Indefinite metric, quantum axiomatics, and the Markov property

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Abstract. In answer to a remark of Jauch, a set of axioms for an 'indefinite metric' formulation of quantum electro-dynamics is presented, and the connection with orthocomplementation noted. Here a strict version of the Markov property apparently fails, leading to a novel interpretation.

0. Introduction

This rather amorphous note grew out of the challenge afforded by the brief remark of Jauch in his book ([4] end. sect. 8–4, p. 131) to the effect that the use of an 'indefinite metric' in attempts at a workable precise formulation of quantum electro-dynamics must be superfluous, in the sense that it should be possible to replace such a 'metric' by a definite one as far as physical results are concerned. Jauch draws this conclusion from the well-known elaboration of the striking von Neumann–Birkhoff theorem yielding an essentially unique definite symmetric bilinear functional from an orthocomplementation on a linear vector space. However, we consider this superfluity conclusion invalid or at least misleading, not through any fault in this von Neumann–Birkhoff theorem or its elaboration, but because the usual 'indefinite metric' formulations implicitly violate Jauch's prior assumptions, in particular not having an orthocomplementation defined for the whole linear vector space in question.

In view of the growing interest in quantum axiomatics (see [5], [6], [3], [7]), and in fairness to the possibilities of 'indefinite metric' formulations, a tentative examination of these hypotheses seems desirable. We have no theorems to prove; instead the above implicit violation is made explicit by stating a collection of axioms for an 'indefinite metric' formulation, adding a few physically motivating remarks. These axioms are noted satisfied for a few interesting models, and we also note a few obvious consequences, particularly concerning the Markov property.

1. Axioms and motivation

As explained in Heitler's classic ([2], II sect. 10.2, pp. 90–92), Gupta and Bleuler introduce an 'indefinite metric' (not a mathematical sense metric but an indefinite symmetric bilinear functional) to obtain positive energies for scalar photons; in this exposition ([2] p. 95, 102) scalar and longitudinal photons are not considered directly observable physically, but correspond to different Lorentz gauge choices. The further extension of such an 'indefinite metric' formulation by the author ([1], sect. 8 and 9) is used to describe consistently certain well-known vacuum effects associated with electron-positron pairs; here the added limbotic component spaces are certainly thought to have no direct physical reality. These considerations are thus formalized by the following axioms (see also [8], 1.9–13).

Axiom (1.A) (state space)

There exists a complex Hilbert space X_0, for which a nonnull vector $u \in X_0$ corresponds to a directly physically observed situation or state; thus X_0 is called the state Hilbert space, and a state vector is a $u \in X_0$ having $1 = \|u\| = \sqrt{(u,u)}$ with $(.,.)$ the X_0 inner product.

Axiom (1.B) (including spaces)

The complex Hilbert spaces X_1 and Z satisfy with X_0 of axiom (1.A) the subspace inclusions $X_0 \subseteq X_1 \subseteq Z$ (i.e., for each inclusion the included space is a closed linear manifold of the including space with coincident inner product), and there exists a bounded linear transformation D from X_0 into X_1 satisfying over all $u \in X_0$

$$P_0 Du = u, \tag{1.1}$$

where P_0 is the Z space orthogonal projection operator whose image space is X_0; thus $Q_0 = I - P_0$ and $Y = Q_0 Z$ have $Z = X_0 \oplus Y$, this Y being called the fictitious component space with all $v \in Z$ having $P_0 v = u$ corresponding to the same physical state as $u \in X_0$.

