Local Gauge Models Predicting their own Superselection Rules[1])

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Abstract. By considering local SO(2) or U(1) transformations of a two-component Boson field in two-dimensional space-time, we construct certain non-vacuum representations (now called solitons) of the canonical commutation relations. Superselection rules operate between the spaces of these representations.

I. Introduction

If $\mathfrak{A}$ is the C^*-algebra of observables of the theory, then according to Haag and Kastler [1], states of different 'charge' are assumed to belong to different, i.e. inequivalent, representations of $\mathfrak{A}$. Thus, the charge label appears as a 'Casimir operator' for $\mathfrak{A}$, being diagonal, and constant, on each irreducible part:

$$\mathscr{H} = \bigoplus_q \mathscr{H}_q \qquad \pi(\mathfrak{A}) = \bigoplus_q \pi_q(\mathfrak{A})$$

where

$$\pi(A) = \begin{pmatrix} \pi_1(A) & & & & \\ & \pi_2(A) & & & \\ & & \vdots & & \\ & & & \pi_q(A) & \\ & & & & \vdots \end{pmatrix}$$

Here, $\mathscr{H}$ is the Hilbert space of the whole theory, and $\mathscr{H}_q$ is the space of states of charge q, carrying the representation π_q of $\mathfrak{A}$. If we assume that all the π_q are irreducible,

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and inequivalent, then it can be proved that $\pi(\mathfrak{A})'$ (the set of bounded operators on $\mathscr{H}$ commuting with $\pi(\mathfrak{A})$) is an abelian von Neumann algebra. This, then, is the origin of Wightman's hypothesis of commutative superselection rules.

Haag et al [2] hope to obtain all the $\mathscr{H}_q$ and π_q from the vacuum sector, π_0 acting on $\mathscr{H}_0$, by 'localized automorphisms', (or morphisms if we wish to allow parastatistics). The idea is that if τ is an automorphism of $\mathfrak{A}$ that is *not* unitarily implemented in the representation π_0, and if $\psi \in \mathscr{H}_0$, then we obtain a new state (in the sense of expectation value) by

$$\rho(A) = \langle \psi, \pi_0(\tau(A))\psi \rangle$$

This state is not a vector state in $\mathscr{H}_0$, but lies in some other representation of $\mathfrak{A}$, namely, the representation π_τ defined by $\pi_\tau(A) = \pi_0(\tau(A))$.

Just as Hamilton's equations in classical mechanics are covariant under canonical changes of generalized coordinates, so we would hope that quantum field theory is covariant under the quantum analogue of changes in generalized coordinates, namely, automorphisms of $\mathfrak{A}$. However, the localization in space of an observable has an intrinsic meaning, independent of which fields are used to describe the system. For this reason we only consider automorphisms that preserve the localization. Thus, we consider automorphisms τ such that

1. τ is an automorphism of each $\mathfrak{A}(0)$, the algebra of observables of the open set 0 of space-time.
2. τ is not implementable.
3. The new representation $\pi_\tau(A) = \pi_0(\tau(A))$ on $\mathscr{H}_\tau$ should be covariant; that is the Poincaré automorphisms $L = (\Lambda, a)$ should be implemented by unitary operators $U_\tau(\Lambda, a)$ on the Hilbert space $\mathscr{H}_\tau$, and the energy-momentum spectrum must lie in the forward cone.
4. τ should reduce to the identity on local observables localized outside some compact set.

For models satisfying these axioms, see [3]. By making use of local gauge transformations of the groups SO(2) or U(1), we shall construct models satisfying 1, 2 and 4, and presumably 3 too.

II. The U(1) and SO(2) Gauge Theories in Two Dimensions

Let $\mathscr{h} = \mathscr{h}^{(1)} \oplus \mathscr{h}^{(2)}$ carry two copies of the representation $[m, 0]$ of $\mathscr{P}$ in 2 dimensions. Let $\mathscr{H}$ be the Fock space over $\mathscr{h}$, and $a_1^*(p)$, $a_2^*(p)$ the creation operators in the Fock representation. The definition of a_1^* (and a_2^*) defines and is defined by a distinguished *conjugation* (on $\mathscr{h}^{(1)}$ and $\mathscr{h}^{(2)}$ respectively); namely, to each vector $f \in \mathscr{h}^{(1)}$, Cf is defined by the requirement that $(a^*(f))^* = a(Cf)$, where $a^*(f)$ is the smeared operator $\int \tilde{f}(p) a^*(p)\, dp$. When we write $\mathscr{h}^{(1)} = L_2(\mathbb{R}, dp)$, $\widetilde{Cf}(p)$ means $\bar{\tilde{f}}(-p)$, so 'real' f are real in x-space. In this way, $\mathscr{h}$ is the direct sum of four real L_2 spaces.

