Spin corrections to the two-body eikonal amplitude

Spin corrections to the two-body eikonal amplitude[1])

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Abstract. Using the one-parameter eikonal representation and α-space techniques developed in an earlier paper, the first-order spin corrections are obtained and its high-energy behaviour proved to be non-negligible. In the limit where the range of interaction goes to infinity, bound-states, in the electron positron annihilation region, appear as poles in the s-channel and Regge trajectories in the t-channel, as in the usual eikonal model. Spin corrections are associated with double poles.

1. Introduction

The one-parameter-eikonal-approximation (OPEA) has been introduced by Lévy and Sucher [1] as an alternative form of the usual relativistic eikonal approximation (REA), which has the virtue of involving integration over a real parameter instead of four-dimensional integration over the whole space-time.

Spin corrections to the eikonal approximation have recently been computed by Lévy and Léger [2], who have considered the scattering of a spin-$\frac{1}{2}$ particle by a Coulomb field.

In this work we shall compute the spin corrections for two-body relativistic reactions. We consider an interaction between spin-$\frac{1}{2}$ fermions and 'scalar' photons, as $\mathscr{L}_I = -g\bar{\psi}(x)\psi(x)A(x)$, and the class of ladder Feynman graphs to describe the elastic fermion-fermion amplitude. We shall use a summation procedure simpler than Lévy and Sucher's [3] one. In fact, it can be easily verified that the most economic way of taking into account all topologically different diagrams at a given order n is described in Figure 1, where π means permutations over internal momenta $(K_1, \ldots, K_{l-1}, K_{l+1}, \ldots, K_{n+1})$.

Using the OPEA and the summation procedure just described, it can be shown [4] that the amplitude for the reaction

$$(p_1, \lambda_1) + (p_2, \lambda_2) \to (p'_1, \lambda'_1) + (p'_2, \lambda'_2)$$

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eikonalizes in the following way

$$M_{\text{eik}}(s, t) =$$

$$-ig^2 \int_0^\infty d\lambda \exp[i\lambda(t - \mu^2 + i\varepsilon)] \exp[i\chi(\lambda)]\ \bar{u}_{\lambda'_1}(p'_1)u_{\lambda_1}(p_1)\bar{u}_{\lambda'_2}(p'_2)u_{\lambda_2}(p_2) \quad (1.1)$$

and $\chi(\lambda)$ is the eikonal function

$$\chi(\lambda) = -i(u_1 + u_2) \quad (1.2)$$

where $u_1 = u(p_1, -p_2; \lambda)$, $u_2 = u(p_1, p'_2; \lambda)$ and

$$u(p, p'; \lambda) = (2\text{ mg})^2 \int \frac{d^4K}{(2\pi)^4} D(K) \exp(-2i\lambda q\cdot K + i\lambda K^2)\ \frac{1}{2pK + i\varepsilon}\frac{1}{2p'K + i\varepsilon} \quad (1.3)$$

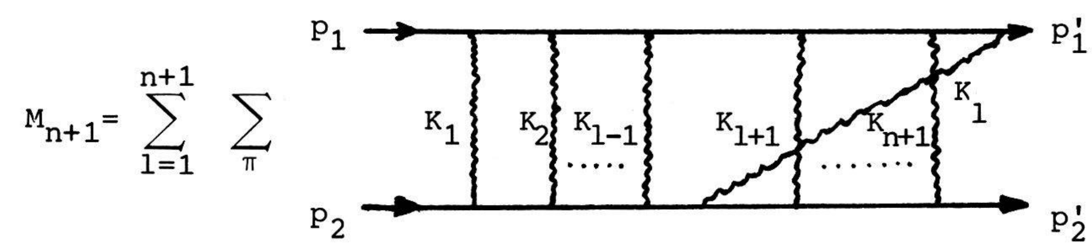

Figure 1
The class of ladder diagrams.

In a precedent paper [4] we have studied the eikonal approximation in the space of Feynman parameters, called α-space, and we have seen that the use of Feynman rules in the α-space [5] was particularly suitable to find some spin corrections to the eikonal approximation in a trivial way. Linearization of fermion propagators implies diagonalization of the Chrisholm determinant $C(\alpha)$, and the factorization in α-space is performed in a similar way to that in K-space. The final expression for M_{eik} is given by (1.1) and (1.2), but this time with u_1 and u_2 defined by

$$u_1 = \int_0^\infty d\rho(s, t) \qquad u_2 = \int_0^\infty d\rho(u, t) \quad (1.4)$$

and the integration measures defined by

$$d\rho(s, t) = i\left(\frac{2\text{ mg}}{4\pi}\right)^2 d\delta\, d\gamma\, d\beta \frac{1}{(\beta + \lambda)^2} \exp\left(-i\beta\mu^2 + \frac{b^2(s, t)}{\beta + \lambda}\right) \quad (1.5)$$

$$b(s, t) = \lambda^2 t + (\gamma^2 + \delta^2)m^2 - \lambda(\gamma + \delta)t - \delta\lambda t - \gamma\delta(s - 2m^2)$$

The equivalence between (1.3) and (1.4) is easily proved by integration of (1.3). However the amplitude defined by (1.4) shows explicitly the $s - u$ symmetry. Throughout this paper we shall use the notations and conventions of I.

Let $G(K)$ be the spinor factor coming from the rationalized fermion propagators. We shall identify the spin effects of order n with monomials of degree n in γK coming from the development of $G(K)$.

