(s), \qquad \mathrm{Re}(s) > 0. \tag{5.16}$$

The derivative $\dot{k}(t)$ possess only simple discontinuities and $k(t)$ is continuous and of bounded variation for $t > 0$. Then we may define the inverse Laplace transform

$$k(t) = \int_{(c)} h^{-1}(s)e^{st}\, ds, \qquad t > 0, \tag{5.17}$$

where (c) is a vertical contour in the right half-plane $\mathrm{Re}(s) > 0$. That is, $k(t)$ can be represented as a sum of residues. We split $k(t)$ according to

$$k(t) = c_1 e^{\lambda t} + k_1(t), \tag{5.18a}$$

where $\lambda = 0$ in our case. Then

$$c_1 e^{\lambda t} = c_1 = \frac{e^{\lambda t}}{\dot{h}(\lambda)}\bigg|_{\lambda=0} = \frac{1}{m + b\omega}. \tag{5.19}$$

It follows

$$k(t) = \frac{1}{m + m_e} + k_1(t) \tag{5.18b}$$

where $k_1(t)$ only contains exponentials with $\mathrm{Re}(\lambda_i) < 0$, hence $\lim_{t\to\infty} k_1(t) = 0$. From this result, the stability of the free motion follows by a theorem of Bellman and Cooke [5, page 115], taking into account that the zeros of (5.15a) satisfy $\mathrm{Re}(\lambda_i) \leqslant 0$ and that $\lambda_0 = 0$ is a simple zero. From the splitting (5.18a), it can be easily deduced that all solutions of (5.3) are uniformly bounded in $0 \leqslant t \leqslant \infty$ if $\mathbf{f}(t) \in L_1[0, \infty)$:

$$\|\mathbf{u}(t)\|_{\mathbb{R}^3} \leqslant \|\mathbf{u}_0\|_{\mathbb{R}^3} + \|k\|_{\mathbb{R}^3} \sup_{i=1,2,3} \int_0^\infty |f_i(t_1)|\, dt_1, \tag{5.20}$$

where $\|k\|_{\mathbb{R}^3}$ is the norm of the kernel $k(t)$. This shows explicitly that no run-away solutions exist.

Now let us investigate the limit $a \to 0$ to a point particle for the kernel $k(t)$. We consider $k(t)$ for $t = N\omega$; putting

$$\gamma = \frac{4e^2}{3c^3 mt}, \tag{5.21}$$

it follows

$$k_\omega(t) = \frac{1}{m}\sum_{l=0}^{N} \frac{1}{l!}\left[\gamma N^2\left(1 - \frac{l}{N}\right)\right]^l \cdot e^{-\gamma N^2(1-l/N)}, \qquad t > 0. \tag{5.22}$$

Theorem 5.3. If we define $k_\infty(t)$ to be

$$\lim_{N\to\infty} \frac{1}{m}\sum_{l=0}^{N} \frac{1}{l!}\left[\gamma N^2\left(1 - \frac{l}{N}\right)\right]^l e^{-\gamma N^2(1-l/N)}, \qquad t > 0$$

then

$$k_\infty(t) = \begin{cases} \dfrac{1}{m}, & t = 0 \\ 0, & t > 0. \end{cases}$$

Proof. Let

$$S_l(N) = \frac{1}{l!}\left[\gamma N^2\left(1 - \frac{l}{N}\right)\right]^l \cdot e^{-\gamma N^2(1-l/N)};$$

then $k_\omega(t) \leqslant N \max_{l\in[0,N-1]} S_l(N)$. In the ratio

$$\frac{S_{l+1}(N)}{S_l(N)} = \frac{\gamma N^2}{l}\left(1 - \frac{l+1}{N}\right)\left(1 - \frac{1}{N-l}\right)^l e^{\gamma N},$$

we put $l = N - k$ and obtain

$$\frac{S_{l+1}(N)}{S_l(N)} = \frac{\gamma N}{(N-k)}\cdot(k-1)\cdot\left(1 - \frac{1}{k}\right)^N\cdot\frac{e^{\gamma N}}{(1-1/k)^k}$$

$$= \frac{\gamma N(k-1)}{(N-k)(1-1/k)^k}\cdot\left[\left(\frac{k-1}{k}\right)\cdot e^\gamma\right]^N.$$

