Zeitschrift: Helvetica Physica Acta
Band: 50 (1977)
Heft: 6

Artikel: The regular external field problem in quantum electrodynamics
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DOI: https://doi.org/10.5169/seals-114890

The regular external field problem in quantum electrodynamics[1])

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(2.VII.1977)

Abstract. The quantized electron-positron field in interaction with an external classical static electromagnetic field is considered. The external potential is restricted in such a way that a dressed vacuum and dressed electron-positron states exist in Fock space. In this case the Furry picture has a mathematically well-defined meaning. A large class of such regular external fields is found which, however, contains no static magnetic field.

1. Introduction

One of the simplest systems of quantum electrodynamics is the quantized electron-positron field interacting with an external classical electromegnetic field. Although the physics of this system is well understood and rather simple, the mathematical character of the theory is not at all trivial and requires a careful investigation.

We will consider the system described by the formal Hamiltonian

$$\mathbb{H}_{\text{formal}} = \int d^3x \colon \psi^+(\mathbf{x}, t)\left(-i\sum_{k=1}^{3}\alpha_k \frac{\partial}{\partial x_k} + m\beta + eV(\mathbf{x})\right)\psi(\mathbf{x}, t)\colon \tag{1.1}$$

where $\psi(\mathbf{x}, t)$ is the electron-positron field operator satisfying equal-time Fermi anti-commutation relations

$$\{\psi_a(\mathbf{x}, t), \psi_b^+(\mathbf{x}', t)\}_+ = \delta_{ab}\delta(\mathbf{x}\text{-}\mathbf{x}'); \quad a, b = 1, \ldots, 4 \tag{1.2}$$

with all other anticommutators vanishing. $\mathbb{H}_{\text{formal}}$ (1.1) as it stands is not a well-defined operator in Fock space. Nevertheless, the Heisenberg equation of motion derived from (1.1)

$$\frac{\partial\psi(\mathbf{x}, t)}{\partial t} = -i\left(-i\sum_{k=1}^{3}\alpha_k \frac{\partial}{\partial x_k} + m\beta + eV(\mathbf{x})\right)\psi(\mathbf{x}, t) \tag{1.3}$$

makes sense and is immediately solved by

$$\psi(f, t) = \psi(e^{iHt}f). \tag{1.4}$$

Here

$$\psi(f, t) = \int d^3x f_a(\mathbf{x})^* \psi_a(\mathbf{x}, t), f_a \in L^2(\mathbb{R}^3) \tag{1.5}$$

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[1]) Work supported by the Swiss National Science Foundation.

is the spacially smeared out field operator and

$$\psi(f) = \psi(f, 0) \tag{1.6}$$

is the initial value at time $t = 0$, H is the one-particle Dirac operator in $(L^2(\mathbb{R}^3))^4$.

We are now interested in the question whether there exists a selfadjoint Hamiltonian $\mathbb{H}$ in Fock space (some renormalized version of (1.1)) which generates the time evolution automorphism (1.4) as an inner automorphism

$$\psi(e^{iHt}f) = e^{i\mathbb{H}t}\psi(f)\, e^{-i\mathbb{H}t}. \tag{1.7}$$

It has been shown by Bongaarts [1] that $\mathbb{H}$ exists if and only if

$$P_+^0\, e^{-iHt} P_-^0 \in \text{H.S.} \tag{1.8}$$

is a Hilbert–Schmidt operator (H.S.) in $(L^2(\mathbb{R}^3))^4$ for all t, where $P_\pm^0$ are the projection operators on the positive and negative part of the spectrum of the free Dirac operator H_0, respectively (see next section). This condition can be satisfied probably only with scalar potentials (see Section 4). For this reason, we have restricted ourselves to scalar external potentials from the very beginning (1.1). Bongaarts gives no construction of the renormalized Hamiltonian $\mathbb{H}$. On the other hand, Friedrichs [2] long ago constructed such a $\mathbb{H}$ under the conditions

$$P_+ P_-^0 \in \text{H.S.}, \qquad P_+^0 P_- \in \text{H.S.} \tag{1.9}$$

where $P_\pm$ are now the projections on the positive and negative spectral part of the Dirac operator H with potential. The conditions (1.9) imply (1.8), and if (1.9) holds the method of Friedrichs gives a very simple construction of the renormalized Hamiltonian $\mathbb{H}$. Very likely the conditions (1.8) and (1.9) are actually equivalent. We have proven this at the moment only for a restricted class of external potentials.

The condition (1.9), or the equivalent one

$$P_+ - P_+^0 \in \text{H.S.} \tag{1.10}$$

has the more direct physical meaning. It guarantees the existence of a dressed vacuum and dressed electron-positron states in Fock space, that means, the Furry picture is mathematically well-defined. This will be discussed in Section 3. Most results of this section have been obtained by several authors [2, 3, 4, 5], so that our contribution is merely the clarification of some details and certain simplifications of the proofs. In Section 4, we discuss the class of regular potentials which is defined by the property that condition (1.10) is fulfilled. Our characterization of this class is not complete in the sense that we do not know the most general condition on the potential implying (1.10). The physical consequences of these results are discussed in the following section. It turns out that the existence of the dressed states enables one to construct the renormalized Hamiltonian and charge density operators and give a rigorous discussion of the vacuum polarization.

2. Preliminaries

The free Dirac operator

$$H_0 = -i\sum_{k=1}^{3} \alpha_k \frac{\partial}{\partial x_k} + m\beta, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

gives rise to a self-adjoint operator in $(L^2(\mathbb{R}^3))^4$. Its spectral decomposition is most conveniently described in terms of the (generalized) eigenfunctions

$$u_s^0(\mathbf{p}, \mathbf{x}) = u_s(\mathbf{p})\, e^{i\mathbf{px}}, \qquad v_s^0(\mathbf{p}, \mathbf{x}) = v_s(\mathbf{p})\, e^{-i\mathbf{px}} \qquad s = \pm 1 \tag{2.2}$$

where

$$u_s(\mathbf{p}) = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \chi_s \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \end{pmatrix} \chi_s \tag{2.3}$$

$$v_s(\mathbf{p}) = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m} \\ \chi_s \end{pmatrix} \chi_s$$

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E = +\sqrt{\mathbf{p}^2 + m^2}. \tag{2.4}$$

They satisfy

$$\begin{aligned} H_0 u_s^0(\mathbf{p}, \mathbf{x}) &= E u_s^0(\mathbf{p}, \mathbf{x}) \\ H_0 v_s^0(\mathbf{p}, \mathbf{x}) &= -E v_s^0(\mathbf{p}, \mathbf{x}) \end{aligned} \tag{2.5}$$

and the orthogonality relations

$$\begin{aligned} u_s(\mathbf{p})^+ u_{s'}(\mathbf{p}) &= \delta_{ss'} = v_s(\mathbf{p})^+ v_{s'}(\mathbf{p}) \\ u_s(\mathbf{p})^+ v_{s'}(-\mathbf{p}) &= v_s(\mathbf{p})^+ u_{s'}(-\mathbf{p}) = 0. \end{aligned} \tag{2.6}$$

For any $\Phi(\mathbf{x}) \in (L^2)^4$ let

$$\hat{\Phi}_+(\mathbf{p}, s) = (2\pi)^{-3/2}\, \text{l.i.m.} \int d^3x\, u_s^0(\mathbf{p}, \mathbf{x})^+ \Phi(\mathbf{x})$$

$$\hat{\Phi}_-(\mathbf{p}, s) = (2\pi)^{-3/2}\, \text{l.i.m.} \int d^3x\, v_s^0(\mathbf{p}, \mathbf{x})^+ \Phi(\mathbf{x}). \tag{2.7}$$

Then the spectral projection $P^0(\Omega)$ for a Borel set

$$\Omega \subset (-\infty, m] \cup [m, +\infty) \tag{2.8}$$

is given by

$$(P^0(\Omega)\Phi)(\mathbf{x}) = (2\pi)^{-3/2}\, \text{l.i.m.} \left[\int_{\Omega+} d^3p\, \hat{\Phi}_+(\mathbf{p}, s) u_s^0(\mathbf{p}, \mathbf{x}) + \int_{\Omega-} d^3p\, \hat{\Phi}_-(\mathbf{p}, s) v_s^0(\mathbf{p}, \mathbf{x}) \right] \tag{2.9}$$

where

$$\Omega_\pm = \{\mathbf{p} \in \mathbb{R}^3 \mid \pm E(\mathbf{p}) \in \Omega\}. \tag{2.10}$$

In particular, we have the eigenfunction expansion

$$\Phi(\mathbf{x}) = (2\pi)^{-3/2}\, \text{l.i.m.} \int d^3p [\hat{\Phi}_+(\mathbf{p}, s) u_s^0(\mathbf{p}, \mathbf{x}) + \Phi_-(\mathbf{p}, s) v_s^0(\mathbf{p}, \mathbf{x})] \tag{2.11}$$

which defines a unitary transformation of $(L^2)^4$ onto itself, completely analogous to the Fourier transformation in ordinary L^2. The first member on the right-hand side of (2.11) is the projection $P_+^0\Phi$ on the electron subspace, the second member $P_-^0\Phi$ on the positron subspace.