In Section 2 we shall study the zero-th order term, which coincides with the eikonal amplitude (1.1). First for very small scattering angle and then in the limit where the range of the interaction goes to infinity, $R \sim 1/\mu \to \infty$. The result (2.15) resembles what one finds in the usual eikonal approximation, showing the Coulomb phase, the forward ($t = 0$) singularity and simple poles in the complex energy plane corresponding to bound-states.

In Section 3 we evaluate the first order correction terms in $\gamma \cdot K$. At high energy the leading term behaves as a constant. It is, in this way, comparable to the eikonal amplitude. In the limit $\mu \to 0$ it shows, equation (3.38), the Coulomb phase, the forward singularity, an additional factor μ and double poles in the energy plane located at the same positions as the former ones. One can interpret [2] this term as responsible for the fine structure energy levels of positronium. Unfortunately, these levels would be proportional to μ, and so, they would go to zero with μ.

Appendix A shows an application of the Mellin transform to compute the behaviour at t fixed and $s \to \infty$ of Feynman integral in α-space.

In Appendix B we explicitly compute the terms not considered in Section 3 (non-leading terms). They factorize in a little more complicated way, equation (B.13). Lévy and Léger [2] have conjectured that these terms did not contribute in the high-energy region. By application of the methods described in Appendix A we can prove that this is indeed the case and these terms behave as $\log s/s^2$ when $s \to \infty$.

2. The zero-th order correction term: the eikonal amplitude

Let us consider the amplitude corresponding to n-loop diagrams, Figure 1. The spinor factor $G(K)$ coming from the two fermion lines can be written

$$G(K)_{l,n} = G_1(K)_{l,n} G_2(K)_{l,n} \tag{2.1}$$

with

$$G_1(K)_{l,n} = \bar{u}_{\lambda_1'}(p_1') \prod_{\substack{\alpha=1 \\ (\alpha \neq l)}}^{n+1} (\not{p}_1 + m + \not{L}_\alpha) u_{\lambda_1}(p_1) \tag{2.2}$$

$$G_2(K)_{l,n} = \bar{u}_{\lambda_2'}(p_2') \prod_{\beta=n+1}^{l+1} (\not{p}_2' + m + \underline{\not{K}}_\beta) \prod_{\alpha=l-1}^{1} (\not{p}_2 + m - \bar{\not{K}}_\alpha) u_{\lambda_2}(p_2) \tag{2.3}$$

where L_α, $\bar{K}_\alpha$ and $\underline{K}_\beta$ are combinations of internal momenta, defined by equations (4.4)–(4.6) of I. The factor $G(K)$ can be developed as

$$G(K)_{l,n} = \sum_{j=0}^{2n} G^{(j)}(K)_{l,n} \tag{2.4}$$

from the partial developments

$$G_{(1,2)}(K)_{l,n} = \sum_{j=0}^{n} G_{(1,2)}^{(j)}(K)_{l,n} \tag{2.5}$$

when $G^{(j)}$ are polynomials of degree j in $\gamma \cdot K$, in such a way that $G^{(0)}(K)_{l,n}$ and $G^{(1)}(K)_{l,n}$ give the spin effects of order zero and one, respectively. As to the zero order terms they are given by

$$G^{(0)}(K)_{l,n} = (2m)^{2n} \bar{u}_{\lambda_1'}(p_1') u_{\lambda_1}(p_1) \bar{u}_{\lambda_2'}(p_2') u_{\lambda_2}(p_2) \tag{2.6}$$

and we get the amplitude (1.1). Because the eikonal approximation is essentially a small angle approximation, we shall compute the limit of (1.1) for small values of θ; θ being the center-of-mass scattering angle in the s-channel.

In this way the limit $\theta \to 0$ is easily considered doing $p'_i \to p_i$ $(i = 1, 2)$. With regard to $q = p'_1 - p_1 = \mathbf{q} + 0(\theta^2)$ we only keep the transverse part and we neglect the rest, $0(\theta^2)$.

Let $\chi_0(\lambda) = \lim_{\theta \to 0} \chi(\lambda)$, and using the formal identity

$$\frac{1}{x \pm i\varepsilon} = P\frac{1}{x} \mp i\pi\,\delta(x) \tag{2.7}$$

we can write

$$\chi_0(\lambda) = \\ -(2\,\mathrm{mg})^2 \int \frac{d^4K}{(2\pi)^3} D(K) \exp(2i\lambda\mathbf{qK} + i\lambda K^2)\,\delta(2p_2K)\left\{P\frac{1}{2p_1K} - i\pi\,\delta(2p_1K)\right\} \tag{2.8}$$

We can integrate over K_0 in (2.8) using the function $\delta(2p_2K)$. Once this is done, $2p_1K$ becomes an odd function of K_3 and, because $D(K)$ remains an even function, the integral overK_3 of the term containing $P(1/2p_1K)$ vanishes. There only remains the term $\delta(2p_1K)$. Let us note that this is a consequence of having extracted in $q \cdot K$ the transverse part $\mathbf{qK}$ which is K_0 and K_3-independent. Let us finally remark the origin of $P(1/2p_1K)$ comes from the fact that we have used a summation procedure slightly different and simpler than Lévy and Sucher's one [3]. We have only symmetrized the lower line but not the upper one. As we have said this is the most economic way of considering all different configurations at a given order, and the two methods coincide in the limit $\theta \to 0$. In this way we can write (2.8) as

$$\chi_0(\lambda) = -\tfrac{1}{2}\rho(s)\exp(i\lambda\mu^2)\int_{\mu^2\lambda}^{\infty}\frac{d\alpha}{\alpha}\exp\left(-i\alpha - i\frac{\mu^2\lambda^2}{\alpha}t\right) \tag{2.9}$$