Axiom (1.C) ('indefinite metric')

Additionally to the Z inner product there exists another symmetric bilinear functional on Z denoted $(.:.)$ (i.e. $g(u, v) = (u:v)$ defines g as a complex valued function on $Z \otimes Z$ which is linear in the first place and conjugate linear in the second and has $(u:v) = \overline{(v:u)}$ and $|(u:v)| \leq M\|u\| \|v\|$ over $u, v \in Z$ for some constant $M \in (0, +\infty)$), which is indefinite (i.e. real $(u:u)$ takes over $u \in Z$ both positive and negative values) and on X_1 coincides with the Z inner product (i.e. over $u, v \in X_1$

$$(u:v) = (u, v) \tag{1.2}$$

holds); here $(.:.)$ is called the physical product, and analogous to state vectors in axiom (1.A) a condition vector is a $u \in Z$ having $(u:u) = 1$.

Axiom (1.D) (dynamical observables)

A set Γ of linear operators in Z has each $B \in \Gamma$ be densely defined and physically symmetric,

$$(Bu:v) = (u:Bv) \tag{1.3}$$

over all u, v in the B domain D_B with $\bar{D}_B = Z$; this Γ is called the set of dynamical observables.

Aziom (1.E) (Hamiltonian)

There exists $H \in \Gamma$ of axiom (1.D) such that

$$V_t = e^{-itH} \tag{1.4}$$

suitably defined (see below) over t real has for each $u \in X_0$ with $w_0 = Du \in X_1$ of axiom (1.B) satisfying $(w_0, w_c) = 1$ that

$$w_t = V_t w_0 \tag{1.5}$$

describes the time evolution of the system starting with w_0, in which for $B \in \Gamma$ whenever $w_t \in D_B$ then

$$(Bw_t : w_t) \tag{1.6}$$

is real and is taken to be the expectation value at time t of the dynamical observable denoted by B; of course H is called the Hamiltonian.

To comment on the V_t definition in equation (1.3), if (as in the usual atomic quantum mechanics with no field quantization) $Z = X_1$ and so $(u:v) = (u, v)$ on Z by equation (1.2) (contradicting the (1.C) indefiniteness), then equation (1.3) would make H be symmetric and hence to be taken self-adjoint, and V_t in equation (1.4) could then be defined unitary by the usual spectral integral. However, $Z \neq X_1$ here under axiom (1.C) and the definition of V_t in equation (1.4) is not clear in general; fortunately in our above formulation [1], there H becomes bounded on Z due to momentum cutoffs, and for such of course we define $e^{-itH} = I + \sum_{p=1}^{\infty} (p!)^{-1}(-it)^P H^P$ convergent in operator norm. Thus for V_t in equation (1.4) we can expect in equation (1.6) by equation (1.3) for H

$$\frac{d}{dt}(Bw_t : w_t) = -i([BH - HB]w_t : w_t) \tag{1.7}$$

as usual, showing by taking $B = I$ that equation (1.5) has by equation (1.2)

$$(w_t : w_t) = (w_0 : w_0) = (w_0, w_0) = 1 \tag{1.8}$$

over all real t (making w_t a condition vector (1.C)) and confirming the expectation value interpretation of equation (1.6).

2. Orthocomplementation

As in axiom (1.A), X_0 is considered to contain all the directly physically observable data concerning our system. Thus as in [4], propositions concerning the physically real systems are identified with subspaces (i.e. closed linear manifolds) of X_0, intuitively the proposition being considered certainly true for the situation described by a state vector in the subspace. Since propositions here are concerned only with physical reality, and likewise for their negatives, they apparently must correspond to objects in X_0 rather than the including Z. Hence we now take the negative of a proposition to correspond to the orthogonal complement in X_0 (*not* in Z) of the X_0 subspace corresponding to the original proposition. This of course defines an orthocomplementation on X_0, but not on Z, so explaining the breakdown of Jauch's remark mentioned in the introduction.

We remark here that, except for P_0 in equation (1.1) and the resulting $Z = X_0 \oplus Y$, the Z inner product outside X_0 and thus on Y is not supposed to have any direct physical significance, being merely a mathematical convenience yielding the topology for the operators there. Thus no physically reasonable orthocomplementation on Z is to be expected.

