Zeitschrift: Helvetica Physica Acta
Band: 50 (1977)
Heft: 6

Artikel: Vacuum polarization in Fock space
Autor: Klaus, M. / Scharf, G.
DOI: https://doi.org/10.5169/seals-114891

Helvetica Physica Acta, Vol. 50 (1977), Birkhäuser Verlag, Basel

Vacuum polarization in Fock space[1])

by **M. Klaus** and **G. Scharf**

Institut für Theoretische Physik der Universität Zürich, Schönberggasse 9, 8001 Zürich

(2. VII. 1977)

Abstract. The regular external field problem in quantum electrodynamics which can be treated by ordinary Fock space methods is further investigated. The renormalized Hamiltonian is discussed and the charge density $\rho(x)$ is constructed as a local limit $x' \to x$ of a non-local quantity $\rho(x, x')$. This construction involves no formal manipulations for the class of regular external potentials. The expectation value of $\rho(x)$ in the dressed vacuum Ω' gives the usual vacuum polarization potential. In strong external fields, the vacuum Ω' may become charged. This phenomenon has recently attracted large interest in connection with heavy ion collisions.

1. Introduction

In this paper we continue the discussion of the regular external field problem in quantum electrodynamics by investigating the physical consequences of the results of the foregoing article (referred to as I).

We have found that for regular external fields the Furry picture is mathematically well-defined: the free (bare) electron-positron states are changed into dressed states which are still vectors in Fock space. In particular, the dressed vacuum Ω' can be expressed in terms of the bare quantities in the following form (I. 3.47)

$$\Omega' = C b_1^+ \cdots b_N^+ \, d_1^+ \cdots d_{N'}^+ \exp\left(\sum_{\substack{p > N \\ q > N'}} A_{pq} b_p^+ \, d_q^+ \right) \Omega, \tag{1.1}$$

where the matrix operator A_{pq}, the finite numbers N, N' and the normalization factor C are uniquely determined by the external potential (I. 3.33). Ω' is the ground state of the renormalized Hamiltonian $\mathbb{H}$

$$\mathbb{H} = \int dp \, E(p)[b_s'^+(\mathbf{p}) b_s'(\mathbf{p}) + d_s'^+(\mathbf{p}) \, d_s'(\mathbf{p})], \tag{1.2}$$

which is discussed in the next section. Here b', d' are the dressed emission and absorption operators which are given in terms of the bare operators by a linear (Bogoliubov) transformation (I. 3.8)

$$\begin{aligned} b_j' &= b'(\varphi_+^j) = (\varphi_+^j, \varphi_{0+}^k) b_k + (\varphi_+^j, \varphi_{0-}^k) \, d_k^+ \\ d_j'^+ &= d'(\varphi_-^j) = (\varphi_-^j, \varphi_{0+}^k) b_k + (\varphi_-^j, \varphi_{0-}^k) \, d_k^+, \end{aligned} \tag{1.3}$$

[1]) Work supported by the Swiss National Science Foundation.

where $\varphi_{\pm}^{j}$ and $\varphi_{0\pm}^{k}$ is a basis in the electron and positron sub-spaces $\mathfrak{h}_{\pm}$ and $\mathfrak{h}_{\pm}^{0}$, defined by the one-particle Hamiltonians $H = H_0 + V$ and H_0, respectively.

The most sensitive test of Ω' is its expectation value for the charge density $\rho(x)$, which should give the well-known result for the vacuum polarization potential. The main difficulty here is to find a definition for $\rho(x)$. The usual formal procedure [1] of constructing $\rho(x)$ as a local limit $x' \to x$ of a non-local quantity $\rho(x, x')$ can be put on a sound mathematical basis in the case of regular external potentials. This has the practical advantage (besides the principle one) that, since all expressions are well-defined, we are no longer forced to take relativistic covariance as the guide in the construction which simplifies the actual calculations.

Finally, we will investigate the physical meaning of the N (respectively N') extra bare emission operators in Ω' (1.1). It turns out that these operators appear in strong external fields if eigenvalues move from the positive to the negative part of the spectrum (or *vice versa*) of the one-particle Hamiltonian H. If this happens, the total charge of the dressed vacuum Ω' (1.1) becomes different from 0. This is precisely the phenomenon that was discussed by W. Greiner and others [2] in connection with heavy ion collisions.

2. The renormalized Hamiltonian and Vacuum Polarization

Regular external fields defined by the property (I. 4.2)

$$P_+ - P_+^0 \in \text{H.S.} \tag{2.1}$$

satisfy the condition of Bongaarts [3] (see I. 4.49)

$$P_+^0 \, e^{-iHt} P_-^0 \in \text{H.S.} \quad \text{for all } t.$$

Consequently, according to the results of Bongaarts, there must exist a unique Hamiltonian $\mathbb{H}$ in Fock space which generates the time evolution of the field operators (I. 1.7)

$$\Psi(e^{iHt} f) = e^{i\mathbb{H}t} \Psi(f) \, e^{-i\mathbb{H}t}. \tag{2.2}$$

In fact, $\mathbb{H}$ can be easily written down in terms of the dressed operators (1.3)