where the function $\rho(s)$ is given by

$$\rho(s) = \frac{g^2}{8\pi}\frac{(2m)^2}{\sqrt{s(s-4m^2)}} \tag{2.10}$$

From (2.9) we deduce

$$\chi_0\left(\frac{\lambda}{2\mu\sqrt{-t}}\right) = -\rho(s)\left\{K_0(\lambda) + 0\left(\lambda\frac{\mu}{\sqrt{-t}}\right)\right\} \tag{2.11}$$

where $K_0(\lambda)$ is a modified Bessel function of third kind. In this way the amplitude (1.1) becomes

$$M^{(0)}(s, t) = -ig^2\frac{1}{2\mu\sqrt{-t}}\int_0^{\infty} d\lambda \exp\left[i\lambda\left(\frac{\sqrt{-t}}{2\mu} - \frac{\mu}{2\sqrt{-t}} + i\varepsilon'\right)\right] \cdot \\ \cdot \exp[-i\rho K_0(\lambda)]\left[1 + 0\left(\lambda\frac{\mu}{\sqrt{-t}}\right)\right] \tag{2.12}$$

where we have factorized the helicity conservation factor $\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}$, which we do not write, and performed the transformation of variables $\lambda \to 1/(2\mu\sqrt{-t})\lambda$. In the limit $\mu \to 0$, $\mu/\sqrt{-t}$ is negligible with respect to $\sqrt{-t}/\mu$, which grows indefinitely. The factor $\exp\{i\lambda(\sqrt{-t}/2\mu)\}$ oscillates very quickly and the integral (2.12) is dominated by the values $\lambda \sim 0$. In this way, using the behaviour $K_0(\lambda) \sim -\log\lambda$, when $\lambda \to 0$, we can write (2.12) as

$$M^{(0)}(s, t) = -ig^2 \frac{1}{2\mu\sqrt{-t}} \int_0^\infty d\lambda \lambda^{i\rho} \exp\left[i\lambda\left(\frac{\sqrt{-t}}{2\mu} + i\varepsilon'\right)\right] + R(\mu) \tag{2.13}$$

Let us note the presence of $\varepsilon' = \varepsilon/(2\mu\sqrt{-t})$ which is necessary to assure the convergence of the integral. One must take the limit $\varepsilon' \to 0$ after the integration. Using the integral [6]

$$\int_0^\infty \exp(-\alpha x)\, x^{s-1}\, dx = \alpha^{-s}\Gamma(s), \qquad \mathrm{Re}\,\alpha > 0; \qquad \mathrm{Re}\, s > 0 \tag{2.14}$$

one gets

$$M^{(0)}(s, t) \underset{\mu\to 0}{\sim} g^2 \frac{1}{-t} \exp\left(-\frac{\pi}{2}\rho\right) \Gamma(1 + i\rho) \left(\frac{-t}{4\mu^2}\right)^{-(i/2)\rho} \tag{2.15}$$

This formula is slightly different from the one obtained by Lévy and Sucher [1] in the usual eikonal approximation, but it keeps the main attributes: behaviour as $1/-t$, infinite Coulomb phase and poles at $\rho = in$ ($n = 1,2\ldots$). The terms contained into $R(\mu)$ are easy to evaluate.

Using the series representation for $K_0(\lambda)$ we can write

$$\exp\{-i\rho K_0(\lambda)\} = \lambda^{i\rho}\left\{1 + \sum_{\substack{n=2,m=0\\(n\geqslant 2m)}}^{\infty} A_{nm}\lambda^n \log^m \lambda\right\} \tag{2.16}$$

and $R(\mu)$ can be written as the series

$$R(\mu) = \sum_{\substack{n=2,m=0\\(n\geqslant 2m)}}^{\infty} A_{nm}R_{nm} \tag{2.17}$$

where

$$R_{nm} = -ig^2 \frac{1}{2\mu\sqrt{-t}} \int_0^\infty d\lambda \lambda^{i\rho+n} \log^m \lambda \exp\left[i\lambda\left(\frac{\sqrt{-t}}{2\mu} + i\varepsilon'\right)\right] \tag{2.18}$$

From (2.14) one finds

$$R_{n0} = g^2 \frac{1}{-t}\left(\frac{2\mu}{\sqrt{-t}}\right)^n \exp\left[-\frac{\pi}{2}(\rho - in)\right]\left(\frac{-t}{4\mu^2}\right)^{-(i/2)\rho} \Gamma(1 + n + i\rho) \tag{2.19}$$

and

$$R_{nm} = \left(\frac{1}{i}\frac{d}{d\rho}\right)^m R_{n0} \tag{2.20}$$

From (2.19)–(2.20) we can see that (2.17) is dominated, at $\mu \to 0$, by R_{21} which behaves as $\mu^2 \log \mu$. So

$$R(\mu) \underset{\mu \to 0}{\sim} \mu^2 \log \mu \tag{2.21}$$

which proves the statement that the values $\lambda \sim 0$ dominate the integral (2.12).