Similar results are true for the Dirac operator

$$H = H_0 + V(\mathbf{x}) \tag{2.12}$$

for a large class of static scalar potentials which contains the class of potentials we are going to consider in the following. Also in this case, the spectral projections $P(\Omega)$ can be expressed in terms of eigenfunctions $u_s(\mathbf{p}, \mathbf{x})$, $v_s(\mathbf{p}, \mathbf{x})$ as (2.9) [6], with the only difference that in addition to the continuous spectrum (2.8) there are in general discrete eigenvalues, and we have no explicit expression for the eigenfunctions like (2.3). The Fourier transform analogous to (2.11) reads

$$\Phi(\mathbf{x}) = (2\pi)^{-3/2}\ \text{l.i.m.} \int dp[\tilde{\Phi}_+(\mathbf{p}, s)u_s(\mathbf{p}, \mathbf{x}) + \tilde{\Phi}_-(\mathbf{p}, s)v_s(\mathbf{p}, \mathbf{x})] \tag{2.13}$$

where $\int dp$ is a short notation for $\int d^3p$ and a possible sum over the discrete eigenvalues.

Let us now construct the Fock space $\mathscr{F}$ using

$$\mathfrak{h}_1 = (L^2(\mathbb{R}^3))^4$$

as the one-particle subspace. To have a concise notation we write

$$p = (\mathbf{p}, s, \varepsilon)$$

where $\varepsilon = \pm 1$ distinguishes the electron and positron subspaces (2.7). The n-particle subspace $\mathfrak{h}_n$ then consists of functions $\Phi_n(p_1 \cdots p_n) \in ((L^2)^4)^{\otimes n}$, antisymmetric in the arguments $p_j = (\mathbf{p}_j, s_j, \varepsilon_j)$. $\mathfrak{h}_n$ contains electron and positron states together, only the total number of particles ($= n$) is specified. The Fock space $\mathscr{F}$ then is

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathfrak{h}_n$$

with $\mathfrak{h}_0 = \mathbb{C}$.

The symbolic absorption and emission operators are given by

$$(b_s(\mathbf{p})\Phi)_n(p_s \cdots p_n) = \sqrt{n+1}\,\Phi_{n+1}(\mathbf{p}, s, 1; p_s \cdots p_n)$$

$$(d_s(\mathbf{p})\Phi)_n(p_s \cdots p_n) = \sqrt{n+1}\,\Phi_{n+1}(\mathbf{p}, s, -1; p_s \cdots p_n)$$

$$(b_s^+(\mathbf{p})\Phi)_n(p_s \cdots p_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-)^{j-1} \delta(\mathbf{p} - \mathbf{p}_j)\delta_{ss_j}\delta_{s\varepsilon_j}$$
$$\Phi_{n-1}(p_s \cdots \not{\mathbf{p}}_j \not{s}_j \not{\varepsilon}_j \cdots p_n)$$

$$(d_s^+(\mathbf{p})\Phi)_n(p_s \cdots p_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-)^{j-1} \delta(\mathbf{p} - \mathbf{p}_j)\delta_{ss_j}\delta_{-1\varepsilon_j}$$
$$\Phi_{n-1}(p_s \cdots \not{\mathbf{p}}_j \not{s}_j \not{\varepsilon}_j \cdots p_n) \tag{2.14}$$

where the crossed out arguments p_j have to be omitted. If $f(x) \in (L^2)^4$ is in $P^0_+ \mathfrak{h}_1$ then

$$b(\hat{f}_+) = \int d^3p \hat{f}_+(\mathbf{p}, s)^* b_s(\mathbf{p})$$

$$b^+(\hat{f}_+) = \int d^3p \hat{f}_+(\mathbf{p}, s) b_s^+(\mathbf{p}) = b(\hat{f}_+)^+ \tag{2.15}$$

are well-defined (even bounded) operators in $\mathscr{F}$:

$$(b(\hat{f}_+)\Phi)_n(p_s \cdots p_n) = \sqrt{n+1} \int d^3p \hat{f}_+(\mathbf{p}, s)^* \Phi_{n+1}(\mathbf{p}, s, 1, \ldots)$$

$$(b^+(\hat{f}_+)\Phi)_n(p_s \cdots p_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-)^{j-1} \delta_{1\varepsilon_j} \hat{f}_+(\mathbf{p}_j, s_j) \Phi_{n-1}(p_s \cdots \not{p}_j \not{s}_j \not{\varepsilon}_j \cdots p_n) \tag{2.16}$$

and similarly for

$$d^+(\hat{f}_-) = \int d^3p \hat{f}_-(\mathbf{p}, s)^* d_s^+(\mathbf{p})$$

$$d(\hat{f}_-) = \int d^3p f_-(\mathbf{p}, s) d_s(\mathbf{p}). \tag{2.17}$$

They satisfy the anti-commutation rules

$$\{b(\hat{f}_+), b^+(\hat{g}_+)\}_+ = (\hat{f}_+, \hat{g}_+)$$
$$\{d^+(\hat{f}_-), d(\hat{g}_-)\}_+ = (\hat{f}_-, \hat{g}_-). \tag{2.18}$$

with all other anti-commutators vanishing.

The field operators smeared out in space are defined by

$$\Psi(f) = b(\hat{f}_+) + d^+(\hat{f}_-)$$
$$\Psi^+(f) = b^+(\hat{f}_+) + d(\hat{f}_-) \tag{2.19}$$

which corresponds to the formal expressions

$$\Psi(f) = \int d^3x f_a^*(x) \Psi_a(x)$$

$$\Psi(x) = (2\pi)^{-3/2} \int d^3p [u_s^0(\mathbf{p}, \mathbf{x}) b_s(\mathbf{p}) + v_s^0(\mathbf{p}, \mathbf{x}) d_s^+(\mathbf{p})]. \tag{2.20}$$

3. The dressed electron-positron states

Let

$$\mathfrak{h}_1 = \mathfrak{h}_+^0 \oplus \mathfrak{h}_-^0, \qquad \mathfrak{h}_\pm^0 = P_\pm^0 \mathfrak{h}_1 \tag{3.1}$$

be the decomposition of $\mathfrak{h}_1$ into the electron and positron subspaces defined by the free Dirac operator H_0 and similarly let

$$\mathfrak{h}_1 = \mathfrak{h}_+ \oplus \mathfrak{h}_-, \qquad \mathfrak{h}_\pm = P_\pm \mathfrak{h}_1 \tag{3.2}$$

be the decomposition defined by the positive and negative spectral parts of H (2.12). We call $\mathfrak{h}_\pm$ the dressed electron and positron subspaces, respectively, $\mathfrak{h}^0_\pm$ are the bare electron and positron subspaces. For $f(x) \in \mathfrak{h}_+$ we now define the dressed emission and absorption operators by

$$\begin{aligned} b'(\tilde{f}_+) &= b(\widehat{P^0_+ f}) + d^+(\widehat{P^0_- f}) \\ b'^+(\tilde{f}_+) &= b'(\tilde{f}_+)^+ \end{aligned} \tag{3.3}$$

and similarly for $f(x) \in \mathfrak{h}_-$

$$\begin{aligned} d'^+(\tilde{f}_-) &= b(\widehat{P^0_+ f}) + d^+(\widehat{P^0_- f}) \\ d'(\tilde{f}_-) &= d'^+(\tilde{f}_-)^+, \end{aligned} \tag{3.4}$$

where $\tilde{f}_+(\mathbf{p}, s) \in \tilde{\mathfrak{h}}_\pm$ is the Fourier–Dirac transform of f (2.13) defined by H. From (2.18) we get canonical anti-commutation rules for the dressed operators

$$\begin{aligned} \{b'(\tilde{f}_+), b'^+(\tilde{g}_+)\} &= (\tilde{f}_+, \tilde{g}_+) \\ \{d'^+(\tilde{f}_-), d'(\tilde{g}_-)\} &= (\tilde{f}_-, \tilde{g}_-) \end{aligned} \tag{3.5}$$

and from (2.19) the decomposition of the field operators $\Psi(f)$ for arbitrary $f(x) \in (L^2)^4$

$$b'(\tilde{f}_+) + d'^+(\tilde{f}_-) = b(\widehat{P^0_+ f}) + d^+(\widehat{P^0_- f}) = \Psi(f). \tag{3.6}$$

As in the case of free fields (2.20), the dressed operators can be expressed in terms of operator-valued distributions $b'_s(\mathbf{p})$, $d'_s(\mathbf{p})$ etc.

It is convenient, for what follows, to write the linear transformation (3.3), (3.4) in matrix notation. Let $\varphi^j_+(\mathbf{p}, s)$, $\varphi^k_-(\mathbf{p}, s)$ be a complete orthonormal system in $\tilde{\mathfrak{h}}_\pm$ and $\varphi^j_{0+}(\mathbf{p}, s)$, $\varphi^k_{0-}(\mathbf{p}, s)$ in $\tilde{\mathfrak{h}}^0_\pm$ respectively (which are all separable Hilbert spaces). Introducing

$$b'_j = b'(\varphi^j_+), \qquad d'^+_k = d'^+(\varphi^k_-) \tag{3.7}$$

and similarly for the bare operators, we can write (3.3), (3.4) as follows

$$\begin{aligned} b'_j &= (\varphi^j_+, \varphi^k_{0+})\, b_k + (\varphi^j_+, \varphi^k_{0-}) d^+_k \\ d'^+_j &= (\varphi^j_-, \varphi^k_{0+})\, b_k + (\varphi^j_-, \varphi^k_{0-}) d^+_k. \end{aligned} \tag{3.8}$$