We note that for N large enough the sequence is increasing up to $l = N - k_0$ with $k_0 > e^\gamma/(e^\gamma - 1)$, $\gamma > 0$. Therefore we have $k_\omega(t) \leqslant NS_{N-k_0}(N)$. We compute $S_{N-k_0}(N)$ using the Stirling formula and obtain

$$S_{N-k_0}(N) = \left[\frac{e\gamma N k_0}{e^{\gamma k_0}(N-k_0)}\right]^N\cdot\left[\frac{N-k_0}{\gamma N k_0 l}\right]^{k_0}\cdot\frac{1}{2\pi(N-k_0)^{1/2}},$$

where $k_0 = [e^\gamma/(e^\gamma - 1)]_+$, $\gamma > 0$. For N large enough the terms with exponents N and k_0 are less than one and therefore the theorem follows.

The theorem (5.3) shows that the action of the external force $\mathbf{f}(t)$ is reduced to zero for $a \to 0$. The physical reason for this is the fact that the total mass $m_T = m + m_e$ becomes infinite, because $m_e = 2e^2/3ac^2 \to \infty$. The limit $a \to 0$ with m_T fixed will be considered in Section 7. We know further from (5.6) that the free motion $\mathbf{u}(t) = \mathbf{v}_0$, $t \in [0, \omega]$ is the only initial motion which gives a continuous acceleration at $t = \omega$ if no forces are present. This remains true, of course, in the limit $a \to 0$.

We now give some examples with different initial conditions. From (5.10) we have, with $\mathbf{f}(t) = \mathbf{0}$,

$$\mathbf{u}(t) = m\mathbf{g}(0)k(t) + \int_0^\omega [m\dot{\mathbf{g}}(t_1) + b\mathbf{g}(t_1)]k(t - t_1)\,dt_1 \tag{5.23}$$

if $g(t) \in C^1[0, \omega]$. Since we integrate only over $[0, \omega]$, the asymptotic behaviour of $\mathbf{u}(t)$ for large t is obtained by substitution of

$$k(t) = \frac{1}{m_e + m} = \frac{1}{m_T}.$$

Then we obtain

$$\mathbf{u}(t) = \frac{m}{m_T}\mathbf{g}(\omega) + \frac{b}{m_T}\int_0^\omega \mathbf{g}(t_1)\,dt_1 \tag{5.24}$$

Let us compute $\mathbf{u}(t)$ for some special initial conditions:

$$\begin{aligned}
&\text{(E1)}\quad g(t) = \mathbf{v}_0\cdot e^{-t}, \qquad t \in [0, \omega] \quad \text{gives} \quad \lim_{a\to 0}\mathbf{u}(t) = v_0.\\
&\text{(E2)}\quad g(t) = \mathbf{v}_0\cdot t, \qquad t \in [0, \omega] \quad \text{gives} \quad \lim_{a\to 0}\mathbf{u}(t) = 0.\\
&\text{(E3)}\quad g(t) = \mathbf{v}_0\cdot\sin\left(\pi\frac{c}{a}t\right), \qquad t \in [0, \omega] \quad \text{gives} \quad \mathbf{u}(t) = \mathbf{0},
\end{aligned} \tag{5.25}$$

a finite.

6. Motion of the Charged Shell in the Presence of a Field (non-linear problem)

We follow the same procedure as for the linear case. We write the equation (3.12) in the following vector form:

$$m\frac{d}{dt}\begin{bmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{bmatrix}\begin{bmatrix}\mathbf{z}(t)\\ \dot{\mathbf{z}}(t)\end{bmatrix} = \begin{bmatrix}0 & \mathbb{1}\\ 0 & \mathbb{1}\end{bmatrix}\begin{bmatrix}\mathbf{z}(t)\\ \dot{\mathbf{z}}(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & b\cdot\mathbb{1}\end{bmatrix}\begin{bmatrix}\mathbf{z}(t-\omega)\\ \dot{\mathbf{z}}(t-\omega)\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{F}_{\text{ext}}\end{bmatrix},$$

$$\begin{bmatrix}\mathbf{z}(t)\\ \dot{\mathbf{z}}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{g}_1(t)\\ \mathbf{g}_2(t)\end{bmatrix} \qquad \text{for } t \in [0, \omega]. \tag{6.1}$$

Here m is the bare mass and $\mathbf{F}_{\text{ext}}$ is defined according to (3.7) and (3.8); $\mathbb{1}$ is the 3×3 unit matrix.