There are two distinct localizations of this theory, the charged field and the SO(2) doublet. These define different Borchers classes ([4]); they have gauge groups called U(1) and SO(2) respectively.

1. *The charged field*

$$\phi(x) = (2\pi)^{-1/2} \int \frac{a_1^*(p)e^{-ip\cdot x} + a_2(p)e^{ip\cdot x}}{\sqrt{2\omega}}\, dp.$$

2. *The SO(2) doublet*

$$\phi_1(x) = (2\pi)^{-1/2} \int \frac{a_1^*(p)e^{-ip\cdot x} + a_1(p)e^{ip\cdot x}}{\sqrt{2\omega}}\, dp$$

$$\phi_2(x) = (2\pi)^{-1/2} \int \frac{a_2^*(p)e^{-ip\cdot x} + a_2(p)e^{ip\cdot x}}{\sqrt{2\omega}}\, dp.$$

We subject ϕ to local U(1) gauge transformations ($x \in \mathbb{R}$)

$$\phi'(x, 0) = e^{i\theta(x)}\phi(x, 0); \qquad \pi'(x, 0) = e^{i\theta(x)}\pi(x, 0)$$
$$\phi'^*(x, 0) = e^{-i\theta(x)}\phi^*(x, 0); \qquad \pi'^*(x, 0) = e^{-i\theta(x)}\pi(x, 0)$$

where $\pi = \dot{\phi}$; and we subject ϕ_1 and ϕ_2 to local SO(2) rotations:

$$\phi_1'(x, 0) = \cos\theta(x)\phi_1(x, 0) + \sin\theta(x)\phi_2(x, 0)$$
$$\phi_2'(x, 0) = -\sin\theta(x)\phi_1(x, 0) + \cos\theta(x)\phi_2(x, 0)$$

and the same for π's. These are canonical transformations, and can be extended to the c^*-algebra generated by the field. To get the charged sectors, we must choose the transformations so that they are *not* implemented in the vacuum representation.

III. The Shale Criterion [5]

Let $\hbar$ be a complex Hilbert space, with a complex conjugation C so that $\hbar = H + iH$, where H consists of real vectors. Let $T: \hbar \to \hbar$ be a real symplectic transformation; then there is a unique C^*-automorphism τ of the C^*-algebra generated by the Weyl operators $W(f)$, such that $\tau(W(f)) = W(Tf)$. This τ is implemented in the Fock representation if and only if $T^tT - I$ is a (real) Hilbert–Schmidt operator on $\hbar$. See [6] for a generalization to Weyl systems based on smaller classes of test-functions, e.g. $\mathscr{S}$.

In our models, we can explicitly compute $I - T^tT$, and so check the implementability of τ. Let γ be the operator $\gamma: \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ defined by $\widetilde{\gamma f}(p) = \sqrt{p^2 + m^2}\tilde{f}(p)$, and let s and c be the operators $sf(x) = \sin\theta(x)\cdot f(x)$, $cf(x) = \cos\theta(x)\cdot f(x)$. Then we find T in the two cases:

Case 1, with U(1) symmetry.

$$T_1 = \tfrac{1}{2}\gamma^{-1}\begin{pmatrix} c & s & c & -s \\ -s & c & -s & -c \\ c & s & c & -s \\ s & -c & s & c \end{pmatrix}\gamma + \tfrac{1}{2}\gamma\begin{pmatrix} c & s & -c & s \\ -s & c & s & c \\ -c & -s & c & -s \\ -s & c & s & c \end{pmatrix}\gamma^{-1}.$$

Case 2, with SO(2) symmetry.