The anti-commutation rules (3.4) take the familiar form

$$\{b'_j, b'^+_k\} = \delta_{jk}, \qquad \{d'^+_j, d'_k\} = \delta_{jk}$$

and 0 otherwise. The matrix

$$W = \begin{pmatrix} (\varphi^j_+, \varphi^k_{0+}) & (\varphi^j_+, \varphi^k_{0-}) \\ (\varphi^j_-, \varphi^k_{0+}) & (\varphi^j_-, \varphi^k_{0-}) \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} \tag{3.9}$$

occurring in (3.8) can be considered as a unitary operator in $\mathfrak{h}_1$. This follows from the properties

$$\begin{aligned} W_1 W_1^+ + W_2 W_2^+ &= 1 \qquad & W_3 W_3^+ + W_4 W_4^+ &= 1 \\ W_1 W_3^+ + W_2 W_4^+ &= 0 \qquad & W_1^+ W_2 + W_3^+ W_4 &= 0, \end{aligned} \tag{3.10}$$

and we have the additional relations

$$W_1^+ W_1 + W_3^+ W_3 = 1 \qquad W_2^+ W_2 + W_4^+ W_4 = 1. \tag{3.11}$$

Here $W_1, \ldots, W_4$ are considered as transformations on the following subspaces

$$W_1 : \hat{\mathfrak{h}}_+^0 \longrightarrow \tilde{\mathfrak{h}}_+ \qquad W_2 : \hat{\mathfrak{h}}_-^0 \longrightarrow \tilde{\mathfrak{h}}_+$$
$$W_3 : \tilde{\mathfrak{h}}_+^0 \longrightarrow \tilde{\mathfrak{h}}_- \qquad W_4 : \hat{\mathfrak{h}}_-^0 \longrightarrow \tilde{\mathfrak{h}}_- . \tag{3.12}$$

These mappings can be trivially extended to the whole of $\mathfrak{h}_1$. Omitting $\hat{}$ and $\tilde{}$ from now on, because only the p-space is used in the further discussion, we have

$$\overline{W}_1 f = (\varphi_+^j, \varphi_{0+}^k)(\varphi_{0+}^k, f)\varphi_+^j = (\varphi_+^j, P_+^0 f)\varphi_+^j = P_+ P_+^0 f$$
$$\overline{W}_2 f = (\varphi_+^j, \varphi_{0-}^k)(\varphi_{0-}^k, f)\varphi_+^j = P_+ P_-^0 f$$
$$\overline{W}_3 f = (\varphi_-^j, \varphi_{0+}^k)(\varphi_{0+}^k, f)\varphi_-^j = P_- P_+^0 f$$
$$\overline{W}_4 f = (\varphi_-^j, \varphi_{0-}^k)(\varphi_{0-}^k, f)\varphi_-^j = P_- P_-^0 f, \quad f \in \mathfrak{h}_1, \tag{3.13}$$

where the bar denotes the extended transformations on $\mathfrak{h}_1$.

It is of central importance to know whether there exists a dressed vacuum Ω', that is a vector $\Omega' \in \mathscr{F}$ satisfying

$$b_j' \Omega' = v \tag{3.14}$$

$$d_j' \Omega' = v \tag{3.15}$$

for all j. If there exists a unique Ω', then the dressed electron-positron operators realize a Fock representation of the anti-commutation relations (3.5). Consequently, there must exist a unitary dressing transformation U relating the dressed and bare operators

$$b_j' = U b_j U^{-1}, \qquad d_j' = U d_j U^{-1}. \tag{3.16}$$

If Ω' does not exist, the representation is inequivalent to the Fock representation. We call the first possibility ‘regular’ and the second ‘singular’, and we are going to discuss the regular case in detail.

Let us expand Ω' into bare states

$$\Omega' = \sum_{m,n=0}^{\infty} \sum_{\substack{p_1 < \cdots < p_m \\ q_1 < \cdots < q_n}} A^{mn}_{p_1 \cdots p_m q_1 \cdots q_n} b^+_{p_1} \ldots b^+_{p_m} d^+_{q_1} \ldots d^+_{q_n} \Omega. \tag{3.17}$$

Inserting this expression into (3.14) and using (3.8), we get the following recursion relation

$$\sum_{p_0} (\varphi_+^p, \varphi_{0+}^{p_0}) A^{mn}_{p_0 p_1 \cdots p_{m-1} q_1 \cdots q_n} = \sum_{k=1}^{n} (-)^{m-1+k} (\varphi_+^n, \varphi_{0-}^{q_k}) A^{m-1,n-1}_{p_1 \cdots p_{m-1} q_1 \cdots \not{q}_k \cdots q_n} \tag{3.18}$$

$$m = 1, 2, \ldots, n = 0, 1, 2, \ldots$$

where the coefficients A satisfy

$$A^{m,-1} = A^{-1,n} = 0$$

$$\|\Omega'\|^2 = \sum_{m,n} \sum_{\substack{p_1 < \cdots < p_m \\ q_1 < \cdots < q_n}} |A^{mn}_{p_1 \cdots q_n}|^2 < \infty. \tag{3.19}$$

Forming the absolute square of both sides of (3.18) and summing over p and q_1, all terms on the left-hand side are finite because of (3.19). The same is obviously true for all terms on the right-hand side except the first one where the summation index q_1 does not appear under the indices of $A^{m,n-1}$. This leads to the necessary condition

$$|A^{m-1,n-1}_{p_1\cdots p_{m-1},q_2\cdots q_n}|^2 \sum_{pq_1} |(\varphi^p_+, \varphi^{q_1}_{0-})|^2 < \infty.$$

For a non-trivial solution some $A^{m-1,n-1}$ must be different from 0, consequently

$$\sum_{pq} |(\varphi^p_+, \varphi^q_{0-})|^2 = \| W_2 \|^2_{\text{H.S}} < \infty \tag{3.20}$$

that means

$$\overline{W}_2 = P_+ P^0_- \in \text{H.S.} \tag{3.21}$$

must be a Hilbert–Schmidt operator (H.S.). From the equation (3.15) we obtain in the same way the recursion relation

$$\sum_{q_0} A^{mn}_{p_1\cdots p_m,q_0\cdots q_{n-1}} (\varphi^q_{0-}, \varphi^q_-)$$

$$= - \sum_{k=1}^{m} (-)^{m-1+k} A^{m-1,\,n-1}_{p_1\cdots \not{p}_k \cdots p_m,q_1\cdots q_{n-1}} (\varphi^{pk}_{0+}, \varphi^q_-)$$

$$m = 0, 1, 2, \ldots, n = 1, 2, \ldots \tag{3.22}$$

and the second necessary condition

$$\overline{W}^+_3 = P_- P^0_+ \in \text{H.S.} \quad \text{or} \quad \overline{W}_3 = P^0_+ P_- \in \text{H.S.} \tag{3.23}$$

It follows from (3.23) that $W_3{}^+ W_3$ is a positive trace-class operator. Consequently the kernel $\mathfrak{n}$ of

$$W^+_1 W_1 = 1 - W^+_3 W_3$$

$$\mathfrak{n} = \text{Ker } W^+_1 W_1 = \{ f \in \mathfrak{h}^0_+ \mid W^+_1 W_1 f = 0 \} = \text{Ker } W_1 \tag{3.24}$$

is finite-dimensional.

We first consider the case $\mathfrak{n} = 0$ (a 'weak' Bogoliubov transformation in the terminology of Labonté [5]). Then W_1 is bounded away from 0

$$\| W_1 f \| \geq \varepsilon \| f \|, \quad \varepsilon > 0 \tag{3.25}$$

so that there exists a bounded inverse W_1^{-1}. A solution of the recursion relation (3.18) is now given by

$$A^{00} = 1 \tag{3.26}$$

$$A^{11}_{pq} = - (W_1^{-1})_{pk} (W_2)_{kq} \stackrel{\text{def}}{=} A_{pq} \tag{3.27}$$

$$A^{nn}_{p_1\cdots q_n} = \sum_{\pi} (-)^{\pi} A_{p_1 q\pi_1} \cdots A_{p_n q\pi_n} \tag{3.28}$$

where the sum runs over all permutations π of the symmetric group S_n. Other solutions of (3.17) are obtained by choosing, instead of (3.26), different initial conditions, namely

$$A^{m_0 0} \neq 0 \quad \text{for some } m_0 = 1, 2, \ldots, \tag{3.29}$$

or

$$A^{0n_0} \neq 0 \quad \text{for some } n_0 = 1, 2, \ldots . \tag{3.30}$$

The general solution is a linear combination of those solutions. If W_1^{-1} exists, it follows from (3.18) for $n = 0$, $m = m_0$ that the first possibility (3.29) is excluded. This leads to the following expression for the dressed vacuum

$$\Omega' = \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\sum_{pq} A_{pq} b_p^+ d_q^+\right)^n\right)\Phi \tag{3.31}$$

where Φ is an arbitrary Fock vector containing only bare positrons.