We first study the characteristic equation corresponding to (6.1). Labelling the matrices occurring in (6.1) by A_0, B_0, B_1 (from the left to the right), the characteristic equation becomes

$$\det H(s) = \det \sum_{i=0}^{1} (A_i \cdot s + B_i)e^{-\omega_i \cdot s}, \tag{6.2}$$

where $\omega_0 = 0$, $\omega_1 = \omega$ and $A_1 = 0$. We write down explicitly $H(s)$ and $H^{-1}(s)$ because we will need these matrices later for the study of the asymptotic behaviour of the kernel.

$$H(s) = \begin{bmatrix} ms\cdot\mathbb{1} & \mathbb{1}\\ 0 & \mathbb{1}\cdot(ms + b - be^{-\omega s})\end{bmatrix}, \tag{6.3}$$

$$H^{-1}(s) = \begin{bmatrix} \dfrac{1}{ms}\cdot\mathbb{1} & \dfrac{1}{ms(ms + b - be^{-\omega s})}\cdot\mathbb{1}\\ 0 & \dfrac{1}{ms + b - be^{-\omega s}}\cdot\mathbb{1}\end{bmatrix}. \tag{6.4}$$

Let us now define the kernel $K(t)$ to be a 6×6 matrix with the following properties:

$$K(t) = 0, \qquad t < 0; \tag{6.5a}$$

$$K(0) = A_0^{-1}; \tag{6.5b}$$

$$\sum_{i=0}^{1} A_i K(t - \omega_i) \in C^0[0, \infty); \tag{6.5c}$$

$$\sum_{i=0}^{1} [A_i K(t - \omega_i) + B_i K(t - \omega_i)] = 0, \qquad \text{for } t > 0 \tag{6.5d}$$

and $t \notin S_+$ where $S_+ = E \cap (0, \infty)$; E is the set $\{t \mid t = \sum_{i=0}^{1} j_i\omega_i, j_i \in \mathbb{N}\}$. By Laplace transformation we obtain

$$K(t) = \int_{(c)} e^{ts} H^{-1}(s)\, ds \qquad \text{for } t > 0, t \notin S_+, \tag{6.6}$$

and the inverse formula

$$H^{-1}(s) = \int_0^{\infty} K(t)e^{-st}\, dt, \qquad \operatorname{Re}(s) > 0. \tag{6.7}$$

The equations (6.6) and (6.7) are useful in order to find the asymptotic behaviour of $K(t)$.

We now compute $K(t)$; the equation (6.5d) gives

$$m\begin{bmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{bmatrix}\begin{bmatrix}\dot{k}_1(t) & \dot{k}_2(t)\\ \dot{k}_3(t) & \dot{k}_4(t)\end{bmatrix}+\begin{bmatrix}0 & \mathbb{1}\\ 0 & b\cdot\mathbb{1}\end{bmatrix}\begin{bmatrix}k_1(t) & k_2(t)\\ k_3(t) & k_4(t)\end{bmatrix}$$

$$+\begin{bmatrix}0 & 0\\ 0 & -b\cdot\mathbb{1}\end{bmatrix}\begin{bmatrix}k_1(t-\omega) & k_2(t-\omega)\\ k_3(t-\omega) & k_4(t-\omega)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix};$$

here $k_i(t)$ $(i = 1, 2, 3, 4)$ are 3×3 matrices. Using conditions (6.5a, b, c) we obtain the kernel as follows:

$$K(0) = A_0^{-1} \quad \text{gives} \quad (k_3)_{ij} = 0, \qquad i, j = 1, 2, 3;$$