3. The first order correction terms

What we name spin corrections of first order is contained in the term $G^{(1)}(K)$, (2.4). We can keep a fermion propagator on the line 2, a contribution as $G_1^{(0)}(K)G_2^{(1)}(K)$, or on the line 1, a contribution as $G_1^{(1)}(K)G_0^{(2)}(K)$. The two contributions must be equal because passing from one to the other is only doing the change $1 \leftrightarrow 2$, or $s \leftrightarrow s$, $u \leftrightarrow u$ and $t \leftrightarrow t$, and the amplitude remains unchanged. Thus $G^{(1)}(K)$ will give rise to the term $M^{(1)}(s, t)$ of the decomposition

$$M(s, t) = \sum_{j=0}^{\infty} M^{(j)}(s, t) \tag{3.1}$$

according to (2.4). Let us consider in the following the contribution of $G_1^{(0)}(K)G_2^{(1)}(K)$ to $M^{(1)}(s, t)$, that is to say the spin effects on the line 2. Explicitly from (2.3),

$$G_2^{(1)}(K)_{ln} = G_2^L(K)_{ln} + G_2^R(K)_{ln} \tag{3.2}$$

where G^L corresponds to keep $\not{K}$ to the left from K_1, and G^R to the right. In this way

$$G_2^L(K)_{ln} = \sum_{i=1}^{l-1} G_{iln}^L; \qquad G_{iln}^L = (2m)^{n-l+i}\bar{u}_{\lambda_2'}(p_2')(\not{p}_2 + m)^{l-i-1}(-\not{K}_i)u_{\lambda_2}(p_2) \tag{3.3}$$

$$G_2^R(K)_{ln} = \sum_{i=l+1}^{n+1} G_{iln}^R; \qquad G_{iln}^R = (2m)^{n-l+i}\bar{u}_{\lambda_2'}(p_2')\not{K}_i(\not{p}_2' + m)^{i-l-1}u_{\lambda_2}(p_2) \tag{3.4}$$

or graphically in Figure 2.

Accordingly, the amplitude is decomposed as

$$\tfrac{1}{2}M^{(1)}(s, t) = M_L^{(1)}(s, t) + M_R^{(1)}(s, t) \tag{3.5}$$

with

$$\begin{aligned} M_L^{(1)}(s, t) &= M_L'(s, t) + M_L(s, t) \\ M_R^{(1)}(s, t) &= M_R'(s, t) + M_R(s, t) \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} M_L'(s, t) &= \sum_{n=2}^{\infty} \sum_{l=3}^{\infty} \sum_{i=1}^{l-2} (M_L^{(1)})_{iln} \\ M_L(s, t) &= \sum_{n=1}^{\infty} \sum_{l=2}^{n+1} (M_L^{(1)})_{l-1,l,n} \\ M_R'(s, t) &= \sum_{n=2}^{\infty} \sum_{l=1}^{n-1} \sum_{i=l+2}^{n+1} (M_R^{(1)})_{iln} \\ M_R(s, t) &= \sum_{n=1}^{\infty} \sum_{l=1}^{n} (M_R^{(1)})_{l+1,l,n} \end{aligned} \tag{3.7}$$

$(M_L^{(1)})_{iln}$ and $(M_R^{(1)})_{iln}$ being the amplitudes corresponding to the non-crossed diagrams of Figure 2. The cases $i=l-1$ and $i=l+1$ have been isolated because they play a very special role as we shall see later. In fact they are, at high energy, of the same order as $M^{(0)}(s, t)$ while the others are dominated by them. Let us compute the left-amplitude $M_L^{(1)}(s, t)$ using α-space techniques as we have described in I

$$(M_L^{(1)})_{iln} = -ig^2 \int_0^\infty d\lambda \exp[i\lambda(t - \mu^2)](M_L^{(1)}(\lambda))_{iln} \tag{3.8}$$

and

$$(M_L^{(1)}(\lambda))_{iln} = g^{2n} \int_0^\infty \prod_j \{d\alpha_{2j}\, d\alpha_{2j-1}\, d\beta_j \exp(-iB_j)\} L_{iln} \tag{3.9}$$

$$L_{iln} = \int \prod_j \left\{\frac{d^4 q_j}{(2\pi)^4} \exp(-i\beta_j q_j^2)\right\} G_1^{(0)}(K(q)) G_2^L(K(q))_{iln} \tag{3.10}$$

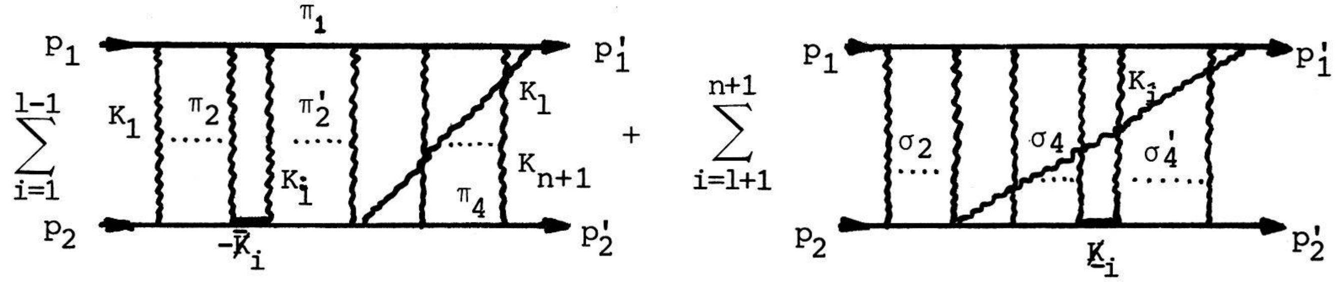

Figure 2
First-order correction terms.