We must still take the second recursion relation (3.22) into account. Now it follows from (3.21) that the kernel $\mathfrak{n}'$ of

$$W_4^+ W_4 = 1 - W_2^+ W_2$$

$$\mathfrak{n}' = \text{Ker } W_4^+ W_4 = \text{Ker } W_4 \tag{3.32}$$

is finite-dimensional. Let us first assume $\mathfrak{n}' = 0$. Then W_4^+ has a bounded right-inverse and consequently (3.22) is solved by

$$A^{00} = 1$$

$$A_{pq}^{11} = (W_3^+)_{pk}(W_4^{+\,-1})_{kq} = A_{pq}$$

$$A_{p_1 \cdots q_n}^{nn} = \sum_{\pi} (-)^{\pi} A_{p_1 q\pi_1} \times \ldots \times A_{p_n q\pi_n}, \tag{3.33}$$

which is consistent with (3.27) because of (3.10). Furthermore, it follows from (3.22), for $m = 0$, that a different initial condition of the form (3.30) is excluded, that means Φ in (3.31) must be the bare vacuum

$$\Omega' = \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\sum_{pq} A_{pq} b_p^+ d_q^+\right)^n\right)\Omega$$

$$= \exp\left(\sum_{pq} A_{pq} b_p^+ d_q^+\right)\Omega. \tag{3.34}$$

We have still to check whether Ω' has a finite norm. This is best done with the original expression (3.17)

$$\Omega' = \Omega + \sum_{n=1}^{\infty} \sum_{\substack{p_1 < \cdots < p_n \\ q_1 < \cdots < q_n}} \sum_{\pi} (-)^{\pi} A_{p_1 q\pi_n} \times \ldots \times A_{p_n q\pi_1} b_{p_1}^+ \ldots b_{p_n}^+ d_{q_1}^+ \ldots d_{q_n}^+ \Omega$$

$$= \Omega + \sum_{n=1}^{\infty} \sum_{\substack{p_1 < \cdots < p_n \\ q_1 < \cdots < q_n}} \det(A_{p_j q_k}) b_{p_1}^+ \ldots d_{q_n}^+ \Omega,$$

where the determinant is formed by the $n \times n$ elements $A_{p_1 q_n}, \ldots, A_{p_n q_1}$. Then we get

$$\|\Omega'\|^2 = 1 + \sum_{n=1}^{\infty} \sum_{\substack{p_1 < \cdots < p_n \\ q_1 < \cdots < q_n}} |\det(A_{p_j q_k})|^2.$$

The determinants are transformed as follows

$$\|\Omega'\|^2 = 1 + \sum_n \sum_{\substack{p_1<\cdots<p_n\\ q_1<\cdots<q_n}} \det(A_{p_jq_k})(\det A^*_{p_jq_k})$$

$$= 1 + \sum_n \sum_{\substack{p_1<\cdots<p_n\\ q_1<\cdots<q_n}} \det\left(\sum_{l=1}^{n} A_{p_jq_l}A^*_{p_kq_l}\right)$$

$$= 1 + \sum_n \sum_{\substack{p_1<\cdots<p_n\\ q_1<\cdots<q_n}} \sum_{l_1\cdots l_n=1}^{n} \det(A_{p_jq_{l_j}}A^*_{p_kq_{l_k}})$$

$$= 1 + \sum_n \sum_{\substack{p_1<\cdots<p_n\\ q_1\cdots\cdots q_n}} \det(A_{p_jq_j}A^*_{p_kq_j})$$

$$= 1 + \sum_n \sum_{p_1<\cdots<p_n} \det(A^+A)_{p_kp_j} = \det(1 + A^+A). \tag{3.35}$$

This is finite because A^+A is a trace class operator.

Let us now consider the general situation

$$\dim \mathfrak{n} = N \qquad \dim \mathfrak{n}' = N'$$

where N and N' are finite. We have the direct decomposition

$$\mathfrak{h}^0_+ = \operatorname{Ker} W_1 \oplus \operatorname{Ran} W_1^+ = \mathfrak{n} \oplus \operatorname{Ran} W_1^+ \tag{3.36}$$

both subspaces being closed invariant subspaces for $W_1^+W_1$ and $W_3^+W_3$. That $\operatorname{Ran} W_1^+ = \operatorname{Ran} P^0_+P_+$ is closed follows from the fact that the operator P^0_+ as a mapping from $\mathfrak{h}_+$ to $\mathfrak{h}^0_+$ obeys $\|P^0_+f\| \geq \delta\|f\|$ for some $\delta>0$ and all $f\in\mathfrak{h}_+\cap(\operatorname{Ker} W_1^+)^\perp$, which is a simple consequence of (3.21). Since $f\in \operatorname{Ker} W_1$ implies both $P^0_+f=f$ and $P_-f=f$ and the same is true for $\operatorname{Ker} W_4^+$, it follows

$$\operatorname{Ker} W_1 = \operatorname{Ker} W_4^+ = \mathfrak{n} = \mathfrak{h}_- \cap \mathfrak{h}^0_+. \tag{3.37}$$

Consequently, we have the following direct decomposition of $\mathfrak{h}_-$

$$\mathfrak{h}_- = \mathfrak{n} \oplus \operatorname{Ran} W_4. \tag{3.38}$$

Applying the same arguments to $\mathfrak{h}^0_-$ and $\mathfrak{h}_+$, we get

$$\mathfrak{h}_+ = \mathfrak{n}' \oplus \operatorname{Ran} W_1 \tag{3.39}$$

$$\mathfrak{n}' = \mathfrak{h}_+ \cap \mathfrak{h}^0_-. \tag{3.40}$$

We choose now the basis vectors $\varphi^j_{0\pm}$ in $\mathfrak{h}^0_\pm$ and $\varphi^j_\pm$ in $\mathfrak{h}_\pm$ in such a way that

$$\mathfrak{n} = \{\varphi^1_{0+},\cdots\varphi^N_{0+}\}, \quad \varphi^k_{0+} = \varphi^k_-, k = 1,\ldots N \tag{3.41}$$

$$\mathfrak{n}' = \{\varphi^1_{0-},\cdots\varphi^{N'}_{0-}\}, \quad \varphi^k_{0-} = \varphi^k_+, k = 1,\ldots N'. \tag{3.42}$$

The transformation (3.8) then assumes the following form

$$b'_j = d^+_j \quad j = 1,\ldots N' \tag{3.43}$$

$$b'_j = \sum_{k=N+1}^{\infty} W_1^{jk}b_k + \sum_{k=N'+1}^{\infty} W_2^{jk}d^+_k, \quad j = N'+1,\cdots\infty \tag{3.44}$$

$$d'^+_j = b_j, \quad j = 1,\ldots N \tag{3.45}$$

$$d_j'^{+} = \sum_{k=N+1}^{\infty} W_3^{jk} b_k + \sum_{k=N'+1}^{\infty} W_4^{jk} d_k^{+}, \quad j = N+1, \ldots \infty. \tag{3.46}$$

Since W_1 and W_4 are invertible on $\mathfrak{n}'^{\perp} = \text{Ran } W_1$ and $\mathfrak{n}^{\perp} = \text{Ran } W_4$, respectively, the vacuum for (3.44) and (3.45) is of the form (3.31)

$$\Omega' = \exp\left(\sum_{pq} A_{pq} b_p^{+} d_q^{+}\right) \Phi. \tag{3.47}$$

Here A_{pq} is given by (3.27) or (3.33) using for W_1^{-1} or W_4^{+-1} the restricted inverse operators defined on Ran W_1 or Ran W_4^{+}, respectively, and Φ is an arbitrary vector containing only bare electrons and positrons from $\mathfrak{n}$ and $\mathfrak{n}'$. In order to satisfy (3.14), (3.15) for the remaining operators (3.43), (3.45), Φ must be of the form

$$\Phi = b_1^{+} \cdots b_N^{+} d_1^{+} \cdots d_{N'}^{+} \Omega. \tag{3.48}$$

Consequently, the dressed vacuum (3.47) is uniquely determined (up to normalization). Summing up, we have shown that the conditions (3.21), (3.23) are necessary and sufficient for the existence of a unique dressed vacuum, or equivalently, for the dressed electron-positron operators forming a Fock representation. This answers a question raised in Reference [7]. If $N \neq N'$, the dressed vacuum Ω' becomes charged. This interesting phenomenon occurs in strong fields and is discussed in the following paper.

In the rest of this section, we will construct the unitary dressing transformation U (3.16) explicitly. This has been done for Bose fields by Friedrichs [2]. The result in the Fermi case was given by Labonté [5] without proof. We shall prove it by very simple Fock space methods. U maps bare states

$$\Phi_{mn} = d_{q_m}^{+} \cdots d_{q_1}^{+} b_{p_1}^{+} \cdots b_{p_n}^{+} \Omega \tag{3.49}$$

on the corresponding dressed states

$$\Phi'_{mn} = U\Phi_{mn} = d_{q_m}'^{+} \cdots d_{q_1}'^{+} b_{p_1}'^{+} \cdots b_{p_n}'^{+} \Omega' \tag{3.50}$$

where Ω' is the normalized vacuum from now on. On the bare vacuum Ω, U operates as follows

$$\Omega' = C_0 b_1^{+} \cdots b_N^{+} d_1^{+} \cdots d_{N'}^{+} e^{A_1} \Omega \tag{3.51}$$

where

$$A_1 = \sum_{pq} A_{pq} b_p^{+} d_q^{+} \tag{3.52}$$

and the normalization factor C_0 is given by (3.35).

On the one-electron states $q > N$ we have

$$\begin{aligned} b_q'^{+} \Omega' &= \left(\sum_{k=N+1}^{\infty} W_1^{qk*} b_k^{+} + \sum_{k=N'+1}^{\infty} W_2^{qk*} d_k \right) C_0 b_1^{+} \cdots d_{N'}^{+} e^{A_1} \Omega \\ &= C_0 b_1^{+} \cdots d_{N'}^{+} (-)^{N+N'} e^{A_1} \sum_{k=N+1}^{\infty} W_1^{qk*} b_k^{+} \Omega \\ &\quad + C_0 b_1^{+} \cdots d_{N'}^{+} (-)^{N+N'} \sum_{k=N'+1}^{\infty} W_2^{qk*} [d_k, e^{A_1}] \Omega. \end{aligned} \tag{3.53}$$

We note that exp A_1 is bounded on the vectors Φ_{mn} (3.49)

$$\|e^{A_1}\Phi_{mn}\|^2 \leq \exp \sum_{pq} |A_{pq}|^2, \tag{3.54}$$

which justifies the manipulations in (3.53) and in the following. Since

$$[d_k, e^{A_1}] = e^{A_1}[d_k, A_1] = -e^{A_1} \sum_{p=N+1}^{\infty} A_{pk} b_p^+, \tag{3.55}$$

we get

$$b_q'^+\Omega' = C_0 b_1^+ \cdots d_{N'}^+ (-)^{N+N'} e^{A_1} \times \sum_{pq'=N+1}^{\infty} \left(W_1^{q'p*} - \sum_{k=N'+1}^{\infty} W_2^{q'k*} A_{pk} \right) b_p^+ b_{q'} b_q^+ \Omega. \tag{3.56}$$

Writing

$$B = \pm (W_1^+ - AW_2^+)_{\mathrm{Ran}\, W_1} = \pm (W_1^+ + W_1^{-1} W_2 W_2^+) = \pm W_1^{-1} \tag{3.57}$$

$+$ for $N + N'$ even, $-$ for $N + N'$ odd.