$$\mathbb{H} = \sum_n E_n^+ b_n'^+ b_n' + \sum_m E_m^- \, d_m'^+ d_m' + \int d^3p E_p [b_s'^+(\mathbf{p}) b_s'(\mathbf{p}) + d_s'^+(\mathbf{p}) \, d_s'(\mathbf{p})], \tag{2.3}$$

where

$$E_p = +\sqrt{p^2 + m^2}$$

and E_n^+ and $-E_m^-(E_m \geq 0)$ are the positive and negative point eigenvalues of H. The expression (2.3) makes sense as a quadratic form on an appropriate domain of dressed states and gives rise to a positive self-adjoint operator in $\mathscr{F}$ in exactly the same way as in the case of the free Dirac field (see for example [4] Vol. 2, p. 218). It is easy to verify equation (2.2) or the corresponding infinitesimal relation

$$\Psi(Hf) \supseteq [\mathbb{H}, \Psi(f)], \quad f \in D(H).$$

The term 'renormalized Hamiltonian' for $\mathbb{H}$ (2.3) comes from the connection with the formal expression (I. 1.1)

$$\mathbb{H}_{\text{formal}} = \int d^3x :\Psi^+(x)H\Psi(x):.$$

Substituting (see I. 3.6)

$$b'^+_s(\mathbf{p}) = (2\pi)^{-3/2} \int d^3x \Psi^+(x)u_s(\mathbf{p}, x)$$

$$b'_s(\mathbf{p}) = (2\pi)^{-3/2} \int d^3x u^+_s(\mathbf{p}, x)\Psi(x)$$

and similarly for d'_s, d'^+_s, into (2.3), we obtain in the sense of quadratic forms

$$\mathbb{H} = \int d^3x \Psi^+(x)H\Psi(x) - (2\pi)^{-3} \int d^3x \int dp\, v^+_s(p, x)Hv_s(p, x),$$

or after normal ordering (with respect to bare operators) in the first term

$$= \int d^3x :\Psi^+(x)H\Psi(x): + (2\pi)^{-3} \int d^3x \int dp$$
$$\{v^+_{0s}(p, x)Hv_{0s}(p, x) - v^+_s(p, x)Hv_s(p, x)\}.$$

Now the first term is just $\mathbb{H}_{\text{formal}}$ which becomes a well-defined operator only after the addition of an infinite renormalization constant.

A dense set of vectors in the domain of $\mathbb{H}$ is given by the following dressed states

$$\prod_{n=1}^{N} b'^+(f_n) \prod_{m=1}^{M} d'^+(g_m)\Omega', \quad f_n, g_m \in D(H). \tag{2.4}$$

However, no such vector is in the domain of the free Hamiltonian $\mathbb{H}_0$, in general. In order to see this, it is sufficient to consider the dressed vacuum Ω' and the simple form (I. 3.34) of the dressing transformation. Then

$$\mathbb{H}_0\Omega' = \mathbb{H}_0 e^A\Omega = [\mathbb{H}_0, e^A]\Omega \tag{2.5}$$

with

$$A = \int dp \int dq A(p, q)b^+(p)d^+(q). \tag{2.6}$$

The commutator (2.5) can be computed by means of the formula

$$[\mathbb{H}_0, e^A] = e^A([\mathbb{H}_0, A] + \tfrac{1}{2}[[\mathbb{H}_0, A], A] + \cdots).$$

Since

$$[\mathbb{H}_0, A] = \int dp \int dq A(p, q)(E_p + E_q)b^+(p)d^+(q), \tag{2.7}$$

all higher commutators vanish

$$[[\mathbb{H}_0, A], A] = 0 \quad \text{etc.}$$

Consequently, we get from (2.5)

$$\|\mathbb{H}_0\Omega'\|^2 \geq \|[\mathbb{H}_0, A]\Omega\|^2 = \int dp \int dq|A(p, q)|^2 (E_p + E_q)^2. \tag{2.8}$$

This integral, in general, is diverging, because only the integral without the energy factor is convergent for regular external fields (I 3.52), which shows that $\Omega' \notin D(\mathbb{H}_0)$. It follows from these findings that the set of vectors in $D(\mathbb{H}) \cap D(\mathbb{H}_0)$ is highly not dense. For this reason, it is impossible to define the interaction Hamiltonian

$$e \int d^3x V(\mathbf{x})\psi^+(x)\psi(x) \tag{2.9}$$

by the difference $\mathbb{H} - \mathbb{H}_0$. On the other hand, it is very desirable to define the charge density $\rho(x)$. To do this, we therefore proceed in a different way.