The function $K(q)$ is given by

$$\begin{aligned} K_j &= q_j + \frac{\lambda}{\beta_j + \lambda} q + \frac{\gamma_j}{\beta_j + \lambda} p_2 - \frac{\delta_j}{\beta_j + \lambda} p_1 \quad (j < l) \\ K_j &= q_j + \frac{\lambda}{\beta_j + \lambda} q - \frac{\gamma_j}{\beta_j + \lambda} p_2' - \frac{\delta_j}{\beta_j + \lambda} p_1 \quad (j > l) \end{aligned} \tag{3.11}$$

and the function $B_j \equiv B_j(\lambda_j, \beta_j, \lambda; s, t, \mu)$ is given by

$$B_j = \beta_j \mu^2 + \frac{b_j^2}{\beta_j + \lambda} \tag{3.12}$$

with

$$\begin{aligned} b_j &= -\lambda q - \gamma_j p_2 + \delta_j p_1 \equiv b_j(s, t) \quad (j < l) \\ b_j &= -\lambda q + \gamma_j p_2' + \delta_j p_1 \equiv b_j(u, t) \quad (j > l) \end{aligned} \tag{3.13}$$

We have introduced integral representations

$$\frac{i}{x + i\varepsilon} = \int_0^\infty d\alpha \exp[i\alpha(x + i\varepsilon)] \tag{3.14}$$

for each propagator denominator. The Feynman parameters β_j are attached to photon propagators while α_j's are attached to fermion propagators, as is indicated in Figure 3. The variables γ_j and δ_j are linear combinations of the variables α_j, as indicated in equation (4.28) of I.

To get the amplitude (3.8) we have used the α-space approximation

$$(C(\alpha))_{ij} = (\beta_j + \lambda)\,\delta_{ij} \tag{3.15}$$

as given in equation (4.24)–(4.27) of I.

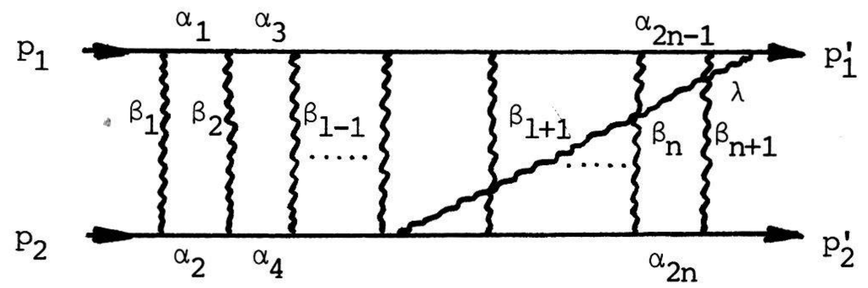

Figure 3
Definition of Feynman parameters in ladder graphs.

Let us compute now the term corresponding to $i = l - 1$, which we have called $M_L(s, t)$ and which corresponds to the configuration of Figure 2, where $\not{K}_i$ is joined to K_1. This will be the high-energy leading term, comparable to the eikonal approximation $M^{(0)}(s, t)$, while, as we shall see later, $M'_L(s, t)$ will be non-leading and so negligible at high energy.

3.1. *The high-energy leading contribution*

From (3.3), (3.8) and (3.13) we get

$$\begin{aligned}(M_L^{(1)}(\lambda))_{l-1,l,n} = {} & \left(-2im\frac{g^2}{16\pi^2}\right)^n (2m)^{n-1} \int_0^\infty \prod_j \left\{ d\alpha_{2j}\, d\alpha_{2j-1}\, d\beta_j \frac{e^{-iB_j}}{(\beta_j+\lambda)^2} \right\} \\ & \times \bar{u}_{\lambda'_1}(p'_1) u_{\lambda_1}(p_1) \bar{u}_{\lambda'_2}(p'_2) \\ & \times \left\{ -\left(\sum_{i=1}^{l-1} \frac{\gamma_i}{\beta_i+\lambda}\right) m + \left(\sum_{i=1}^{l-1} \frac{\delta_i}{\beta_i+\lambda}\right) \not{p}_1 \right\} u_{\lambda_2}(p_2)\end{aligned} \tag{3.16}$$

The factorization method we shall use in the following will be the same as the one described in I, that is we perform the permutations over loop momenta

$$(M_L^{(1)}(\lambda))_{l-1,l,n} \to \frac{1}{(l-1)!}\frac{1}{(n-l+1)!} \sum_{\pi_1}\sum_{\pi_2}\sum_{\pi_4} (M^{(1)}(\lambda))_{l-1,l,n} \tag{3.17}$$

The factor in front of the sum is due to the multiple counting of diagrams. The effects of permutations over the function $B_j(\gamma_j, \delta_j, \lambda)$ are to change $\delta_j \to \delta_{\pi_1(j)}$ and $\gamma_j \to \gamma_{\pi_2(j)}$, if $j < l$, or $\gamma_j \to \gamma_{\pi_4(j)}$ if $j > l$. As for the spinor factor we have seen that also $K_j \to K_{\pi_2(j)}$, so that under π_2-permutations $\delta_j \to \delta_{\pi_2(j)}$, $\beta_j \to \beta_{\pi_2(j)}$ and $\gamma_j \to \gamma_{\pi_2\pi_2(j)}$, for $j < l$. In this way π_1 and π_4 permutations can be accomplished as in $M^{(0)}$, leading to

$$\sum_{\pi_1}\int_0^\infty \prod_{j\neq l}\{d\alpha_{2j-1}\} = \int_0^\infty \prod_{j\neq l} d\delta_j$$
$$\sum_{\pi_4}\int_0^\infty \prod_{j=l+1}^{n+1}\{d\alpha_{2j}\} = \int_0^\infty \prod_{j=l+1}^{n+1} d\gamma_j \tag{3.18}$$