U operates on the one-electron states ($j > N$) as

$$C_0 b_1^+ \cdots d_{N'}^+ e^{A_1} \sum_{pq} B_{pq} b_p^+ b_q \tag{3.58}$$

and on the vacuum Ω and the one-electron states simultaneously as

$$C_0 b_1^+ \cdots d_{N'}^+ e^{A_1} \left\{ \mathbb{1} + \sum_{pq} (B_{pq} - \delta_{pq}) b_p^+ b_q \right\}. \tag{3.59}$$

By induction, we obtain on arbitrary electron states ($j > N$)

$$U_1 = C_0 b_1^+ \cdots d_{N'}^+ e^{A_1} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{p_1 \cdots p_n \\ q_1 \cdots q_n}} (B_{p_1 q_1} - \delta_{p_1 q_1}) \cdots (B_{p_n q_n} - \delta_{p_n q_n}) b_{p_1}^+ \ldots b_{p_n}^+ b_{q_n} \ldots b_{q_1}$$
$$= C_0 b_1^+ \ldots d_{N'}^+ e^{A_1} :\exp \sum_{pq} (B_{pq} - \delta_{pq}) b_p^+ b_q:. \tag{3.60}$$

Now we add positrons ($j > N'$)

$$d_q'^+ \prod_k b_k'^+ \Omega' = \left(\sum_{p=N+1}^{\infty} W_3^{qp} b_p + \sum_{p=1}^{\infty} W_4^{qp} d_p^+ \right) U_1 \prod_k b_k^+ \Omega$$
$$= U_1 (-)^{N+N'} \left[\sum_{p=N+1}^{\infty} W_3^{qp} b_p + \sum_{pp'=N+1}^{\infty} W_3^{qp'} (B_{p'p} - \delta_{p'p}) b_p + \sum_{\substack{p=N+1 \\ q'=N'+1}}^{\infty} W_3^{qp} A_{pq'} d_{q'}^+ + \sum_{p=N'+1}^{\infty} W_4^{qp} d_p^+ \right] \prod_k b_k^+ \Omega. \tag{3.61}$$

Introducing on Ran W_4 the matrix operators

$$C = \pm (AW_3 + W_4)^T_{\mathrm{Ran}\,W_4} = \pm (W_4^{-1} W_3^+ W_3 + W_4)^T = \pm W_4^{-1*}$$

$$D = \pm (W_3 B)^T = \pm (W_3 W_1^{-1})^T,$$

$$+ \text{ if } N + N' \text{ even}, - \text{ if } N + N' \text{ odd}, \tag{3.62}$$

U operates on the states (3.61) as

$$U_1 \sum_{pq} (C_{pq} d_p^+ d_q + D_{pq} b_p d_q), \tag{3.63}$$

and combined with pure electron states ($q > N$) as

$$U_1 \left\{ 1 + \sum_{pq} [(C_{pq} - \delta_{pq}) d_p^+ d_q + D_{pq} b_p d_q] \right\}. \tag{3.64}$$

Adding successively further positrons, we get

$$\begin{aligned} U_2 &= U_1 :\exp \sum_{pq} [(C_{pq} - \delta_{pq}) d_p^+ d_q + D_{pq} b_p d_q]: \\ &= U_1 :\exp \sum_{pq} (C_{pq} - \delta_{pq}) d_p^+ d_q \exp \sum_{pq} D_{pq} b_p d_q: \\ &= C_0 b_1^+ \ldots d_{N'}^+ \exp \sum_{pq} A_{pq} b_p^+ d_q^+ :\exp \sum_{pq} (B_{pq} - \delta_{pq}) b_p^+ b_q: \\ &\quad :\exp \sum_{pq} (C_{pq} - \delta_{pq}) d_p^+ d_q :\exp \sum_{pq} D_{pq} b_p d_q. \end{aligned} \tag{3.65}$$

Finally, we have to include the finitely many exceptional states $p \leq N$ respectively $q \leq N'$. Since, including these states, the above construction does not change, it is sufficient to consider a typical example:

$$\begin{aligned} d_4'^+ \Omega' &= \sum_{k=1}^{N} W_3^{qk} b_k C_0 b_1^+ \ldots b_N^+ d_1^+ \ldots d_{N'}^+ e^{A_1} \Omega \\ &= C_0 e^{A_1} \sum_{k=1}^{N} W_3^{qk} (-)^{k-1} b_1^+ \ldots \not{b}_k^+ \ldots b_N^+ d_1^+ \ldots d_{N'}^+ \Omega \\ &= C_0 e^{A_1} \sum_{p=1}^{N} \sum_{q'=1}^{N'} W_3^{qp} (-)^{p-1} b_1^+ \ldots \not{b}_p^+ \ldots b_N^+ d_1^+ \ldots d_{N'}^+ d_{q'} d_q^+ \Omega \\ &= C_0 e^{A_1} \sum_{pq'} W_3^{q'p} (-)^{N+N'-1} :b_1^+ \ldots d_{q'} \ldots b_N^+ d_1^+ \ldots d_{N'}^+ : d_q^+ \Omega. \end{aligned} \tag{3.66}$$

From this, the general form of the factor U_0 transforming the exceptional states is obvious

$$\begin{aligned} U_0 = &:\left(b_1^+ \mp \sum_q W_3^{q1} d_q\right) \ldots \left(b_N^+ \mp \sum_q W_3^{qN} d_q\right) \\ &\times \left(d_1^+ \mp \sum_p W_2^{p1*} b_p\right) \ldots \left(d_{N'}^+ \mp \sum_p W_2^{pN'*} b_p\right): \end{aligned}$$

$$- \text{ if } N + N' \text{ even}, + \text{ if } N + N' \text{ odd}. \tag{3.67}$$

Then we have the following final result for the dressing transformation U

$$U = C_0 U_0 \exp \sum_{pq} A_{pq} b_p^+ d_p^+ :\exp \sum_{pq} (B_{pq} - \delta_{pq}) b_p^+ b_q: \\ :\exp \sum_{pq} (C_{pq} - \delta_{pq}) d_p^+ d_q :\exp \sum_{pq} D_{pq} b_p d_q. \tag{3.68}$$

Since after construction, U maps the basis of bare states (3.49) on the basis of dressed states (3.50), it extends to a unitary transformation on all of $\mathscr{F}$. The result (3.68) is in normal ordered form.[2]) The normal ordered exponentials can be transformed into ordinary exponentials by means of the following lemma, which may be of its own interest.

Lemma. *Let* $H = (H_{pq})$ *be a matrix operator and*

$$\tilde{H} = \sum_{pq} H_{pq} b_p^+ b_q.$$

Then

$$\exp \alpha\tilde{H} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{p_1 \cdots p_n \\ q_1 \cdots q_n}} (e^{\alpha H} - 1)_{p_1 q_1} \ldots (e^{\alpha H} - 1)_{p_n q_n} \\ \times b_{p_n}^+ \ldots b_{p_1}^+ b_{q_1} \ldots b_{q_n}. \tag{3.69}$$

Proof. We show that the right-hand side of (3.69) satisfies the differential equation

$$\frac{d}{d\alpha} e^{\alpha\tilde{H}} = e^{\alpha\tilde{H}} \tilde{H}. \tag{3.70}$$

$$e^{\alpha\tilde{H}} \tilde{H} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{p_1 \cdots p_{n+1} \\ q_1 \cdots q_{n+1}}} (e^{\alpha H} - 1)_{p_1 q_1} \ldots (e^{\alpha H} - 1)_{p_n q_n} H_{p_{n+1} q_{n+1}} \\ \times b_{p_n}^+ \ldots b_{p_1}^+ b_{q_1} \ldots b_{q_n} b_{p_{n+1}}^+ b_{q_{n+1}}.$$

This we have to order normally

$$= \sum_n \frac{1}{n!} \sum_{\substack{p_1 \ldots p_n \\ q_1 \ldots q_{n+1}}} \sum_{m=1}^{n} (-)^{m-n} (e^{\alpha H} - 1)_{p_1 q_1} \ldots (e^{\alpha H} - 1)_{p_m q_m} H_{q_m q_{n+1}} \ldots \\ b_{p_n}^+ \ldots b_{p_1}^+ b_{q_1} \ldots \not{b}_{q_m} \ldots b_{q_n} b_{q_{n+1}} \\ + \sum_n \frac{1}{n!} \sum_{\substack{p_1 \ldots p_{n+1} \\ q_1 \ldots q_{n+1}}} (e^{\alpha H} - 1)_{p_1 q_1} \ldots (e^{\alpha H} - 1)_{p_n q_n} H_{p_{n+1} q_{n+1}} \\ \times b_{p_{n+1}}^+ \ldots b_{p_1}^+ b_{q_1} \ldots b_{q_{n+1}}.$$

$$= \sum_n \frac{1}{n!} \sum_{\substack{p_1 \ldots p_n \\ q_1 \ldots \not{q}_m \ldots q_{n+1}}} \sum_{m=1}^{n} (-)^{m-n} \ldots (e^{\alpha H} H)_{p_m q_{n+1}} \ldots \\ b_{p_n}^+ \ldots b_{p_1}^+ b_{q_1} \ldots \not{b}_{q_m} \ldots b_{q_n} b_{q_{n+1}}$$