Guided by the well-known formal procedure of quantum electrodynamics [1], we start with the bounded operator in Fock space

$$\rho(f, g) = \tfrac{1}{2} \int d^3x \int d^3x' f(\mathbf{x}) g(\mathbf{x}') \sum_a \{\psi_a^+(\mathbf{x}')\psi_a(\mathbf{x}) - \psi_a(\mathbf{x})\psi_a^+(\mathbf{x}') \tag{2.10}$$

where $f, g \in L^2(\mathbb{R}^3)$. We will study its expectation value in the dressed vacuum Ω'

$$(\Omega', \rho(f, g)\Omega') = -\tfrac{1}{2}(f, Sp(P_+ - P_-)g), \tag{2.11}$$

where the trace is taken in $\mathbb{C}^4$. For test functions $f, g \in S(\mathbb{R}^3)$ the L^2-scalar product (2.11) can be looked at as a bilinear functional $Sp(P_+ - P_-)$ on $S(\mathbb{R}^3) \times S(\mathbb{R}^3)$, continuous in both arguments. According to the nuclear theorem, there exists a tempered kernel distribution $\rho(\mathbf{x}, \mathbf{x}') \in S'(\mathbb{R}^6)$ such that

$$(\Omega', \rho(f, g)\Omega') = \int d^3x f(\mathbf{x})^* \int d^3x' \rho(\mathbf{x}, \mathbf{x}') g(\mathbf{x}'). \tag{2.12}$$

For regular potentials we can say even more: Since

$$Sp(P_+ - P_-) = Sp(P_+ - P_+^0 + P_-^0 - P_-)$$

is a Hilbert–Schmidt operator because of (2.1), the kernel $\rho(\mathbf{x}, \mathbf{x}')$ is given in $L^2(\mathbb{R}^6)$

$$\int d^3x \int d^3x' |\rho(\mathbf{x}, \mathbf{x}')|^2 < \infty. \tag{2.13}$$

This kernel $\rho(\mathbf{x}, \mathbf{x}')$ is the basic object to be studied.

The projections $P_\pm$ are again expressed by the resolvent (I. 4.4)

$$P_+ - P_- = \frac{1}{\pi} \int_{-\infty}^{+\infty} R(i\eta)\, d\eta.$$

Using for R the perturbation expansion (I. 4.6), the first non-vanishing contribution to $\rho(\mathbf{x}, \mathbf{x}')$ comes from the first order term $R_0 V R_0$, namely

$$\rho_1(\mathbf{x}, \mathbf{x}') = \frac{1}{\pi} Sp \int_{-\infty}^{+\infty} d\eta \int d^3x_1 R_0(i\eta; \mathbf{x}, \mathbf{x}_1) V(\mathbf{x}_1) R_0(i\eta; \mathbf{x}_1, \mathbf{x}') \tag{2.14}$$

where

$$R_0(i\eta; \mathbf{x}, \mathbf{y}) = (H_0 + i\eta)\frac{1}{4\pi}\frac{\exp -\sqrt{\eta^2 + m^2}|\mathbf{x} - \mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} \tag{2.15}$$

is the kernel of the free resolvent. Let us first consider

$$\begin{aligned} K_1(\mathbf{x}, \mathbf{x}'; \mathbf{x}_1) &= \frac{1}{\pi} Sp \int d\eta R_0(i\eta; \mathbf{x}, \mathbf{x}_1) R_0(i\eta; \mathbf{x}_1, \mathbf{x}') \\ &= \frac{4}{\pi}\int d\eta(\vec{\partial}_x \vec{\partial}_{x'} + m^2 - \eta^2)\frac{\exp -\sqrt{\eta^2 + m^2}|\mathbf{x} - \mathbf{x}_1|}{4\pi|\mathbf{x} - \mathbf{x}_1|} \\ &\qquad \frac{\exp -\sqrt{\eta^2 + m^2}|\mathbf{x}_1 - \mathbf{x}'|}{4\pi|\mathbf{x}_1 - \mathbf{x}'|} \end{aligned} \tag{2.16}$$

The distribution K_1 depends only on the variables

$$\xi = \mathbf{x} = \mathbf{x}' \qquad \mathbf{y} = \tfrac{1}{2}(\mathbf{x} + \mathbf{x}') - \mathbf{x}_1 \tag{2.17}$$

$$\begin{aligned} K_1(\xi, \mathbf{y}) &= \frac{4}{\pi}\int d\eta(\tfrac{1}{4}\Delta_y - \Delta_\xi + m^2 - \eta^2)\frac{\exp -\sqrt{\eta^2 + m^2}|\mathbf{y} + \tfrac{1}{2}\xi|}{4\pi|\mathbf{y} + \tfrac{1}{2}\xi|} \\ &\qquad \frac{\exp -\sqrt{\eta^2 + m^2}|\mathbf{y} - \tfrac{1}{2}\xi|}{4\pi|\mathbf{y} - \tfrac{1}{2}\xi|} \end{aligned} \tag{2.18}$$