Let us perform the transformations of variables

$$\alpha_{2j}\to\gamma_j\to\gamma'_j=\gamma_{\pi_2(j)}$$

with the integration domain defined by $\infty \geqslant \gamma_1 \geqslant \cdots \geqslant \gamma_{l-1} \geqslant 0$. In this way

$$\sum_{\pi_2}\int_{\gamma_1\geqslant\cdots\geqslant\gamma_{l-1}\geqslant 0}^{\infty}\prod_{j=1}^{l-1}\left\{\frac{\exp[-iB_j(\gamma_{\pi_2(j)})]}{(\beta_j+\lambda)^2}d\gamma_j\right\}\left\{-m\left(\sum_{k=1}^{l-1}\frac{\gamma_{\pi_2\pi_2}(K)}{\beta_{\pi_2(K)}+\lambda}\right)\right.$$
$$\left.+\not{p}_1\left(\sum_{k=1}^{l-1}\frac{\delta_{\pi_2(K)}}{\beta_{\pi_2(K)}+\lambda}\right)\right\} = \sum_{\pi_2}\int_{\gamma'^{-1}_{\pi_2(1)}\geqslant\cdots\geqslant\gamma'^{-1}_{\pi_2(l-1)}\geqslant 0}^{\infty}\prod_{j=1}^{l-1} \tag{3.19}$$
$$\left\{\frac{\exp[-iB_j(\gamma'_j)]}{(\beta_j+\lambda)^2}d\gamma'_j\right\}\left\{-m\left(\sum_{K=1}^{l-1}\frac{\gamma'_k}{\beta_K+\lambda}\right)+\not{p}_1\left(\sum_{K=1}^{l-1}\frac{\delta_k}{\beta_K+\lambda}\right)\right\}$$

and being the integrand π_2-independent

$$\sum_{\pi_2}\int_{\gamma'^{-1}_{\pi_2(1)}\geqslant\cdots\geqslant\gamma'^{-1}_{\pi_2(1)}\geqslant 0}^{\infty}\prod_{j=1}^{l-1}d\gamma'_j = \int_0^\infty\prod_{j=1}^{l-1}d\gamma'_j \tag{3.20}$$

Using (3.18)–(3.20) it is evident how $(M_L^{(1)}(\lambda))_{l-1,l,n}$ factorizes and the sum over l and n results in an exponential. The result can be written as

$$M_L(s,t) = -ig^2\int_0^\infty d\lambda\,\exp[i\lambda(t-\mu^2)]\,M_L(\lambda) \tag{3.21}$$

and

$$M_L(\lambda) = \tfrac{1}{2}a_1(\lambda)\exp[i\chi(\lambda)]\,\bar{u}_{\lambda'_1}(p'_1)u_{\lambda_1}(p_1)\bar{u}_{\lambda'_2}(p'_2)u_{\lambda_2}(p_2)$$
$$+\frac{1}{2m}b_1(\lambda)\exp[i\chi(\lambda)]\,\bar{u}_{\lambda'_1}(p'_1)u_{\lambda_1}(p_1)\bar{u}_{\lambda'_2}(p'_2)\not{p}_1u_{\lambda_2}(p_2) \tag{3.22}$$

where the functions a_1 and b_1 are defined by

$$a_1(\lambda) = \int_0^\infty\frac{\gamma}{\beta+\lambda}d\rho(s,t) \qquad b_1(\lambda) = -\int_0^\infty\frac{\delta}{\beta+\lambda}d\rho(s,t) \tag{3.23}$$

The amplitude $M_R(s,t)$ can be calculated in the same way. One gets

$$M_R(\lambda) = \tfrac{1}{2}a_2(\lambda)\exp[i\chi(\lambda)]\,\bar{u}_{\lambda'_1}(p'_1)u_{\lambda_1}(p_1)\bar{u}_{\lambda'_2}(p'_2)u_{\lambda_2}(p_2)$$
$$-\frac{1}{2m}b_2(\lambda)\exp[i\chi(\lambda)]\,\bar{u}_{\lambda'_1}(p'_1)u_{\lambda_1}(p_1)\bar{u}_{\lambda'_2}(p'_2)\not{p}_1u_{\lambda_2}(p_2) \tag{3.24}$$

with

$$a_2(\lambda) = \int_0^\infty\frac{\gamma}{\beta+\lambda}d\rho(u,t) \qquad b_2(\lambda) = -\int_0^\infty\frac{\delta}{\beta+\lambda}d\rho(u,t) \tag{3.25}$$

In the region of very high energy we have

$$M_L(\lambda) + M_R(\lambda) \underset{s\to\infty}{\sim} \frac{s}{(2m)^2}(b_1(\lambda) - b_2(\lambda)) \exp[i\chi(\lambda)]\, \delta_{\lambda_1'\lambda_1}\, \delta_{\lambda_2'\lambda_2} \tag{3.26}$$

where we have used

$$b_1(\lambda) - b_2(\lambda) \underset{s\to\infty}{\sim} \frac{g^2}{8\pi^2}\frac{(2m)^2}{s} G_3(\lambda, t) \tag{3.27}$$

and we have that $M_L(\lambda) + M_R(\lambda) \underset{s\to\infty}{\sim}$ constant, while the other terms from (3.22) and (3.24) are very strongly dominated at high energy because $a_1(\lambda) + a_2(\lambda) \sim 1/s^2$. The value of $M_L'(s, t) + M_R'(s, t)$ can be found in Appendix B. In the high energy region it is shown that it behaves as $\log s/s^2$ so we shall neglect this contribution in the following. From (3.5) and (3.26–28) one can write

$$M^{(1)}(s, t) \sim -i\frac{g^4}{4\pi^2}\int_0^\infty d\lambda \exp[i\lambda(t - \mu^2)]\, G_3(\lambda, t) \exp\{i\chi(\lambda)\}\, \delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2} \tag{3.28}$$

and, Appendix A,

$$G_3(\lambda, t) =$$

$$\frac{\mu}{m}\exp[i\lambda\mu^2]\int_0^\infty d\gamma \exp\left[i\lambda\frac{\mu}{m}\gamma t\right]\int_{\mu^2\lambda}^\infty \frac{d\beta}{\beta}\exp\left[-i\frac{\mu^2\lambda^2}{\beta}t - i\beta(1 + \gamma^2)\right] \tag{3.29}$$

3.2. *Application to electrodynamics*

Let us compute the limit $(\mu/\sqrt{-t}) \to 0$ in (3.28) transforming $\lambda \to (\lambda/2\mu\sqrt{-t})$ and writing for θ small, $\chi_0(\lambda/2\mu\sqrt{-t}) = -\rho(s)\{K_0(\lambda) + 0[(\mu/\sqrt{-t})\lambda]\}$.