$$T_2 = \gamma^{-1}\begin{pmatrix} c & 0 & s & 0 \\ 0 & 0 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\gamma + \gamma\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c & 0 & s \\ 0 & 0 & 0 & 0 \\ 0 & -s & 0 & c \end{pmatrix}\gamma^{-1}.$$

In each case, the 4×4 matrix acts on the direct sum

$$\mathscr{h} = \mathscr{h}_r^{(1)} \oplus i\mathscr{h}_r^{(1)} \oplus \mathscr{h}_r^{(2)} + i\mathscr{h}_r^{(2)},$$

where $\mathscr{h}_r = L_2(\mathbb{R}, dx)$ (r standing for real-valued functions). To prove these forms for T, consider the Fock representation of the CCR over $L_2^{\mathbb{C}}(\mathbb{R}, dx)$, relative to the conjugation $Cf = \bar{f}$. Let $q(x)$, $p(x)$ be the canonical fields, and let $a^*(x) = (1/\sqrt{2})(q + ip)$ be the creation operator. Then $a\psi_0 = 0$, where ψ_0 is the Fock vacuum. We see that the charged field, ϕ, is given by

$$\phi = \gamma^{-1}\left(\frac{q_1 + ip_1}{\sqrt{2}} + \frac{q_2 - ip_2}{\sqrt{2}}\right)$$

$$\pi = \frac{\gamma}{i}\left(\frac{q_1 + ip_1}{\sqrt{2}} - \frac{q_2 - ip_2}{\sqrt{2}}\right).$$

Thus,

$$\operatorname{Re}\phi = \gamma^{-1}\frac{q_1 + q_2}{\sqrt{2}}, \qquad \operatorname{Im}\phi = \gamma^{-1}\frac{p_1 - p_2}{\sqrt{2}}$$

$$\operatorname{Re}\pi = \gamma\frac{p_1 + p_2}{\sqrt{2}}, \qquad \operatorname{Im}\pi = \gamma\frac{-q_1 + q_2}{\sqrt{2}}.$$

Hence

$$\begin{pmatrix} \operatorname{Re}\phi \\ \operatorname{Im}\phi \\ \operatorname{Re}\pi \\ \operatorname{Im}\pi \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \gamma^{-1} & 0 & \gamma^{-1} & 0 \\ 0 & \gamma^{-1} & 0 & -\gamma^{-1} \\ 0 & \gamma & 0 & \gamma \\ -\gamma & 0 & \gamma & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix} = \Gamma\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}.$$

We find Γ^{-1} by solving for q_1, q_2, p_1, p_2:

$$\Gamma^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \gamma & 0 & 0 & -\gamma^{-1} \\ 0 & \gamma & \gamma^{-1} & 0 \\ \gamma & 0 & 0 & \gamma^{-1} \\ 0 & -\gamma & \gamma^{-1} & 0 \end{pmatrix}.$$

Thus the transformation on $\mathscr{h}$, in this basis, is

$$\begin{pmatrix} q_1' \\ p_1' \\ q_2' \\ p_2' \end{pmatrix} = \Gamma^{-1}\begin{pmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{pmatrix}\Gamma = T_1.$$

Actually, the action on the test-functions is got by replacing θ by $-\theta$ in the above T. This does not affect the implementability of τ. The formula for T_2 is derived quickly from

$$\phi_1 = \gamma q_1, \qquad \phi_2 = \gamma q_2, \qquad \pi_1 = \gamma^{-1} p_1, \qquad \pi_2 = \gamma^{-1} p_2.$$

Thus, $I - T^t T$ is a 4×4 matrix of pseudo-differential operators; it is of Hilbert–Schmidt class if and only if each entry is. The cross terms in $T^t T$ cancel, leaving

$$A = T_1^t T_1 = \tfrac{1}{4}\left\{\gamma^{-1}\begin{pmatrix} c & -s & c & s \\ s & c & s & -c \\ c & -s & c & s \\ -s & -c & -s & c \end{pmatrix}\gamma^2\begin{pmatrix} c & s & c & -s \\ -s & c & -s & -c \\ c & s & c & -s \\ s & -c & s & c \end{pmatrix}\gamma^{-1}\right.$$