This expression is most easily handled using its (distributional) Fourier transform

$$\begin{aligned} \hat{K}_1(\mathbf{q}, \mathbf{k}) &= \frac{4}{\pi}\int d\eta \frac{\mathbf{q}^2 - \tfrac{1}{4}\mathbf{k}^2 + m^2 - \eta^2}{[(\mathbf{q} + \tfrac{1}{2}\mathbf{k})^2 + m^2 + \eta^2][(\mathbf{q} - \tfrac{1}{2}\mathbf{k})^2 + m^2 + \eta^2]} \\ &= \frac{2}{\pi}\int d\eta \frac{(\mathbf{q} + \tfrac{1}{2}\mathbf{k})^2 + m^2 + \eta^2 + (\mathbf{q} - \tfrac{1}{2}\mathbf{k})^2 + m^2 + \eta^2 - \mathbf{k}^2 - 4\eta^2}{[(\mathbf{q} + \tfrac{1}{2}\mathbf{k})^2 + m^2 + \eta^2][(\mathbf{q} - \tfrac{1}{2}\mathbf{k})^2 + m^2 + \eta^2]}. \end{aligned} \tag{2.19}$$

The charge density is obtained from $\rho(\mathbf{x}, \mathbf{x}')$ by the local limit $\mathbf{x}' \to \mathbf{x}$, i.e. $\xi \to 0$. For this reason, we compute

$$K_1(\xi, \mathbf{k}) = \int d^3q\, e^{-iq\cdot\xi}\hat{K}_1(\mathbf{q}, \mathbf{k}) \tag{2.20}$$

(always in the sense of tempered distributions). The integrations in (2.19) and (2.20) now combine to a 4-dimensional integral with respect to the *euclidean* 4-vector $q = (\eta, \mathbf{q})$. Introducing the two other euclidean 4-vectors

$$k = (0, \mathbf{k}), \qquad \xi = (\xi_0, \boldsymbol{\xi}), \tag{2.21}$$

we arrive at

$$\begin{aligned} K_1(\xi, k) = \frac{2}{\pi}\int d^4q\, e^{-iq\cdot\xi}\Bigg\{&\frac{1}{(q - \tfrac{1}{2}k)^2 + m^2} + \frac{1}{(q + \tfrac{1}{4}k)^2 + m^2} \\ &- \frac{\mathbf{k}^2 + 4q_0^2}{[(q - \tfrac{1}{2}k)^2 + m^2][(q + \tfrac{1}{2}k)^2 + m^2]}\Bigg\}. \end{aligned} \tag{2.22}$$

The remaining 4-dimensional euclidean Fourier transforms are much more simple to compute than the usual covariant integrals, for example

$$\int d^4q \frac{\exp - iq\xi}{(q - \frac{1}{2}k)^2 + m^2} = e^{-ik\xi/2} \int d^4x \frac{\exp - ix\xi}{x^2 + m^2}$$

$$= e^{-ik\xi/2} \frac{4\pi^2}{|\xi|} \int_0^\infty d|x| \frac{x^2}{x^2 + m^2} J_1(|x||\xi|)$$

$$= - \frac{4\pi^2}{|\xi|} e^{-ik\xi/2} \frac{d}{d|\xi|} \int_0^\infty d|x| \frac{|x|}{x^2 + m^2} J_0(|x||\xi|)$$

$$= - 4\pi^2 \frac{m}{|\xi|} e^{-ik\xi/2} K_0'(m|\xi|),$$

where J_ν are Bessel functions and K_ν modified Bessel functions, the prime denotes the derivative with respect to the argument. In the last integral in (2.22) Feynmann's trick

$$\frac{1}{(x^2 + m^2)[(x + k)^2 + m^2]} = \int_0^1 d\alpha (x^2 + 2\alpha x \cdot k + \alpha k^2 + m^2)^{-2}$$

is used. In this way we obtain

$$K_1(\xi, \mathbf{k}) = - 16 \frac{m}{|\xi|} \cos \tfrac{1}{2}\mathbf{k} \cdot \xi K_0'(m|\xi|)$$

$$- 4\pi \mathbf{k}^2 e^{-1/2i\mathbf{k}\cdot\xi} \int_0^1 d\alpha e^{i\alpha\mathbf{k}\cdot\xi} K_0(|\xi| \sqrt{k^2(\alpha - \alpha^2) + m^2})$$

$$+ \frac{16\pi}{|\xi|} e^{-1/2i\mathbf{k}\cdot\xi} \int_0^1 d\alpha e^{i\alpha\mathbf{k}\cdot\xi} \sqrt{k^2(\alpha - \alpha^2) + m^2}$$

$$\times K_0'(|\xi| \sqrt{k^2(\alpha - \alpha^2) + m^2}). \tag{2.23}$$

Now let us consider the local limit $\xi \to 0$. Expanding the modified Bessel functions and performing the remaining integrals we get

$$K_1(\xi, \mathbf{k}) = - \frac{4}{3}\pi \mathbf{k}^2 \left(\log m\xi - \log 2 + C - \frac{3}{2} \right) - \frac{4}{3}\pi \frac{(\mathbf{k}\cdot\xi)^2}{\xi^2}$$

$$+ \frac{4}{3}\pi \mathbf{k}^2 \left\{ \left(1 - \frac{2}{a}\right) \sqrt{1 + \frac{4}{a}} \log \frac{\sqrt{1 + \frac{4}{a}} + 1}{\sqrt{1 + \frac{4}{a}} - 1} + \frac{4}{a} - \frac{5}{3} \right\} + 0(\xi) \tag{2.24}$$

with $a = \mathbf{k}^2/m^2$, C is Euler's constant.
The last member in (2.24) is the well-known finite part of the vacuum polarization tensor; the constant $- \frac{5}{3}$ in the bracket is uniquely determined by the requirement that the bracket has to vanish in the static limit $k \to 0$, i.e., $a \to 0$. The first terms $\sim \mathbf{k}^2$ (choosing ξ parallel to $\mathbf{k}$) are the logarithmically divergent and finite charge renormalization terms $K_1^r(\xi, \mathbf{k})$, which have to be subtracted.