The development of the function $\exp\{i\lambda(\sqrt{-t}/2m)\gamma\}$ in a power series, and the integral [6]

$$\int_0^\infty K_\nu\{\alpha(x^2 + z^2)^{1/2}\}\frac{x^{2\rho+1}}{(x^2 + z^2)^{\nu/2}}\,dx = \frac{2^\rho\Gamma(\rho + 1)}{\alpha^{\rho+1}z^{\nu-\rho-1}}K_{\nu-\rho-1}(\alpha z) \tag{3.30}$$

$(\mathrm{Re}\,\alpha, \mathrm{Re}\,z > 0 \qquad \mathrm{Re}\,\rho > -1)$

give an expression of the amplitude $M^{(1)}(s, t)$ as

$$M^{(1)}(s, t) = -i\frac{g^4}{4\pi^2}\frac{1}{m\sqrt{-t}}\sum_{n=0}^\infty C_n(t)I_n(s, t) \tag{3.31}$$

with

$$C_n(t) = \frac{1}{n!}\left(i\frac{\sqrt{-t}}{2m}\right)^n 2^{(n-1)/2}\Gamma\left(\frac{n + 1}{2}\right) \tag{3.32}$$

$$I_n(s, t) = \int_0^\infty d\lambda \exp\left[i\lambda\left(\frac{\sqrt{-t}}{2\mu} - \frac{\mu}{2\sqrt{-t}} + i\varepsilon\right)\right]\lambda^{(n/2)-(1/2)+i\rho}K_{(n/2)+(1/2)}(\lambda) \tag{3.33}$$

where we have already supposed that the integral (3.33) is dominated at $\mu \to 0$ by small values of λ and we have written $\exp[-i\rho K_0(\lambda)] = \lambda^{i\rho}$. The other terms of the series (2.16) will be negligible in the limit $\mu \to 0$ and this statement will be proved after having analyzed the integral (3.33). This integral can be explicitly calculated and one gets

$$I_n(s, t) = \frac{\sin\left\{\pi\left(\frac{n}{2} + i\rho - \frac{1}{2}\right)\right\}\Gamma(i\rho)}{\sin\{\pi(n + i\rho)\}} \tag{3.34}$$

$$\left\{-\left(\frac{\sqrt{-t}}{2\mu} - \frac{\mu}{2\sqrt{-t}}\right)^2 - 1\right\}^{-(1/2)((n/2)+(1/2)+i\rho)}$$

$$\times\ Q^{(n/2)+(1/2)}_{(n/2)-(1/2)+i\rho}\left[\frac{-i\left(\frac{\sqrt{-t}}{2\mu} - \frac{\mu}{2\sqrt{-t}}\right)}{\left\{-\left(\frac{\sqrt{-t}}{2\mu} - \frac{\mu}{2\sqrt{-t}}\right)^2 - 1\right\}^{1/2}}\right]$$

where $Q^{(n/2)+(1/2)}_{(n/2)-(1/2)+i\rho}$ is a Legendre function of second kind. In the limit $(\mu/\sqrt{-t}) \to 0$ the argument of this function behaves as $-1 + (\mu^2/t)$ and using well-known properties of Legendre functions we can write

$$Q^{(n/2)+(1/2)}_{(n/2)-(1/2)+i\rho}\left(-1 + \frac{\mu^2}{t}\right) \sim$$

$$\exp\left[i\frac{\pi}{2}(n + 1)\right]2^{-i\rho-(1/4)(n+1)}\sqrt{\pi}\,\frac{\Gamma(n + i\rho + 1)}{\Gamma((n/2) + i\rho + 1)}\left(\frac{\mu^2}{-t}\right)^{1/4\,(n+1)}$$

$$\times\ F\left(\frac{n}{2} + 1 + \frac{i}{2}\rho, \frac{n}{2} + \frac{1}{2} + \frac{i}{2}\rho, \frac{n}{2} + 1 + i\rho; 1\right) \tag{3.35}$$

From the point of view of μ-dependence

$$C_n I_n \underset{\mu\to 0}{\sim} \mu^{n+1} \tag{3.36}$$

and the amplitude $M^{(1)}(s, t)$ can be written in a series as

$$M^{(1)}(s, t) = \sum_{n=0}^{\infty} \mu^{n+1} f(s, t, \mu) F_n(s, t) \tag{3.37}$$

The term $n = 0$ will dominate the amplitude in the limit $\mu \to 0$. The other terms of the series (2.16) will give contributions as $\lambda^p \log^q \lambda$ with $p = 2, 3, \ldots$ and $p \geqslant 2q$. The result will be to change n by $n + 2p$ in (3.33) and their contribution to $M^{(1)}(s, t)$ will be

$$\mu^{n+2p+1} f(\mu, s, t) \log^q \mu F_n(s, t)$$

The leading term of $M^{(1)}(s, t)$, in the limit $(\mu/\sqrt{-t}) \to 0$, will be

$$M^{(1)}(s, t) = -\frac{g^4}{4\pi^2\rho}2^{-(1/4)}\exp\left[i\frac{\pi}{4} + \frac{\pi}{2}\rho\right]\frac{1}{-t}\frac{\mu}{m}\cosh(\pi\rho)\Gamma(1 - i\rho)$$

$$\times\ \Gamma^2(1 + i\rho)\left(\frac{-t}{4\mu^2}\right)^{-(i/2)\rho} \tag{3.38}$$