$$\left. +\gamma\begin{pmatrix} c & -s & -c & -s \\ s & c & -s & c \\ -c & s & c & s \\ s & c & -s & c \end{pmatrix}\gamma^{-2}\begin{pmatrix} c & s & -c & s \\ -s & c & s & c \\ -c & -s & c & -s \\ -s & c & s & c \end{pmatrix}\gamma\right\}$$

$$B = T_2^t T_2 = \gamma\begin{pmatrix} c & 0 & -s & 0 \\ 0 & 0 & 0 & 0 \\ s & 0 & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\gamma^{-2}\begin{pmatrix} c & 0 & s & 0 \\ 0 & 0 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\gamma$$

$$+\gamma^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c & 0 & -s \\ 0 & 0 & 0 & 0 \\ 0 & s & 0 & c \end{pmatrix}\gamma^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c & 0 & s \\ 0 & 0 & 0 & 0 \\ 0 & -s & 0 & c \end{pmatrix}\gamma^{-1}.$$

We note that if γ commuted with c and s, these would be I, i.e. SO(2) rotations applied locally to $p_j(x)$, $q_j(x)$ are orthogonal and therefore implementable. In the relativistic localization, γ does not commute with c and s. We list the diagonal and upper components of A and B.

$$\begin{aligned}
B_{11} &= B_{33} = \gamma c\gamma^{-2}c\gamma + \gamma s\gamma^{-2}s\gamma \\
B_{22} &= B_{44} = \gamma^{-1}c\gamma^{2}c\gamma^{-1} + \gamma^{-1}s\gamma^{2}s\gamma^{-1} \\
B_{13} &= \gamma c\gamma^{-2}s\gamma - ys\gamma^{-2}c\gamma \\
B_{24} &= \gamma^{-1}c\gamma^{2}s\gamma^{-1} - \gamma^{-1}s\gamma^{2}c\gamma^{-1} \\
B_{12} &= B_{14} = B_{23} = B_{34} = 0 \\
2A_{ii} &= B_{11} + B_{22} \qquad i = 1, 2, 3, 4 \\
2A_{13} &= -2A_{24} = -B_{11} + B_{22} \\
2A_{12} &= -2A_{34} = B_{13} + B_{24} \\
2A_{14} &= 2A_{23} = B_{13} - B_{24}.
\end{aligned}$$

Hence a necessary and sufficient condition for the implementability (in either U(1) or SO(2)) is that each of $1 - B_{11}$, $1 - B_{22}$, B_{13} and B_{24} should be a Hilbert–Schmidt operator on $L^2(\mathbb{R}, dx)$.

Theorem. Suppose $m > 0$. Suppose $\theta(x)$ is such that $\sin\theta$ and $1 - \cos\theta \in \mathscr{S}$. Then τ is implementable.

Proof. Let $\cos\theta = 1 - t$; then $t \in \mathscr{S}$ and

$$\begin{aligned} B_{13} &= \gamma(1 - t)\gamma^{-2}s\gamma - \gamma s\gamma^{-2}(1 - t)\gamma \\ &= [\gamma^{-1}s\gamma - \gamma s\gamma^{-1}] + [\gamma s\gamma^{-2}t\gamma - \gamma t\gamma^{-2}s\gamma]. \end{aligned}$$

We show that each bracket is H.S. The kernel of $\gamma^{-1}s\gamma - \gamma s\gamma^{-1}$ in momentum space is

$$\begin{aligned} K(p_1, p_2) = {} & (p_1^2 + m^2)^{-1/4}\tilde{s}(p_1 - p_2)(p_2^2 + m^2)^{1/4} \\ & - (p_1^2 + m^2)^{1/4}\tilde{s}(p_1 - p_2)(p_2^2 + m^2)^{-1/4}. \end{aligned}$$

Let $p = (p_1 - p_2)/2$, $q = (p_1 + p_2)/2$. As a function of p, $K \in \mathscr{S}$ and so is in L_2. It remains to show K is L_2 as a function of q. Indeed, expanding by the binomical theorem gives

$$K(p_1, p_2) = -2\tilde{s}(2p)(p/q + 0(1/q^2)), \qquad q \to \infty.$$

This is in L^2. The kernel of $\gamma s\gamma^{-2}t\gamma - \gamma t\gamma^{-2}s\gamma$ is

$$\begin{aligned} K(p_1, p_2) = {} & (p_1^2 + m^2)^{1/4}\int dk[\tilde{s}(p_1 - k)\tilde{t}(k - p_2) - \tilde{t}(p_1 - k)\tilde{s}(k - p_2)] \\ & \times (k^2 + m^2)^{-1/2}(p_2^2 + m^2)^{1/4}. \end{aligned}$$