The next order contribution to $\rho(\mathbf{x}, \mathbf{x}')$ is of the third order in V. This contribution $\rho_3(\mathbf{x}, \mathbf{x}')$ and all higher orders remain finite for $\mathbf{x}' \to \mathbf{x}$, at least for suitably restricted potentials. This follows from estimates given by W. D. Evans [5] which ensure the finiteness of the higher orders for potentials in the Stummel class

$$\int_{|\mathbf{x}-\mathbf{y}|\leq 1} \frac{V(\mathbf{y})^2}{|\mathbf{x}-\mathbf{y}|^{1+\alpha}} d^3y < \infty \quad \text{with} \quad \alpha > \frac{4}{3}.$$

It is therefore not necessary to consider the higher orders in detail. From this discussion, it is now clear that the charge density has to be defined by the subtraction of a C-number renormalization term ρ^r

$$\rho(\mathbf{x}) = w - \lim_{\mathbf{x}'\to\mathbf{x}} \{\psi^+(\mathbf{x}')\psi(\mathbf{x}) - \psi(\mathbf{x})\psi^+(\mathbf{x}') - \rho^r(\mathbf{x}, \mathbf{x}')\}, \tag{2.25}$$

indeed, since ρ is quadratic in the field operators, only C-number counter terms are necessary. The renormalization term $\rho^r(\mathbf{x}, \mathbf{x}')$

$$\rho^r(\mathbf{x}, \mathbf{x}') = \int d^3x_1 K^r(\mathbf{x}, \mathbf{x}'; \mathbf{x}_1) V(\mathbf{x}_1) + \rho_3^r + \rho_5^r + \cdots$$

is fixed by the requirement that the vacuum polarization potential

$$\hat{V}_{vp}(\mathbf{k}) = -\frac{4\pi}{\mathbf{k}^2}(\Omega', \hat{\rho}(\mathbf{k})\Omega') \tag{2.26}$$

vanishes for $k \to 0$. The weak limit in (2.25) means that the local limit $\mathbf{x}' \to \mathbf{x}$ exists for matrix elements between dressed states like (2.4). $\rho(\mathbf{x})$ is a bilinear form on $D \times D$, where

$$D = \left\{ \prod_{n=1}^{N} b'^+(f_n) \prod_{m=1}^{M} d'^+(g_m)\Omega', f_n, g_m \in (S(\mathbb{R}^3))^4, N, M < \infty \right\}. \tag{2.27}$$

Smearing out with a test function $f(\mathbf{x}) \in S(\mathbb{R}^3)$, one gets a well-defined operator in Fock space.

After the construction of the charge density, we return once more to the Hamiltonian (2.3). If $\mathbb{H}$ is considered as an observable, it is unsatisfactory in that its expectation values for dressed one-particle states are the ordinary Dirac energy levels *without* the vacuum polarization correction. This defect can be repaired by adding to $\mathbb{H}$ the Coulomb energy in the form

$$\mathbb{H}_c = \int d^3x \int d^3x' \frac{:\rho(\mathbf{x})\rho(\mathbf{x}'):}{|\mathbf{x}-\mathbf{x}'|}, \tag{2.28}$$

where the normal ordering is meant with respect to the dressed emission and absorption operators. Now, if $b_0'^+\Omega'$ is a one-electron bound state in the external potential with square integrable Dirac wave function $u_s(p_0, \mathbf{x})$, then

$$(b_0'^+\Omega', \mathbb{H}_c b_0'^+\Omega') = \int d^3x u_s^+(p_0, \mathbf{x}) V_{vp}(\mathbf{x}) u_s(p_0, \mathbf{x}) + \text{const},$$

which is the right vacuum polarization correction. This gives a clear physical interpretation to this correction as the Coulomb energy of the dressed one-particle states. It also gives a hint, how the Coulomb term should be defined, if one goes one step further towards full Q.E.D.

3. The charged vacuum in strong external fields

In this section, we want to discuss the question whether the N (respectively N') extra emission operators in Ω' (1.1) actually appear for suitable external potentials $V(\mathbf{x})$ and what is the physical interpretation of this phenomenon.