$$m(\dot{k}_1)_{ij} = 0 \quad \text{gives} \quad (k_2)_{ij} = -\frac{1}{m}\int_0^t [k_4(t_1)]_{ij}\, dt_1,$$

where $[k_4(t)]_{ij} = \delta_{ij} k_s(t)$ and $k_s(t)$ is the scalar kernel for the linear case calculated in the previous section. Therefore $K(t)$ takes the form

$$K(t) = \begin{bmatrix}\frac{1}{m}\cdot\mathbb{1} & \left(-\frac{1}{m}\cdot\int_0^t k_s(t_1)\, dt_1\right)\cdot\mathbb{1}\\ 0 & \mathbb{1}\cdot k_s(t)\end{bmatrix}. \tag{6.8}$$

Let us consider the asymptotic behaviour of $K(t)$. Taking into account (6.6) we can derive an asymptotic formula for $-(1/m)\cdot\int_0^t k_s(t_1)\, dt_1$ by computing the residues of

$$\frac{-e^{\lambda t}}{m\lambda(m\lambda + b - be^{-\omega\lambda})} \qquad \text{at } \lambda = 0.$$

We obtain

$$\text{Res}\left[\frac{-e^{\lambda t}}{m\lambda(m\lambda + b - be^{-\omega\lambda})}\bigg|_{\lambda=0}\right] = \frac{-t}{m(m + b\omega)} - \frac{b\omega^2}{m(m + b\omega)^2}$$

and therefore

$$K(t) = \begin{bmatrix}\frac{1}{m}\cdot\mathbb{1} & -\frac{1}{m}\left(\frac{t}{m + b\omega} - \frac{b\omega^2}{(m + b\omega)^2}\right)\cdot\mathbb{1}\\ 0 & \frac{1}{m + b\omega}\cdot\mathbb{1}\end{bmatrix} + K_1(t), \tag{6.9}$$

for large t, where $K_1(t)$ decreases exponentially to zero.

We will now study the non-linear problem, assuming $\mathbf{f}(\mathbf{z}, \dot{\mathbf{z}}, t) = \mathbf{0}$ in (3.7).

Theorem 6.1. With the assumptions of Lemma (4.1), the solution of (6.1) exists and is unique and continuous for all t; it can be written in the form

$$\mathbf{W}(t) \overset{\text{def}}{=} \begin{pmatrix}\mathbf{z}(t)\\ \dot{\mathbf{z}}(t)\end{pmatrix} = \mathbf{W}_0(t) + \int_\omega^t K(t - t_1)\mathbf{F}_h(\mathbf{W}(t_1), t_1)\, dt_1, \tag{6.10}$$

where $\mathbf{W}_0(t)$ is a solution of the homogeneous equation $(\mathbf{F}_h = \mathbf{0})$.

Proof. This follows from a slight generalisation of a theorem of Dini-Hukuhara [5, page 361], using the fact that the velocity is bounded for all t if $\mathbf{F}_h = 0$ and that the L^∞-norm of the fields decreases like const$/t^{3/2}$ for $t \to \infty$. Note that the kernel is explicitly known.

Remark. In the case $\dot{\mathbf{z}}(t) = \mathbf{v}_0$ for $t \in [0, \omega]$, α may be chosen so that $\mathbf{F}_h = \mathbf{0}$ for $t < \alpha$. Then, from (5.12), we may get additional smoothness of the solution, for example, if $\alpha = n\omega$ then $\dot{\mathbf{z}}(t) \in C^n[0, \infty)$.

7. Discussion

In order to discuss the physical properties of the solutions obtained, we first compute the radiation rate dE_s/dt where E_s is the energy of radiation. This may be accomplished by multiplying the Bohm–Weinstein equation (3.12) by the velocity $\mathbf{v}(t) = \dot{\mathbf{z}}(t)$. We get the following equation of energy conservation:

$$\frac{dE_s}{dt} = \frac{d}{dt}(\tfrac{1}{2}m\mathbf{v}^2) - \mathbf{v}(t)\cdot\mathbf{F}_h(\mathbf{z}(t), \dot{\mathbf{z}}(t), t), \tag{7.1}$$

where

$$\frac{dE_s}{dt} = \frac{e^2}{3a^2c}\mathbf{v}(t)\left[\mathbf{v}\left(t - \frac{2a}{c}\right) - \mathbf{v}(t)\right]. \tag{7.2}$$