3. Remarks and Markov failure

In axioms (1.A) and (1.B), X_0 is supposed to be the subspace of Z where no longitudinal or scalar photons are present, and also no limbotic components as in [1]; X_1 enlarges over X_0 by allowing the addition of only longitudinal photons, the absence of scalar photons yielding no values -1 for the Gupta–Bleuler factor and thus yielding equation (1.2) on X_1. Also D in equation (1.1) is to determine from $u \in X_0$ a solution $Du \in X_1$ of the Lorentz condition satisfying equation (1.1).

Also for the time evolution equation (1.5), note from axiom (1.B) that at time $t = t_1 > 0$ then $\tilde{u} = P_0 w_{t_1}$ apparently describes our physically real state at time t_1, and accordingly starting again in equation (1.5) from $\tilde{w}_0 = \|D\tilde{u}\|^{-1} D\tilde{u}$ (assuming $\|\tilde{u}\| > 0$ and thus $\|D\tilde{u}\| \geq \|\tilde{u}\| > 0$ by (1.1)) we generally expect that $P_0 \tilde{w}_s \neq P_0 w_{s+t_1}$, equivalently

$$\|DP_0 V_{t_1} w_0\|^{-1} P_0 V_s D P_0 V_{t_1} w_0 \neq P_0 V_{s+t_1} w_0 \tag{3.1}$$

generally over $w_0 \in DX_0 \subseteq X_1$ having $\|w_c\| = 1$, and indeed

$$P_0 V_s D P_0 V_{t_1} w \neq \lambda P_0 V_{s+t_1} w \tag{3.2}$$

for any scalar λ generally over $w \in DX_0$. This amounts to the failure of the intuitive Markov property in time, the physical state vector $\|P_0 w_{t_1}\|^{-1} P_0 w_{t_1}$ at time t_1 apparently not containing all the future determining data entering the unstopped time flow $w_t = V_t w_0$ at $t = t_1$.

To accept this situation we must apparently adopt the following view of quantum mechanics under these section 1 axioms. The flow V_t in equation (1.5) yields the following prediction statement: if at time $t = 0$ the initial physical situation is described by the state vector $u_0 \in X_0$, then at a time $t_1 > 0$ the evolved situation is described by the state vector $\|P_0 V_{t_1} D u_0\|^{-1} P_0 V_{t_1} D u_0$. But we are not allowed to ask two questions at once generally and so cannot additionally enquire about the situation $t_2 > t_1$ as well, like the usual failure of independence and disturbance of a state by measurement in quantum mechanics. However, expectation values for $B \in \Gamma$ (perhaps a very restricted set) are given by equation (1.6) over all time t.

REFERENCES

[1] F. H. BROWNELL, *A Limbotic Reformulation of Quantum Electrodynamics and the Lamb Shift Basis*, LDOOO12 (University Microfilms, Ann Arbor, Mich., 1973).

[2] W. HEITLER, *The Quantum Theory of Radiation*, 3rd ed. (Oxford Univ. Press, 1954).

[3] C. A. HOOKER (ed), *The Logico-Algebraic Approach to Quantum Mechanics*, vols. I and II (Reidel, Dordrecht, 1975, 1976).

[4] J. M. JAUCH, *Foundations of Quantum Mechanics* (Addison-Wesley, 1968).

[5] G. W. MACKEY, *Mathematical Foundations of Quantum Mechanics* (Benjamin, 1963).

[6] V. S. VARADARAJAN, *Geometry of Quantum Theory*, vols 1 and 2 (Van Nostrand, 1968, 1970).

[7] A. S. WIGHTMAN, *Hilbert's Sixth Problem: Mathematical Treatment of the Axioms of Physics*, pp. 147–240, *Mathematical Developments Arising from Hilbert Problems*, Proc. Symp. Pure Math. *28*, A.M.S., (1976).

[8] A. S. WIGHTMAN and F. STROCHI, J. Math. Phys. *15*, 2198 (1974).