Again, if $\tilde{s}$, $\tilde{t} \in \mathscr{S}$ the convolution goes to zero rapidly if $p = (p_1 - p_2)/2$ goes to infinity, $q = (p_1 + p_2)/2$ being fixed. K is therefore square-integrable in this variable, and it suffices to consider q. We see that

$$(p_{1,2}^2 + m^2)^{1/4} = (p^2 \pm 2pq + q^2 + m^2)^{1/4} = 0(q^{1/2}), \qquad p \to \infty.$$

By changing the variable of integration to $k - q = k'$ in the first part and $k' = q - k$ in the second, we get

$$\begin{aligned} K(p_1, p_2) &= 0(q^{1/2})\int dk'\tilde{s}(p - k')\tilde{t}(k' + p) \\ & \quad \times \{(k'^2 + q^2 + 2k'q + m^2)^{-1/2}(k'^2 + q^2 - 2k'q + m^2)^{-1/2}\}0(q^{1/2}) \\ &= 0(q)\int dk\tilde{s}(p - k)\tilde{t}(k + p)q^{-1}\left(-\frac{2k}{q} + 0\left(\frac{1}{q^2}\right)\right) \\ &= 0\left(\frac{1}{q}\right) \in L_2. \end{aligned}$$

Therefore B_{13} is of Hilbert–Schmidt class. The operator B_{24} is proved to be H.S. in a similar way.

To deal with B_{22} and B_{33}, we write $1 = c^2 + s^2$ and $c = 1 - t$. Then

$$1 - B_{22} = (1 - t)^2 + s^2 - \gamma^{-1}(1 - t)\gamma^2(1 - t)\gamma^{-1} - \gamma^{-1}s\gamma^2 s\gamma^{-1}.$$

The operators $t - \gamma^{-1}t\gamma$, $t - \gamma t\gamma^{-1}$, $t^2 - \gamma^{-1}t\gamma^2 t\gamma^{-1}$ and $s^2 - \gamma^{-1}s\gamma^2 s\gamma^{-1}$ are separately H.S., by the same argument as given for B_{13} Similarly $I - B_{33}$ is Hilbert–Schmidt, and the theorem is proved.

We conclude that SO(2) and U(1) transformations are locally implementable, and we have a representation of the current group over SO(2) or U(1), as defined in [7]. In [8] the infinitesimal version, i.e. the current algebra over SO(2), was solved.

After this work was essentially complete, we received a preprint of [9], where it is proved that, for the U(1) case, local gauge transformations are not implementable in four-dimensional space-time. We would come to the same conclusion, since $q^{-1} \notin L^2$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. However, we disagree with a casual remark in [9] that the result is independent of dimension.

IV. Appendix; by R. F. Streater

One might attempt to arrive at the 'charged sectors' of [2] by arranging $\theta(x)$ to have a tail, as suggested by the successful theory described in [3]; that is, one might choose $\theta \in C^\infty$, $\partial\theta/\partial x \in \mathscr{D}$ with $\theta(-\infty) = 0$, $\theta(+\infty) = Q \neq 2n\pi$. Since $\tilde{s}(p) \sim (\sin Q/p)\ (p \to 0)$ and $\tilde{c}(p) \sim (\cos Q/p)\ (p \to 0)$ in that case, and one of these at least is not square integrable, one may hope that the corresponding automorphisms fail to be implementable. This is not the case, however; all the 'infra-red' singularities of the kernels of A and B cancel the zeros, at $p_1 = p_2$, of factors like $[\omega(p_1)|\omega(p_2)]^{1/2} - [\omega(p_2)|\omega(p_1)]^{1/2}$.