Let us define

$$\mathbf{T} = \left[\mathbf{v}\left(t - \frac{2a}{c}\right) - \mathbf{v}(t)\right]. \tag{7.3}$$

We first consider the non-relativistic hyperbolic motion

$$\dot{\mathbf{z}}(t) = \frac{\mathbf{F}}{m_T}t, \tag{7.4}$$

which is a solution of (3.12) if $\mathbf{F}_{\text{ext}} = \mathbf{F}$ is a constant. It is easy to see that the uniformly accelerated particle radiates because $\mathbf{T} \neq \mathbf{0}$.

We now compute $\mathbf{v}\cdot\mathbf{T}$ by a Taylor formula

$$\mathbf{v}\cdot\mathbf{T} = -\frac{2e^2}{3ac^2}\mathbf{v}(t)\cdot\dot{\mathbf{v}}(t) + \frac{2e^2}{3c^3}\mathbf{v}(t)\cdot\ddot{\mathbf{v}}(t)$$
$$-\frac{4}{9}\cdot\frac{ae^2}{c^4}\mathbf{v}(t)\cdot\dddot{\mathbf{v}}(t) + \mathbf{R}_a(t);$$

here

$$\mathbf{R}_a(t) = \text{const}\cdot\mathbf{v}^{(\text{IV})}\left(t - \vartheta\frac{2a}{c}\right)\cdot\left(\frac{2a}{c}\right)^2, \quad 0 < \vartheta < 1. \tag{7.5}$$

$\mathbf{R}_a(t)$ is uniformly bounded in t if $\mathbf{v}(t) \in C^4$. The latter condition can be fulfilled by choosing suitable initial conditions.

We integrate equation (7.3) over $0 \leqslant t \leqslant \infty$, using the fact that $\lim_{t\to\infty} \mathbf{v}(t) = \mathbf{v}(\infty)$ exists. (If the motion is periodic due to the presence of an external force $\mathbf{f}(t)$, we integrate over a period.) Then

$$\frac{m}{2}(\mathbf{v}^2(\infty) - \mathbf{v}^2(0)) = \int_0^\infty \mathbf{v}(t)\cdot\mathbf{F}_h\,dt - \frac{2e^2}{3ac^2}\int_0^\infty \mathbf{v}\cdot\dot{\mathbf{v}}\,dt + \frac{2e^2}{3c^3}\int_0^\infty \mathbf{v}.\ddot{\mathbf{v}}\,dt - \frac{4}{9}\cdot\frac{e^2}{c^4}\cdot a\int_0^\infty \mathbf{v}\cdot\dddot{\mathbf{v}}\,dt + \int_0^\infty \mathbf{v}\cdot\mathbf{R}_a\,dt. \tag{7.6}$$

Reminding that $\dot{\mathbf{v}}(\infty) = \ddot{\mathbf{v}}(\infty) = \mathbf{0}$ and choosing the initial time $t = 0$ so that $\dot{\mathbf{v}}(0) = \ddot{\mathbf{v}}(0) = \mathbf{0}$, we obtain by partial integration

$$m_T(\mathbf{v}^2(\infty) - \mathbf{v}^2(0)) = \int_0^\infty \mathbf{F}_h\cdot\mathbf{v}\,dt - \frac{2e^2}{3c^3}\int_0^\infty \dot{\mathbf{v}}^2\,dt + \int_0^\infty \mathbf{v}.\mathbf{R}_a\,dt. \tag{7.7}$$

The limit $a \to 0$ with m fixed to a point particle causes no difficulties here. But this limit is not very interesting since all terms in (7.7) go to zero because of theorem (5.3).