---

[2]) A somewhat different normal ordered form of U was recently given by S. N. M. Ruijsenaars [13].

$$- \sum_n \frac{1}{n!} \sum_{\substack{p_1 \dots p_n \\ q_1 \dots q_m \dots q_{n+1}}} \sum_m (-)^{m-n} \dots (e^{\alpha H} - 1)_{p_m q_m} \dots H_{p_m q_{n+1}}$$

$$\times\, b^+_{p_n} \dots b^+_{pm} \dots b^+_{p_1} b_{q_1} \dots \not{b}_{q_m} \dots b_{q_{n+1}}$$

$$+ \sum_n \frac{1}{n!} \sum_{\substack{p_1 \dots p_{n+1} \\ q_1 \dots q_{n+1}}} (e^{\alpha H} - 1)_{p_1 q_1} \dots (e^{\alpha H} - 1)_{p_n q_n} H_{p_{n+1} q_{n+1}}$$

$$\times\, b^+_{p_{n+1}} \dots b^+_{p_1} b_{q_1} \dots b_{q_{n+1}}$$

$$= \frac{d}{d\alpha} e^{\alpha \tilde{H}} - \sum_n \frac{n}{n!} \sum_{\substack{p_1 \dots p_n \\ q_1 \dots q_n}} (e^{\alpha H} - 1)_{p_1 q_1} \dots (e^{\alpha H} - 1)_{p_{n-1} q_{n-1}} H_{p_n q_n}$$

$$\times\, b^+_{p_n} \dots b_{q_n}$$

$$+ \sum_n \frac{1}{n!} \sum_{\substack{p_1 \dots p_{n+1} \\ q_1 \dots q_{n+1}}} (e^{\alpha H} - 1)_{p_1 q_1} \dots (e^{\alpha H} - 1)_{p_n q_n} H_{p_{n+1} q_{n+1}}$$

$$\times\, b^+_{p_{n+1}} \dots b_{q_{n+1}}.$$

Since the last two members cancel, the lemma is proved.

According to the lemma we have

$$: \exp \sum_{pq} (B_{pq} - \delta_{pq}) b^+_p b_q : = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (B - 1)_{p_1 q_1} \dots$$

$$(B - 1)_{p_1 q_n} b^+_{p_n} \dots b^+_{p_1} b_{q_1} \dots b_{q_n}$$

$$= \exp \sum_{pq} B'_{pq} b^+_p b_q$$

where B' is given by

$$(e^{B'})_{pq} = B_{pq}. \tag{3.71}$$

Then the dressing transformation (3.68) can be written as follows

$$U = C_0 U_0 \exp \sum_{pq} A_{pq} b^+_p d^+_q \exp \sum_{pq} B'_{pq} b^+_p b_q$$
$$\exp \sum_{pq} C'_{pq} d^+_p d_q \exp \sum_{pq} D_{pq} b_p d_q \tag{3.72}$$

with B'_{pq}, C'_{pq} determined by

$$(e^{-B'})_{pq} = (B^{-1})_{pq} = (W_1)_{pq} \qquad (e^{-C'})_{pq} = (C^{-1})_{pq} = (W_4)^*_{pq}. \tag{3.73}$$

This is essentially the form of U given by Labonté [5].

4. Regular External Fields

In this section, we investigate under what assumptions on the external fields the fundamental conditions (3.21) (3.23)

$$P_+ P^0_- \in \text{H.S.} \qquad P^0_+ P_- \in \text{H.S.} \tag{4.1}$$

are satisfied. Both conditions (4.1) are equivalent to the single condition

$$P_+ - P_+^0 \in \text{H.S.} \tag{4.2}$$

In fact, condition (3.2) implies

$$P_- - P_-^0 \in \text{H.S.}$$

and

$$P_+ P_-^0 = (P_+ - P_+^0)P_-^0 \in \text{H.S.}$$
$$P_+^0 P_- = P_+^0(P_- - P_-^0) \in \text{H.S.}$$

Conversely, it follows from (4.1) that

$$P_- P_+^0 \in \text{H.S.}$$

and

$$P_+ P_-^0 - P_- P_+^0 = P_+ P_-^0 - (1 - P_+)P_+^0 = P_+ - P_+^0 \in \text{H.S.}$$

The projection operators in (4.2) are conveniently expressed in terms of the resolvent

$$R(z) = (H - z)^{-1} \tag{4.3}$$

as follows [8, p. 359]

$$P_+ = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(i\eta)\, d\eta. \tag{4.4}$$

We assume that 0 is not an eigenvalue of H, otherwise one must agree upon some convention for the definition of P_+; we return to this problem later on in the discussion of strong fields (see next paper Section 3). Then $R(z)$ is bounded for all $z = i\eta$ and the integral (4.4) is (at least) strongly convergent. Writing

$$H = H_0 + V, \tag{4.5}$$

we have for the resolvent (4.3) the formal equation

$$\begin{aligned} R &= R_0 - R_0VR_0 + R_0VR_0VR_0(1 + VR_0)^{-1} \\ &\stackrel{\text{def}}{=} R_0 + R_1 + R_2 \end{aligned} \tag{4.6}$$

where

$$R_0(z) = (H_0 - z)^{-1}. \tag{4.7}$$

Then we have to consider

$$P_+ - P_+^0 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\eta (R_1 + R_2). \tag{4.8}$$

At first let us look at

$$Q_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\eta R_1(i\eta) \tag{4.9}$$

in the case of a scalar potential $V(\mathbf{x})$. This operator Q_1 acts in p-space as an integral operator

$$(Q_1 f)(\mathbf{p}) = \int d^3 q Q_1(\mathbf{p}, \mathbf{q}) f(\mathbf{q}) \tag{4.10}$$

with the kernel

$$\begin{aligned} Q_1(\mathbf{p}, \mathbf{q}) &= -(2\pi)^{-5/2} \int d\eta \frac{\boldsymbol{\alpha}\cdot\mathbf{p} + m\beta + i\eta}{\mathbf{p}^2 + m^2 + \eta^2} \hat{V}(\mathbf{p} - \mathbf{q}) \frac{\boldsymbol{\alpha}\cdot\mathbf{q} + m\beta + i\eta}{\mathbf{q}^2 + m^2 + \eta^2} \\ &= -(2\pi)^{-5/2}\pi \frac{\hat{V}(\mathbf{p} - \mathbf{q})}{E_p + E_q}\left(\frac{\boldsymbol{\alpha}\cdot\mathbf{p} + m\beta}{E_p}\frac{\boldsymbol{\alpha}\cdot\mathbf{q} + m\beta}{E_q} - 1\right). \end{aligned} \tag{4.11}$$

Writing

$$\frac{\boldsymbol{\alpha}\cdot\mathbf{p} + m\beta}{E_p} = 1 - 2P^0_-(\mathbf{p}) = P^0_+(\mathbf{p}) - P^0_-(\mathbf{p}) \tag{4.12}$$

and using the fact that $P^0_\pm(\mathbf{p})$ are projection operators in $\mathbb{C}^4$, we obtain for the Hilbert–Schmidt norm of Q_1

$$\begin{aligned} \|Q_1\|^2_{\text{H.S.}} &= \int d^3 p \int d^3 q Sp Q_1(p, q)^+ Q_1(p, q) \\ &= (2\pi)^{-3} \int d^3 p \int d^3 q \frac{|V(\mathbf{p} - \mathbf{q})|^2}{(E_p + E_q)^2} Sp[P^0_+(\mathbf{q})P^0_-(\mathbf{p}) + P^0_-(\mathbf{q})P^0_+(\mathbf{p})] \\ &= 2(2\pi)^{-3} \int d^3 p \int d^3 q \frac{|V(\mathbf{p} - \mathbf{q})|^2}{(E_p + E_q)^2}\left(1 - \frac{\mathbf{p}\cdot\mathbf{q} + m^2}{E_p E_q}\right) \\ &\overset{\text{def}}{=} 2(2\pi)^{-3} |||V|||^2_0 \end{aligned} \tag{4.13}$$

where the trace and the adjoint are taken in $\mathbb{C}^4$.
We are now looking for potentials $\hat{V}(\mathbf{p})$ with a finite norm $|||V|||_0$ (4.13).