According to the results of section 3 in I (3.37) (3.40), we must look for the possibility that

$$\mathfrak{n} = \mathfrak{h}_+^0 \cap \mathfrak{h}_- \tag{3.1}$$

or

$$\mathfrak{n}' = \mathfrak{h}_-^0 \cap \mathfrak{h}_+ \tag{3.2}$$

are nontrivial but finite dimensional subspaces (dim $\mathfrak{n} = N$, dim $\mathfrak{n}' = N'$). Let us vary the strength of the external potential by introducing a coupling constant $\lambda > 0$

$$H(\lambda) = H_0 + \lambda V \tag{3.3}$$

$$P_+ = P_+(\lambda) \quad \text{etc.} \tag{3.4}$$

Then for potentials which are H_0-bounded (for example in the class I (4.21)), we have

$$\|P_+(\lambda) - P_+^0\| < 1 \tag{3.5}$$

for sufficiently small λ.
This excludes the possibility of a nontrivial vector $f \in \mathfrak{h}_+ \cap \mathfrak{h}_-^0$ or $f \in \mathfrak{h}_+^0 \cap \mathfrak{h}_-$. On the other hand, if

$$\|P_+(\lambda) - P_+^0\| = 1, \tag{3.6}$$

then at least one of the two subspaces $\mathfrak{n}$ and $\mathfrak{n}'$ is nontrivial (note that $\|P_1 - P_2\| \leq 1$ for any two orthogonal projections P_1, P_2). This can happen only for larger λ, which we are now going to consider.

Since $H(\lambda)$ (3.3) is an analytic family of operators if V is H_0-bounded (with H_0-bound 0) [6, p. 377], one can show that

$$P_+(\lambda) - P_+^0 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [R(i\eta, \lambda) - R(i\eta, 0)]\, d\eta \tag{3.7}$$

is analytic in λ as long as the path of integration in (3.7) has no contact with the spectrum of $H(\lambda)$. In the case of regular potentials considered in I, Section 4, the boundary of the analyticity domain is reached for real λ if a point eigenvalue $\mu_1(\lambda)$ of $H(\lambda)$, which moves continuously in the gap $(-m, +m)$, arrives at 0. Let us assume that this happens for $\lambda = \lambda_1$. Then $P_+(\lambda) - P_+^0$ (3.7) is analytic in a certain complex neighbourhood D_1 of $(0, \lambda_1)$. Furthermore, for regular potentials the operators (3.7) are compact (even H.S.), such that (3.7) is an analytic family of compact operators. Then it follows from the analytic Fredholm theorem [4, Vol. 1, p. 201] that $P_+(\lambda) - P_+^0$ can have the eigenvalues ± 1 only for a discrete set of values λ in D_1. Therefore, in the interval $0 \leq \lambda < \lambda_1$ there is at most a discrete set of λ where we have a non-trivial subspace $\mathfrak{n}$ or $\mathfrak{n}'$. This situation would be very unstable because the nontrivial $\mathfrak{n}$ or $\mathfrak{n}'$ would disappear by a small change of the coupling constant λ. It can be shown by a more detailed analysis of this situation that for regular potentials this possibility is excluded at all.

Now we turn to $\lambda = \lambda_1$ where a point eigenvalue $\mu_1(\lambda)$ goes through 0, say from

above. Then the projection operators $P_\pm(\lambda)$ jump, $P_+(\lambda)$ decreases and $P_-(\lambda)$ increases by the corresponding eigenprojection P_1

$$\begin{aligned} P_+(\lambda_1 + 0) &= P_+(\lambda_1 - 0) - P_1 \\ P_0(\lambda_1 + 0) &= P_-(\lambda_1 - 0) + P_1 . \end{aligned} \tag{3.8}$$

It then follows that

$$\mathfrak{n}(\lambda_1 + 0) = \mathfrak{h}_-(\lambda_1 + 0) \cap \mathfrak{h}_+^0 \tag{3.9}$$

is no longer trivial but has the dimension of the eigenspace

$$N = \dim(P_1 \mathfrak{h}). \tag{3.10}$$

This is the consequence of a simple Lemma proven in the Appendix. After the jump the situation is again stable for $\lambda > \lambda_1$ until some other eigenvalue goes through 0. If, on the other hand, $\mu_1(\lambda)$ moves from below through 0, then $\mathfrak{n}'$ (3.2) becomes different from 0. Note that $\mathfrak{n}(\lambda_1 + 0)$ or $\mathfrak{n}'(\lambda_1 + 0)$, respectively, is not identical with $P_1 \mathfrak{h}$.

In order to understand the physical meaning of this discontinuous change in the structure of the dressed vacuum Ω', we consider the expectation value of the total charge Q

$$(\Omega', Q\Omega') = \int d^3x (\Omega', \rho(x)\Omega') \tag{3.11}$$

which according to (2.11) is formally equal to

$$= -\tfrac{1}{2} \int d^3x \, Sp(P_+ - P_-)(x, x). \tag{3.12}$$

We must not discuss the precise definition of (3.12) as a local limit $x' \to x$ again, because here we are interested only in the change ΔQ of the charge

$$\Delta Q = Q(\lambda_1 + 0) - Q(\lambda_1 - 0) = \int d^3x \, Sp P_1(x, x) \tag{3.13}$$

which is well defined and equal to

$$\Delta Q = \dim(P_1 \mathfrak{h}) = N \tag{3.14}$$

in units of e. Since after construction of the charge operator the vacuum is neutral for $\lambda < \lambda_1$

$$(\Omega', \hat{\rho}(0)\Omega') = 0,$$

it is charged ($= N e$) for $\lambda > \lambda_1$.