The common feature of the successful models of theories predicting their own superselection rules is the presence of a spontaneously broken symmetry. In the Fermion model described in [3], the symmetry $\phi(x) \to \phi(x) + \eta$ is spontaneously broken. Presumably, for the sine-gordon equation, the same is true provided $\eta = 2\pi n$ for $n = \ldots, -1, 0, 1, 2, \ldots$. The Fermions arising in [3] satisfy the dynamics of the Thirring model of zero mass, as was first suggested and proved in [10] and [11]. Coleman [12] shows, at least in perturbation theory, that the sine-gordon equation is equivalent to the massive Thirring model.

An early example [13] is the suggestion that in one-dimensional quantum lattice systems, the boundary point between a region with spins up and a region with spins down, behaves as the coordinate of a particle, called a 'Blochon' because it represents the Bloch wall between magnetic domains. Here, use is made of the spontaneously broken rotation symmetry to obtain the non-implementable automorphism.

These models suggest that, instead of using free fields ϕ_1 and ϕ_2, as in §1–3, we should use two independent copies of $(\phi)_2^4$ with large parameter, for which the symmetry $\phi \to -\phi$ is known to the spontaneously broken [14]. Let us denote the two vacua by ω_+ and ω_-. Then the tensor product of two such field theories has four vacua, $\omega_\pm \oplus \omega_\pm$. These vacua are related to each other by the subgroup of SO(2) generated by a rotation of $\pi/2$: $\phi_1 \to \phi_2$, $\phi_2 \to -\phi_1$, under which the Hamiltonian $H_0^{(1)} + H_0^{(2)} + \lambda{:}\phi_1^4{:} + \lambda{:}\phi_2^4{:}$ is invariant. So, starting with the vacuum $\omega_- \oplus \omega_-$ and performing the SO(2) transformation τ_1, with $\theta(-\infty) = 0$, $\theta(+\infty) = \pi/2$ we arrive at states carrying a quantum number conserved mod 4. A similar idea has been recently put forward by Fröhlich [15], who remarks that these states resemble the soliton solutions of classical non-linear equations.

Since the local SO(2) transformations are locally implemented in the Fock representation, they are also locally implementable in any locally Fock theory such as ϕ_2^4. Indeed, the local implementing operator, '$\exp \frac{1}{2} \int dx\theta(x)(\dot{\phi}_1\phi_2 - \dot{\phi}_2\phi_1)$' lies in the quasi-local C^*-algebra if the local algebras are weakly closed and $\theta \in \mathscr{D}$. A corresponding operator therefore exists in any representation. If $\theta(-\infty) \neq \theta(+\infty)$ however, then the soliton representation is definitely not equivalent to the vacuum representation, as is easily proved using a lemma of Hegerfeldt [16].

The soliton model described so far has several deficiencies: a field theory with two vacua is not a good model for elementary particles; moreover, the theory is structurally unstable under the addition of linear external fields, and the presence of such a coupling, however small, destroys the very existence of the soliton state. In [3], these problems are solved by introducing a superselection rule – only $\nabla\phi$ and not ϕ, are regarded as observables. The broken symmetry group $\phi \to \phi + \eta$ now becomes a broken gauge group, and all the vacua coincide on the algebra of observables. Moreover, in [3], the Fermion states are localized in the sense of Knight [17], since they look like the vacuum, and the transformation $\phi \to \phi + \eta$ reduces to the identity automorphism, outside the support of $\nabla\eta$, as required by [2].

Applying the same remedy to the present model, let us agree that the transformation σ: $\phi_1 \to \phi_2$, $\phi_2 \to -\phi_1$ is to be a gauge transformation; the observable algebra is then defined to be the subalgebra invariant under σ. The transformation τ_1 is then a gauge transformation of the second kind (but also a local morphism in the language of [2]). By agreeing this, we rule out gauge-symmetry breaking terms in the interaction, and the theory becomes, presumably, structurally stable under the allowed perturbations. As before, the vacuum becomes unique and τ_1 reduces to the identity outside a compact set.