Introducing the variables of integration

$$\mathbf{p}_1 = \mathbf{p} - \mathbf{q} \qquad \mathbf{p}_2 = \mathbf{p} + \mathbf{q}, \tag{4.14}$$

we can write (4.13) as follows

$$|||V|||^2_0 = \int d^3 p_1 A(p_1) |\hat{V}(\mathbf{p}_1)|^2, \tag{4.15}$$

where the function $A(p_1)$ can be expressed in terms of complete elliptic integrals of the first and second kind, $K(k)$ and $E(k)$,

$$\begin{aligned} A(p_1) = \text{const}\Bigg\{ &\frac{K(k)}{\sqrt{p_1^2 + 4m^2}}\left(\frac{32m^4}{3p_1^2} - \frac{4}{3}m^2\right) \\ &+ E(k)\sqrt{p_1^2 + 4m^2}\left(\frac{2}{3} - \frac{8}{3}\frac{m^2}{p_1^2}\right)\Bigg\} \end{aligned} \tag{4.16}$$

$$k = \sqrt{\frac{p_1^2}{p_1^2 + 4m^2}} \qquad p = |\mathbf{p}_1|$$

(details are given in the Appendix). Expanding (4.16) for $p_1 \to \infty$ and $p_1 \to 0$, we find

$$A(p_1) \sim p_1 \quad \text{for } p_1 \to \infty$$
$$A(p_1) \sim p_1^2 \quad \text{for } p_1 \to 0 \tag{4.17}$$

which leads to the conditions

$$\int_{|\mathbf{p}| \geq a} d^3p p |\hat{V}(\mathbf{p})|^2 < \infty \tag{4.18}$$

$$\int_{|\mathbf{p}| \leq a} d^3p p^2 |\hat{V}(\mathbf{p})|^2 < \infty \tag{4.19}$$

for some finite $a > 0$. That means $\hat{V}(\mathbf{p})$ can be decomposed into

$$\hat{V}(\mathbf{p}) = \hat{V}_1(\mathbf{p}) + \hat{V}_2(\mathbf{p}) \tag{4.20}$$

with

$$\text{supp } \hat{V}_1 \subset \{|\mathbf{p}| \geq a\}, \qquad \text{supp } \hat{V}_2 \subset \{|\mathbf{p}| \leq a\}$$

and $\hat{V}_1$, $\hat{V}_2$ satisfying (4.18) and (4.19) respectively. We shall denote this class of potentials $V(\mathbf{x})$ by

$$V(\mathbf{x}) \in (L^2_{1/2} + L^2_1))(\mathbb{R}^3) \tag{4.21}$$

according to current terminology [9].

Let us next show the self-adjointness of H on $D(H_0)$ for this class of potentials (4.21). Owing to the well-known theorem of Kato [8, p. 377], it is sufficient to prove that V is H_0- bounded, that means an estimate of the form

$$\|Vf\| \leq a\|H_0 f\| + b\|f\|, \quad f \in D(H_0) \tag{4.22}$$

holds with $a < 1$. This is trivial for V_2, because it follows from (4.19) that $\hat{V}_2 \in L^1(\mathbb{R}^3)$ and $V_2 \in C^0(\mathbb{R}^3)$, hence

$$\|V_2 f\| \leq \|V_2\|_\infty \|f\| \leq \|\hat{V}_2\|_1 \|f\|. \tag{4.23}$$

For $V_1 \in L^2_{1/2}(\mathbb{R}^3)$ more refined estimates are necessary. In this case $V_1(\mathbf{x})$ can be expressed as a convolution of the Bessel potential

$$G_{1/2}(x) = \text{const } e^{-|\mathbf{x}|} \int_0^\infty e^{-t|\mathbf{x}|} \left(t + \frac{t^2}{2}\right)^{3/4} dt \tag{4.24}$$

with an L^2- function $w_1(\mathbf{x})$ [9]

$$V_1 = G_{1/2} * w_1. \tag{4.25}$$

Then, since

$$|G_{1/2}(x)| \leq \text{const}|x|^{-5/2},$$

it follows from the generalized (weak) Young inequality [10, p. 32]

$$\|V_1\|_3 \leq \|G_{1/2}\|_{6/5,w} \|w_1\|_2 \leq \text{const}\| \, |x|^{-5/2} \|_{6/5,w} \|w_1\|_2$$

that

$$V_1(\mathbf{x}) \in L^3(\mathbb{R}^3). \tag{4.26}$$

This implies by Hölder's and Sobolev's inequalities [10, p. 113]

$$\|V_1 f\| \leq \|V_1\|_3 \|f\|_6 \leq \text{const}\|V_1\|_3 \sum_{j=1}^{3} \left\|\frac{\partial f}{\partial x_j}\right\|.$$

Using

$$\|H_0 f\|^2 = \sum_j \left\|\frac{\partial f}{\partial x_j}\right\|^2 + m^2 \|f\|^2,$$

it follows

$$\begin{aligned}\|V_1 f\|^2 &\leq \text{const } \|V_1\|_3^2 3 \sum_j \left\|\frac{\partial f}{\partial x_j}\right\|^2 \\ &\leq \text{const}\|V_1\|_3^2 \|H_0 f\|^2.\end{aligned}$$

This proves (4.22), because $\|V_1\|_3$ can be made arbitrarily small by choosing the decomposition of V (4.20) appropriately. By a slight extension of these arguments [11], it can even be proved that V is H_0-compact, that means

$$V(H_0 - \lambda)^{-1} \text{ is compact} \tag{4.27}$$

for every λ not in the spectrum of H_0. This has the important consequence that H has the same essential spectrum as H_0, i.e. $(-\infty, -m] \cup [m, +\infty)$. Then, in the interval $(-m, m)$ H has only isolated eigenvalues of finite multiplicity. Furthermore, it is possible to define generalized eigenfunctions $u(\mathbf{p}, \mathbf{x})$ and to prove the eigenfunction expansions mentioned in Section 1 [6].

Now we have to consider the remaining term in (4.8)

$$Q_2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\eta R_2(i\eta), \tag{4.28}$$

where

$$R_2 = R_0 V R_0 V R_0 (1 + V R_0)^{-1}. \tag{4.29}$$

We have not succeeded in proving that Q_2 is a Hilbert–Schmidt operator for the whole class of potentials (4.21); we can show this at the moment only for a restricted class.*) The Hilbert–Schmidt norm of Q_2 is estimated as follows

$$\|Q_2\|_{\text{H.S.}} \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\eta \|R_0 V R_0\|_{\text{H.S.}} \|V R_0\| \quad \|(1 + V R_0)^{-1}\|. \tag{4.30}$$

We have

$$\begin{aligned}\|R_0 V R_0\|_{\text{H.S.}}^2 &= \int d^3p \int d^3q \frac{|\hat{V}(\mathbf{p} - \mathbf{q})|^2}{(\mathbf{p}^2 + \eta_0^2)(\mathbf{q}^2 + \eta_0^2)} \\ &= \int d^3k |\hat{V}(\mathbf{k})|^2 \int d^3p \frac{1}{(\mathbf{p}^2 + \eta_0^2)(\mathbf{p} - \mathbf{k})^2 + \eta_0^2)}\end{aligned} \tag{4.31}$$

with

$$\eta_0^2 = \eta^2 + m^2.$$

*) See note added in proof.

The inner integral can be carried out

$$\|R_0 V R_0\|^2_{\text{H.S.}} = \int d^3k |\hat{V}(\mathbf{k})|^2 J(k) \tag{4.32}$$

where

$$J(k) = \frac{2\pi^2}{k}\left(\frac{\pi}{2} - \operatorname{arctg}\frac{2\eta_0}{k}\right). \tag{4.33}$$

Since $J(k) \to \pi^2/\eta_0$ for $k \to 0$, we must require $\hat{V} \in L^2$ for the convergence of (4.32) at $k = 0$, which is a restriction of the infrared condition (4.19). Then, since

$$J(k) \le \frac{\pi^2}{\eta_0} \quad \text{for } k \le 2\eta_0$$

and

$$J(k) \le \text{const}/k \quad \text{for all } k,$$

which implies

$$J(k) \le \text{const}/\eta_0 \quad \text{for all } k, \tag{4.34}$$

we get

$$\|R_0 V R_0\|_{\text{H.S.}} \le \text{const}\|\hat{V}\|_2 (\eta^2 + m^2)^{-1/4}. \tag{4.35}$$

Now for (4.30) being integrable at infinity, we need additional negative powers of η from the second factor under the integral (4.30).

We return to the decomposition (4.20). For V_2 we have

$$\|V_2 R_0\| \le \|V_2\|_\infty \|R_0\| = \|V_2\|_\infty \frac{1}{\sqrt{\eta^2 + m^2}}, \tag{4.36}$$

which decreases rapidly enough for $\eta \to \infty$. On V_1, however, we must impose an additional ultraviolet restriction. Instead of (4.26) let us assume

$$V_1(\mathbf{x}) \in L^{3+\varepsilon}(\mathbb{R}^3). \tag{4.37}$$

Then, following essentially an argument by Prosser [12], Hölder's inequality gives

$$\|V_1 f\| \le \|V_1\|_{3+\varepsilon} \|f\|_s, \quad s = \frac{6 + 2\varepsilon}{1 + \varepsilon}, \tag{4.38}$$

and the Hausdorff–Young inequality implies

$$\|f\|_s \le (2\pi)^{3/2 - 3/r} \|\hat{f}\|_r, \quad r = \frac{6 + 2\varepsilon}{5 + \varepsilon}. \tag{4.39}$$

This can be estimated by Hölder's inequality again

$$\begin{aligned}\|\hat{f}\|_r &= \|(\sqrt{p^2 + m^2} + M)\hat{f}(\sqrt{p^2 + m^2} + M)^{-1}\|_r \\ &\le \|(\sqrt{p^2 + m^2} + M)\hat{f}\| \; \|(\sqrt{p^2 + m^2} + M)^{-1}\|_{s_1}, \quad s_1 = 3 + \varepsilon,\end{aligned} \tag{4.40}$$

with arbitrary M. Since

$$\|(\sqrt{p^2 + m^2} + M)\hat{f}\| \leq \|H_0 f\| + M\|f\| \tag{4.41}$$

and

$$(\sqrt{p^2 + m^2} + M)^{-1}\|_{s_1} \leq \text{const}\, M^{3/s_1 - 1}, \tag{4.42}$$

we finally get

$$\|V_1 f\| \leq \text{const}\|V_1\|_{3+\varepsilon}(M^{-\varepsilon'}\|H_0 f\| + M^{1-\varepsilon'}\|f\|), \tag{4.43}$$

with

$$\varepsilon' = \frac{\varepsilon}{3 + \varepsilon}. \tag{4.44}$$

Taking

$$f = R_0 g,$$

this implies

$$\|V_1 R_0 g\| \leq \text{const}\left\{M^{-\varepsilon'}\left(1 + \frac{|\eta|}{\sqrt{\eta^2 + m^2}}\right) + M^{1-\varepsilon'}\frac{1}{\sqrt{\eta^2 + m^2}}\right\}\|g\|$$

$$\leq \text{const}\left\{2M^{-\varepsilon'} + M^{1-\varepsilon'}\frac{1}{\sqrt{\eta^2 + m^2}}\right\}\|g\|.$$