To illustrate this curious fact, let us assume that a simple eigenvalue moves through 0 from above if $\lambda \uparrow \lambda_1$, the corresponding eigenvector φ_+^1 then changes from $\mathfrak{h}_+$ to $\mathfrak{h}_-$ at $\lambda = \lambda_1$. For $\lambda < \lambda_1$ we have (I. 3.9)

$$\begin{aligned} W_1 &= (\varphi_+^j, \varphi_{0+}^k) \\ W_2 &= (\varphi_+^j, \varphi_{0-}^k), \qquad j \geq 1, \qquad k \geq 1 \\ A &= -W_1^{-1} W_2 \end{aligned} \tag{3.15}$$

and the vacuum is of the following form

$$\begin{aligned}\Omega'(\lambda_1 - 0) &= C \exp \sum_{pq} A_{pq} b_p^+ d_q^+ \Omega \\ &= C \exp \sum_{q} A_{1q} b_1^+ d_q^+ \exp \sum_{\substack{p \geq 2 \\ q \geq 1}} A_{pq} b_p^+ d_q^+ \Omega \\ &= C(1 + \sum_{q} A_{1q} b_1^+ d_q^+) \exp \sum_{\substack{p \geq 2 \\ q \geq 1}} A_{pq} b_p^+ d_q^+ \Omega.\end{aligned} \tag{3.16}$$

For $\lambda > \lambda_1$, on the other hand, the 'over-critical' vacuum assumes the form (I. 3.47)

$$\Omega'(\lambda_1 + 0) = \tilde{C} b_1^+ \exp \sum_{\substack{p \geq 2 \\ q \geq 1}} \tilde{A}_{pq} b_p^+ d_q^+ \Omega \tag{3.17}$$

where

$$\tilde{A} = - \underline{\tilde{W}}_1^{-1} \underline{\tilde{W}}_2 \tag{3.18}$$

and $\underline{\tilde{W}}_1$, $\underline{\tilde{W}}_2$ are obtained from (3.15) by dropping the first row $j = 1$. Since this dropping of the first row changes the inverse $\underline{\tilde{W}}_1^{-1}$ completely, the appearance of the negative charge in (3.17) is accompanied with a complete rearrangement of the bare electron-positron pairs.

In order to see the physical effects of this change, we will express the neutral vacuum $\Omega'(\lambda_1 - 0)$ (3.16) by the charged one $\Omega'(\lambda_1 + 0)$ (3.17) and the corresponding dressed operators (I. 3.43–46)

$$\begin{aligned} b'_j &= \sum_{k=2}^{\infty} \underline{\tilde{W}}_1^{jk} b_k + \sum_{k=1}^{\infty} \underline{\tilde{W}}_2^{jk} d_k^+ \quad j = 1, 2, \ldots \\ d_1'^+ &= b_1 \\ d_j'^+ &= \sum_{k=2}^{\infty} \underline{\tilde{W}}_3^{jk} b_k + \sum_{k=1}^{\infty} \underline{\tilde{W}}_4^{jk} d_k^+ \quad j = 2, 3, \ldots \end{aligned} \tag{3.19}$$

as follows

$$\begin{aligned}\Omega'(\lambda_1 - 0) &= C\left(1 + \sum_{q} A_{1q} b_1^+ d_q^+\right) \exp \sum_{\substack{p \geq 2 \\ q \geq 1}} (A_{pq} - \tilde{A}_{pq}) b_p^+ d_q^+ \\ &\qquad \exp \sum_{\substack{p \geq 2 \\ q \geq 1}} \tilde{A}_{pq} b_p^+ d_q^+ (b_1 b_1^+ \Omega) \\ &= C\tilde{C}^{-1}(1 + A_{1q} b_1^+ d_q^+) \exp \sum_{\substack{p \geq 2 \\ q \geq 1}} (A_{pq} - \tilde{A}_{pq}) b_p^+ d_q^+ \\ &\qquad d_1'^+ \Omega'(\lambda_1 + 0).\end{aligned} \tag{3.20}$$

Here the so-called 'auto-ionization' of a positron [2] shows up. The remaining operators in (3.20) create additional dressed electron-positron pairs. This can be seen by inverting (3.19), using the fact that

$$\underline{\tilde{W}} = \begin{pmatrix} \underline{\tilde{W}}_1 & \underline{\tilde{W}}_2 \\ \underline{\tilde{W}}_3 & \underline{\tilde{W}}_4 \end{pmatrix}$$

is unitary (I. 3.10),

$$b_j = \sum_{k=1}^{\infty} \tilde{W}_1^{+jk} b'_k + \sum_{k=2}^{\infty} \tilde{W}_3^{+jk} d'^{+}_k, \quad j = 2, 3, \ldots$$

$$d_j^{+} = \sum_{k=1}^{\infty} \tilde{W}_2^{+jk} b'_k + \sum_{k=2}^{\infty} \tilde{W}_4^{+jk} d'^{+}_k, \quad j = 1, 2, \ldots .$$