To be sure we still remain with a theory, we must check that τ_1, restricted to the observable algebra, is not implementable. To prove this, note that $A = \sin\phi_1(f_1) \sin\phi_1(f_2) + \sin\phi_2(f_1)\sin\phi_2(f_2)$ is a bounded observable. Now displace f_1 to the left and f_2 to the right. In the vacuum $(\omega_- \oplus \omega_-)$ representation clustering implies that

$$A \to a = 2\,\langle\sin\phi_1(f_1)\rangle\langle\sin\phi_1(f_2)\rangle \text{ weakly}$$

and in the soliton representation,

$$A \to -a \text{ weakly.}$$

Since we may easily choose f_1 and f_2 so that $a \neq 0$, an argument along the lines given in [16], lemma 2.1, shows that the representations are inequivalent.

With this choice of observable algebra, all the various possible soliton states coincide.

The implementability of space translations and the existence of the momentum operator in the soliton state follows easily from the local implementability of SO(2). The rest of the missing axiom (3), Lorentz covariance and the spectrum, is less trivial.

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REFERENCES

[1] R. Haag and D. Kastler, *An Algebraic Approach to Quantum Field Theory*, J. Mathematical Phys. *5*, 848 (1964).
[2] S. Doplicher, R. Haag and J. E. Roberts, *Observables and Gauge Transformations*, Commun.

Math. Phys. *13*, 1 (1969); *15*, 173 (1969); and *Local Observables and Particle Statistics*, Commun. Math. Phys. *23*, 199 (1971).

[3] R. F. STREATER and I. F. WILDE, *Fermion States of a Boson Field*, Nucl. Phys. *B24*, 561 (1970) and R. F. STREATER, *The C*-Approach to Quantum Field Theory*, Phys. Scripta *3*, 5 (1971).

[4] I. F. WILDE, *A Remark on the Localization of a Charged Boson Field*, Commun. Math. Phys. *24*, 37 (1971).

[5] D. SHALE, *Linear Symmetries of Free Boson Fields*, Trans. Amer. Math. Soc. *103*, 149 (1962).

[6] P. KRISTENSON, L. MELJBO and E. T. POULSON, *Tempered Distributions in Infinitely Many Variables III*, Commun. Math. Phys. *6*, 29 (1967).

[7] R. F. STREATER, J. Mathematical Phys. *5*, 581 (1964); also 'Current Commutation Relations, Continuous Tensor Products, and Infinitely Divisible Group Representation,' in *Local Quantum Theory*, Varenna lectures 1968, edited by R. JOST (Academic Press 1969).

[8] M. FITTLESON and R. JOHNSON, Nuclear Phys. *B18*, 151 (1970).

[9] G. F. DELL'ANTONIO, *Are Local Gauge Transformations Implementable?*, J. Mathematical Phys. *12*, 148 (1971).

[10] R. F. STREATER, *Gauge Theories and Superselection Rules*, Acta Phys. Austraica, Supp XI, 317 (1973); Schladming Lectures, Feb. 1973, edited by P. URBAN. (Springer).

[11] R. F. STREATER, 'Charges and Currents in the Thirring Model', in *Physical Reality and Mathematical Description*, edited by ENZ. and MEHRA. D. REIDEL, 1974. See also R. LUTZ, M. Phil. thesis, Bedford College, 1975.

[12] S. COLEMAN, *Quantum Sine-Gordon Equation as the Massive Thirring Model*, Phys. Rev. D, to appear.

[13] R. F. STREATER; Commun. Math. Phys. *6*, 233 (1967); R. F. STREATER, Inaugural Lecture, 1971, published by Bedford College.

[14] J. GLIMM and A. JAFFE, *Existence of Phase Transitions for ϕ_2^4 Quantum Fields*, to appear.

[15] J. FRÖHLICH, *The Pure Phases, the Irreducible Quantum Fields and Dynamical Symmetry Breaking in Symanzik–Nelson-positive Quantum Field Theories*, Princeton preprint.

[16] G. C. HEGERFELDT, *Probability Measures on Distribution Spaces and Quantum Field Theoretical Models*, to be published.

[17] J. M. KNIGHT, *Strict Localization in Quantum Field Theory*, J. Mathematical Phys. *2*, 459 (1961).

[18] After this paper was submitted for typing, J. FRÖHLICH kindly made available typed versions of 'Poetic Phenomena in (Two Dimensional) Quantum Field Theory: Non-Uniqueness of the Vacuum, the Solitons and All That', and 'New Super-selection Sectors ("Soliton States") in Two Dimensional Bose Quantum Field Models', where similar models are treated in detail.