Choosing

$$M = \sqrt{\eta^2 + m^2},$$

we obtain

$$\|V_1 R_0\| \leq \frac{\text{const}}{(\eta^2 + m^2)^{\varepsilon'/2}}. \tag{4.45}$$

This factor produces enough decrease if $\varepsilon' > \frac{1}{2}$, i.e., $\varepsilon > \frac{3}{4}$, that means

$$V_1(x) \in L^{15/4+\delta}(\mathbb{R}^3) \tag{4.46}$$

for some $\delta > 0$. Finally, we must look at the last factor $(1 + VR_0)^{-1}$ in (4.30). It follows from the estimate (4.22) that

$$\|VR_0\| \leq a\left(1 + \frac{|\eta|}{\sqrt{|\eta|^2 + m^2}}\right) + \frac{b}{\sqrt{|\eta|^2 + m^2}},$$

and this becomes < 1 for $|\eta|$ large enough because the constant a can be chosen $< \frac{1}{2}$. Therefore, $(1 + VR_0)^{-1}$ is uniformly bounded on, say, $|\eta| > \eta_0$. For $|\eta| \leq \eta_0$, it is bounded as well: Since VR_0 is compact, $(1 + VR_0)^{-1}$ is meromorphic; the poles are point eigenvalues of H and therefore lie on the real axis. Then $(1 + VR_0)^{-1}$ is bounded on the imaginary axis because we have assumed that 0 is not eigenvalue of H. Hence, $(1 + VR_0)^{-1}$ is uniformly bounded in (4.30), leading to a finite Hilbert–Schmidt norm. Summing up, we have obtained the following class of regular potentials:

$$V(\mathbf{x}) \in (L^{15/4+\delta} + L^2)(\mathbb{R}^3) \tag{4.47}$$

with (4.18)

$$\int_{|\mathbf{p}|\geqslant a} d^3p p|\hat{V}(\mathbf{p})|^2 < \infty. \tag{4.48}$$

As mentioned in the introduction, there is a second definition of regular external fields due to Bongaarts [1]. which in contrast to (4.1) reads

$$P_+^0 e^{-iHt} P_-^0 \in \text{H.S.} \quad \text{for all } t. \tag{4.49}$$

This condition ensures the existence of a Hamiltonian in Fock space (see the following paper) and has, for the time being, nothing to do with the existence of a dressed vacuum. However, the condition (4.1) implies (4.49). This follows simply from

$$P_+^0 e^{-iHt} P_-^0 = P_+^0 e^{-iHt} P_+ P_-^0 + P_+^0 P_- e^{-iHt} P_-^0.$$

We have not succeeded in proving the converse.[3]) That both conditions are actually equivalent is quite plausible from the fact that the first order condition corresponding to (4.49) [1]

$$\int d^3p \int d^3q \frac{|\hat{V}(\mathbf{p}-\mathbf{q})|^2}{(E_p+E_q)^2}\left(1 - \frac{\mathbf{p}\cdot\mathbf{q}+m^2}{E_pE_q}\right)\sin^2\frac{E_p+E_q}{2}t < \infty \quad \text{for all } t \tag{4.50}$$

implies (4.13), which follows by integration over t.

It is not difficult to give examples of scalar potentials which are not regular. The simplest one is the square-well

$$\begin{aligned} V(\mathbf{x}) &= V_0 \quad \text{if } |\mathbf{x}| \le r_0 \\ &= 0 \quad \text{if } |\mathbf{x}| > r_0. \end{aligned}$$

Then condition (4.47) is fulfilled which implies that the higher order term Q_2 (4.28) is a Hilbert–Schmidt operator. But (4.48) is not satisfied, consequently Q_1 (4.9) and therefore $P_+ - P_+^0$ are not Hilbert–Schmidt. To get a regular potential, one has to smooth out the edges of the square-well. Let us finally remark that in the case of a time independent vector potential $\mathbf{A}(\mathbf{x})$ the first order operator Q_1 (4.9) is never a Hilbert–Schmidt operator (unless $\mathbf{A} = 0$). From this, it is quite certain that regular static magnetic fields do not exist. This would be very surprising and requires further investigations.

Note added in proof

In a forthcoming paper by G. Nenciu and G. Scharf it is proved that the class of regular external fields is not larger than (4.21). In particular, no static magnetic field is regular. On the other hand, we can enlarge the class to almost all of (4.21). All potentials satisfying

$$\int_{|\mathbf{p}|\geqslant a} d^3p\, p^{1+\varepsilon}|\hat{V}(\mathbf{p})|^2 < \infty \quad \text{for some } \varepsilon > 0,$$

and (4.19) are regular.

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[3]) Labonté [4] states that this is not too difficult, however, his privately communicated proof is not correct.

Appendix

Here we will give some details of the computation of the integral (4.13)

$$J = \int d^3p \int d^3q \frac{|\hat{V}(\mathbf{p} - \mathbf{q})|^2}{(E_p + E_q)^2}\left(1 - \frac{\mathbf{p}\cdot\mathbf{q} + m^2}{E_p E_q}\right). \tag{A.1}$$

Introducing the integration variables

$$\mathbf{p} - \mathbf{q} = \mathbf{p}_1 \qquad \mathbf{p} + \mathbf{q} = \mathbf{p}_2$$

and integrating over $\mathbf{p}_2$ using spherical coordinates $\mathbf{p}_2 = (p_2, \vartheta, \varphi)$, where ϑ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$ and $\cos\vartheta = z$, we get

$$J = \int d^3p_1\, A(p_1)|\hat{V}(\mathbf{p}_1)|^2 \tag{A.2}$$

with

$$A(p_1) = 2\pi \int_0^\infty dp_2 \int_{-1}^{+1} dz \frac{p_2^2}{(E_p + E_q)^2}\left(1 - \frac{p_2^2 - p_1^2 + 4m^2}{4E_p E_q}\right) \tag{A.3}$$

$$E_p^2 = \tfrac{1}{4}(a + bz) \qquad E_q^2 = \tfrac{1}{4}(a - bz)$$

$$a = p_1^2 + p_2^2 + 4m^2 \qquad b = 2p_1p_2.$$

Next the integral over z in (A.3) can be carried out

$$A_1 = \int_{-1}^{+1} dz \frac{1}{2a + 2(a^2 - b^2z^2)^{1/2}}\left(1 - \frac{a - 2p_1^2}{(a^2 - b^2z^2)^{1/2}}\right) = A_2 + A_3 \tag{A.4}$$

$$A_2 = 2\frac{a - p_1^2}{ab}\left(\frac{a}{b} - \left(\frac{a^2}{b^2} - 1\right)^{1/2}\right) \tag{A.5}$$

$$A_3 = \frac{2}{b}\operatorname{arctg}\left(\frac{a}{b} - \left(\frac{a^2}{b^2} - 1\right)^{1/2}\right). \tag{A.6}$$

The remaining integrals over p_2 lead to elliptic integrals.

We get

$$\begin{aligned}\int_0^M dp_2 p_2^2 A_2 = &-\frac{1}{2p_1^2}\left\{\int_0^M dp_2(p_2^2 + p_1^2 + 4m^2) - \frac{1}{3}\sqrt{P(x)}\right.\\ &\left.-\frac{1}{3}(4m^2 - p_1^2)J_1 - \frac{1}{3}(4m^2 + p_1^2)^2 J_0\right\}\\ &+ \tfrac{1}{2}\{M - \tfrac{1}{2}J_1 - \tfrac{1}{2}(4m^2 - 3p_1^2)J_0 - 2p_1^2(4m^2 + p_1^2)J_3\}\end{aligned} \tag{A.7}$$

where

$$x = M^2, \qquad P(x) = x^3 + x^2(8m^2 - 2p_1^2) + x(4m^2 + p_1^2)^2$$

and

$$J_0 = \int_0^{M^2} \frac{dx}{P(x)^{1/2}} \qquad J_1 = \int_0^{M^2} dx \frac{x}{(P(x))^{1/2}}$$

$$J_3 = \int_0^{M^2} \frac{dx}{(x + 4m^2 + p_1^2)P(x)^{1/2}}. \quad \text{(A.8)}$$

The integral over A_3 (A.6) is first transformed by partial integration and then treated in the same way as A_2.

$$\int_0^M dp_2 p_2^2 A_3 = \tfrac{1}{2}M + \tfrac{1}{4}\{J_1 - 2(p_1^2 + 4m^2)J_0 + 2(p_1^2 + 4m^2)^2 J_3\}. \quad \text{(A.9)}$$

Finally, we express the integrals (A.8) for $M \to \infty$ by complete elliptic integrals of the first and second kind, $K(k)$ and $E(k)$

$$J_0 \to \frac{2}{(p_1^2 + 4m^2)^{1/2}} K(k)$$

$$J_3 \to \frac{1}{(p_1^2 + 4m^2)^{3/2}} K(k)$$

$$J_2 \to 2M + 2(p_1^2 + 4m^2)^{1/2}\{K(k) - 2E(k)\}$$

with

$$k^2 = \frac{p_1^2}{p_1^2 + 4m^2}.$$

Collecting all terms and taking the limit $M \to \infty$, we obtain the result (4.16).

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