Let us now consider the limit $a \to 0$ in the linear Bohm–Weinstein equation itself. The exact equation of motion for the shell is

$$m\dot{\mathbf{u}}(t) = \frac{e^2}{3a^2c}\left[\mathbf{u}\left(t - \frac{2a}{c}\right) - \mathbf{u}(t)\right] + \mathbf{f}(t) \tag{7.8}$$

and we take the free initial condition

$$\mathbf{u}(t) = \mathbf{v}_0, \qquad 0 \leqslant t \leqslant \frac{2a}{c}. \tag{7.9a}$$

We make a Taylor expansion in (7.8) which has only a formal meaning for the moment:

$$m\dot{\mathbf{u}}(t) = -\frac{2e^2}{3ac^2}\dot{\mathbf{u}}(t) + \frac{2e^2}{3c^3}\ddot{\mathbf{u}}(t) + \mathbf{O}(a) + \mathbf{f}(t), \tag{7.10a}$$

$$\mathbf{u}(0) = \mathbf{v}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{0}. \tag{7.9b}$$

Equation (7.10a) is exactly the non-relativistic Lorentz–Dirac equation if we neglect the error term $\mathbf{O}(a)$:

$$m_T\dot{\mathbf{u}}(t) = \frac{2e^2}{3c^3}\ddot{\mathbf{u}}(t) + \mathbf{f}(t), \tag{7.10b}$$

with

$$m_T = m + m_e. \tag{7.11}$$

Since $m_e = 2e^2/3ac^2$ diverges for $a \to 0$,the limit is only meaningful if we let at the same time either (i) $e \to 0$ or (ii) $m \to -\infty$ with m_T fixed (mass renormalisation). The first possibility must be excluded because the particle no longer radiates. There remains the limit (ii) which we will call the Lorentz–Dirac limit.

The general solution of (7.10) with $\mathbf{f}(t) = \mathbf{0}$ is

$$\mathbf{u}(t) = \mathbf{A} + \mathbf{B}e^{m_T t/q} \tag{7.12}$$

where

$$q = \frac{2e^2}{3c^3}. \tag{7.13}$$

Then the free motion $\mathbf{u}(t) = \mathbf{v}_0$ is the unique solution with the initial condition (7.9b); run-away solutions do not occur in this case. However, the condition (7.9b) is not sufficient to assure a bounded solution to (7.10a) if $\mathbf{f}(t) = \mathbf{0}$, because the solution $\dot{\mathbf{u}}(t)$ then reads

$$\dot{\mathbf{u}}(t) = \dot{\mathbf{u}}_0 + \int_0^t e^{m_T(t-s)/q}\mathbf{f}(s)\,ds \tag{7.14}$$

which is not bounded for $t \to \infty$. Further it was proved by Daboul [6] that runaway solutions persist if we add a finite number of terms which are linear in higher derivatives of $\ddot{\mathbf{z}}$ in the case of the Lorentz–Dirac equation.

In order to control the Lorentz–Dirac limit $a \to 0$, $m \to -\infty$, m_T fixed, it must be performed in the solution (5.11) and not in the Bohm–Weinstein equation itself. We use (5.14) for the kernel $k(t)$. The solutions of the characteristic equation (5.15) are

$$s_0 = 0 \quad \text{and} \quad s_1 = \frac{m_T}{q} \tag{7.15}$$

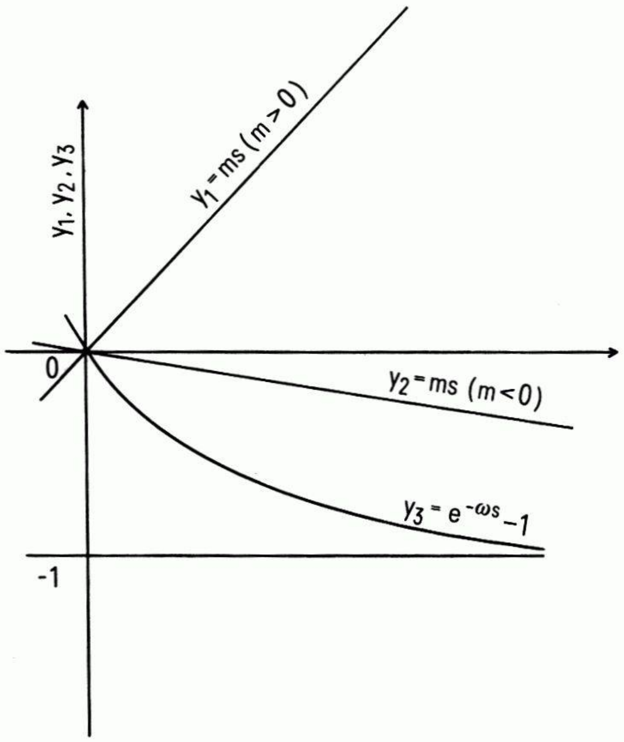
in the Lorentz–Dirac limit. Then the kernel approaches the limit