It should also be kept in mind that the wave function $\psi \in (L^2(\mathbb{R}^3))^4$ of the positron emission operator

$$d'^{+}_1 = d'^{+}(\psi)$$

is not simply the eigenfunction φ^1_+ of the eigenvalue going through 0, but only its part in $\mathfrak{h}^0_+$ in the direct decomposition

$$\varphi^1_+ = \psi_- + \psi, \psi_- \in \mathfrak{h}_-(\lambda_1 - 0), \psi \in \mathfrak{h}^0_+ .$$

This wave function $\psi \in \mathfrak{h}_-(\lambda_1 + 0)$ in general contains a non-vanishing contribution from the positron continuum $(-\infty, -m]$. There is then a finite probability of observing the ionized positron in a scattering state at infinity (although the Fermi energy of quantization is chosen here at 0 and not just above $-m$ as in Ref. [2]). The decay of the neutral vacuum $\Omega'(\lambda_1 - 0)$ into the negatively charged one Ω' $(\lambda_1 + 0)$ plus an additional positron and pairs is obviously not as trivial as one might think on the basis of a naive hole-theory.

On the other hand, we are aware of the limited validity of the external field model. The appearance of the charged vacuum which would manifest in a partial screening of the external charge generating the classical potential can be taken as a hint that a strong static electric field should be treated by quantum theory. For this reason, it seems to be very speculative to make definite predictions about high field effects on the basis of the classical external field model. However, the model may serve for a first qualitative orientation.

Appendix

We prove here the following simple fact about projections

Lemma *Let P_1 and P_2 be two orthogonal projections in a Hilbert space $\mathfrak{h}$ with $\| P_1 - P_2 \| < 1$, projecting on* Ran $P_j = \mathfrak{h}_j, j = 1, 2$. *Let P_3 be another orthogonal projection with finite dimensional range $\mathfrak{h}_3$,* dim $\mathfrak{h}_3 = m < \infty$, and $P_1 P_3 = P_3 P_1 = 0$, *i.e.*, $\mathfrak{h}_3 \perp \mathfrak{h}_1$. *Then*

(*i*) dim $[(\mathfrak{h}_1 \dot{\oplus} \mathfrak{h}_3) \cap \mathfrak{h}_2^{\perp}] = m$

(*ii*) dim $[(\mathfrak{h}_1 \dot{\oplus} \mathfrak{h}_3)^{\perp} \cap \mathfrak{h}_2] = 0$.

Proof. (i) It follows from $\| P_1 - P_2 \| < 1$ that we have the following (non-orthogonal) direct decompositions [6, p. 56]

$$\mathfrak{h} = \mathfrak{h}_1 \dot{\oplus} \mathfrak{h}_2^{\perp} \tag{A.1}$$

$$= \mathfrak{h}_1^{\perp} \dot{\oplus} \mathfrak{h}_2 . \tag{A.2}$$

Let $\varphi_1 \cdots \varphi_m$ be an orthogonal basis in $\mathfrak{h}_3$ which can be uniquely decomposed according to (A.1)

$$\varphi_k = \varphi_k' + \varphi_k'', \qquad \varphi_k' \in \mathfrak{h}_1, \qquad \varphi_k'' \in \mathfrak{h}_2^{\perp} \tag{A.3}$$

with $\varphi_k'' \neq 0$ since $\mathfrak{h}_3 \perp \mathfrak{h}_1$. Then

$$\varphi_k'' = \varphi_k - \varphi_k' \in \mathfrak{h}_2^{\perp} \cap (\mathfrak{h}_1 \oplus \mathfrak{h}_3), \quad k = 1 \cdots m. \tag{A.4}$$

Since the φ_k'' are linearly independent (because the φ_k (A.3) are linearly independent and $\mathfrak{h}_1 \perp \mathfrak{h}_3$), this proves (*i*).

(ii) $(\mathfrak{h}_1 \oplus \mathfrak{h}_3)^{\perp} = \mathfrak{h}_1^{\perp} \cap \mathfrak{h}_3^{\perp} \subset \mathfrak{h}_1^{\perp}$.

Now it follows from (A.2) that

$$\mathfrak{h}_1^{\perp} \cap \mathfrak{h}_2 = 0,$$

therefore

$$(\mathfrak{h}_1 \oplus \mathfrak{h}_3)^{\perp} \cap \mathfrak{h}_2 = 0.$$

REFERENCES

[1] J. G. VALATIN, Proc. Roy. Soc. (Lond) *222A*, 228 (1954).
[2] J. RAFELSKI, B. MÜLLER and W. GREINER, Nucl. Phys. *B.68*, 585 (1974).
[3] P. J. M. BONGAARTS, Annals of Phys. *56*, 108 (1970).
[4] M. REED and B. SIMON, *Methods of Modern Mathematical Physics*, Vols 1 and 2 (Academic Press, New York, London 1975).
[5] W. D. EVANS, Proc. London Math. Soc. *3*, *20*, 546 (1970).
[6] T. KATO, *Perturbation Theory for Linear Operators*, 2nd edn (Springer-Verlag, Berlin, Heidelberg, New York 1976).