$$k_D(t) = a + be^{m_T t/q} \tag{7.16}$$

where

$$a = \frac{3c^3}{4e^2}, \quad a + b = 0, \quad t \text{ fixed}, \tag{7.17}$$

which is precisely the solution of (7.12) of the Lorentz–Dirac equation. The formal limit in the Bohm–Weinstein equation is therefore justified. Now we are able to see



clearly the origin of the run-away solutions: the positive characteristic exponent s_1, (7.15), is due to the fact that the bare mass m becomes negative in the Lorentz–Dirac limit. Indeed, a graphical discussion of (5.15) shows that, as soon as m becomes negative, a second positive real root appears, coming down from $+\infty$. (In fact, infinitely many roots with positive real part.) Run-away solutions exist already for *finite* a, if $m < 0$, which implies $a < 2e^2/3m_Tc^2 = r_e$; r_e is the classical electron radius. Hence, the reason for the run-away solutions is the unphysical negative bare mass due to the mass renormalisation and not the limit to the point particle.

We arrive at the conclusion that none of the limits to a point particle gives a physically meaningful result. The classical electron radius r_e is the lower bound for the extension of the shell without getting into troubles with run-away solutions. On the other hand, a reasonable limit $a \to 0$ would be very welcome for the purpose of a relativistic theory. Since there is no hope in this direction, one has to keep a finite and look for a relativistic theory with charges of finite extension.

One attempt in this direction was made by Nodvik [4] which reduce to our theory in the non-relativistic limit. He finds a covariant equation of motion for the rigid charge distribution by using an action principle. But this equation does not lead to a well-posed Cauchy problem for the coupled system particle-field as we will see in a moment.

Let us therefore discuss the question of causality for the various equations. The equation of Bohm–Weinstein is non-relativistic and in this case we can postulate that impulses may be transmitted faster than light. Then a rigid charge distribution does not contradict causality. The elimination of the fields has introduced a non-local feature in t but the equation remains causal. It is only when we make a Lorentz transformation in the mechanical coordinates that the transmission of impulses faster than light leads to a breakdown of causality, since events which are connected by such impulses may have future and past interchanged as a result of the transformation. That is, the attempts to make a covariant theory of a rigid charge distribution introduce an element of genuine lack of causality, which is not due to the elimination of the fields [3].

Let us show this explicitly in the equation of Nodvik. Neglecting the Thomas precession and the rotations, the translational motion is described by

$$m \frac{d}{ds} v_\mu(s) = \int_{-\infty}^{\infty} d^4x F_{\mu\nu}(x) v_\nu \rho((x - z)^2) \delta[v_\lambda(x_\lambda - z_\lambda)]; \tag{7.18}$$

here $F_{\mu\nu}$ denotes the fields, v_ν ($\nu = 1, 2, 3, 4$) the velocity as a function of the proper time and $(x - z)^2$ means $(x_\lambda - z_\lambda)(x_\lambda - z_\lambda)$. We easily see that the formulation of a Cauchy problem for $t = 0$ requires the fields for time greater than zero. The acausality is of the order of the size of the charge distribution.

A relativistic theory without a rigid charge distribution was given by Sorg [7]. Here the radius of the charge distribution is defined by means of light signals. But some special assumptions are made such that the theory is no longer in the framework of Maxwell's theory.

Added in proof:

A partial result concerning the existence and uniqueness of the one-dimensional motion of a spherical shell has been given long ago by P. Hertz (Gött. Nach. 67, 1903). I am indebted to Prof. R. Jost for bringing this fact to my attention.

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