This result shows double poles at $\rho = in$ ($n = 1, 2, \ldots$) coming from the function $\Gamma^2(1 + i\rho)$. It is similar to the result obtained by Lévy and Léger, analyzing the first order correction terms, in the scattering of a spin-$\frac{1}{2}$ particle by a Yukawa potential in the limit where the range of the interaction goes to infinity.

4. Conclusion

The validity of the eikonal approximation for theories where scalar ψ-particles (mass m) exchange scalar ϕ-photons (mass μ) have been studied by Tiktopoulos and Treiman [9] and also by Cheng and Wu [10]. These authors find that Feynman integrals are not dominated, at high energy, by eikonal paths of integration. This is called the breakdown of the eikonal approximation in the theory $\psi^2(x)\phi(x)$. More recently Banerjee and Mallik [11] have studied in more detail the contributions coming from 3-loop diagrams and shown that non-eikonal terms cancel in the very special (not too interesting) case where the masses m and μ are equal.

Our result, equation (3.28), shows that the inclusion of fermion spin produces also a breaking of the eikonal approximation which will even dominate it from the order g^4. In other words, while the eikonal amplitude give, at an order n, an asymptotic contribution as $g^{2n}s^{1-n}$ ($n = 1, 2, \ldots$), the first order spin corrections contribute as $g^{2n}s^{2-n}$ ($n = 2, 3, \ldots$). In the limit $s \to \infty$ the leading terms are the g^2-Born term coming from the eikonal amplitude, and the g^4-term coming from first order spin corrections.

It is an open problem to know whether higher order spin correction will behave asymptotically as a constant term $g^{2(N+1)}$, N being the corresponding order, and whether the sum over N can be performed in a formal way. This would be in a close relation to the validity of the eikonal approximation in the theory $\bar{\psi}(x)\psi(x)\phi(x)$. One would need to prove that the behaviour at t fixed, $s \to \infty$ of N-loop Feynman integrals agrees with N-order spin corrections.

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Appendix A. An application of the Mellin transformation

We shall use the Mellin transformation to compute the asymptotic behaviour in the s-variable of the Feynman integrals

$$A_n(s, t) = \int_0^\infty d\gamma\, d\delta\, d\beta \frac{1}{(\beta + \lambda)^n} \exp\left[i \frac{\gamma\delta}{\beta + \lambda} s\right] \exp[-iJ] \exp[-\varepsilon(\beta + \gamma + \delta)] \tag{A.1}$$

$$B_n(s, t) = \int_0^\infty d\gamma\, d\delta\, d\beta \frac{\gamma}{(\beta + \lambda)^n} \exp\left[i \frac{\gamma\delta}{\beta + \lambda} s\right] \exp[-iJ] \exp[-\varepsilon(\beta + \gamma + \delta)] \tag{A.2}$$

where the function J is obtained from (1.5) as

$$J(\beta, \gamma, \delta, t) = \beta\mu^2 + \frac{1}{\beta + \lambda} F(\gamma, \delta, t)$$

$$F(\gamma, \delta, t) = \{\lambda^2 - \lambda(\gamma + \delta)\}t + m^2(\gamma + \delta) + 2m^2\gamma\,\delta$$

and the factors $\exp[-\varepsilon(\gamma + \delta + \beta)]$ assure the convergence of the integrals. Let us take the Mellin, M, transform of the function $A_n(s, t)$, over the variable s, as

$$\tilde{A}_n(\tau, t) = \int_0^\infty A_n(s, t)s^{-\tau-1}\,ds \tag{A.3}$$

The s-dependent part of $A_n(s, t)$ is given by $\exp[i(\gamma\delta/\beta + \lambda)s]$, and the integral over s in (A.3) can be performed with the result

$$\Gamma(-\tau)\exp\left[-i\frac{\pi}{2}\tau\right](\gamma\delta)^\tau(\beta + \lambda)^{-\tau}$$

In this way, the integral defining $\tilde{A}_n(\tau, t)$ has an end-point singularity at $\tau = -1$, corresponding to the point $\gamma = \delta = 0$. This singularity can be extracted by a double integration by parts, over γ and δ, so that

$$\tilde{A}_n(\tau, t) = (\tau + 1)^{-2}\Gamma(-\tau)\exp\left[-i\frac{\pi}{2}\tau\right]\int_0^\infty d\beta(\beta + \lambda)^{-n-\tau}$$
$$\int_0^\infty d\gamma\,d\delta(\gamma\,\delta)^{\tau+1}\frac{\partial^2}{\partial\gamma\,\partial\delta}\exp[-iJ] \tag{A.4}$$

One expands the factor multiplying $(\tau + 1)^{-2}$ in a power series around $\tau = -1$ as

$$(\tau + 1)^{-2}f(\tau) = (\tau + 1)^{-2}f(-1) + (\tau + 1)^{-1}f'(-1) + \tfrac{1}{2}f''(-1) + 0(1 + \tau)$$

Taking the inverse Mellin transformation [8]

$$M^{-1}[(\tau + a)^{-b-1}] = \frac{(\log s)^b}{s^a}\frac{1}{\Gamma(b + 1)} \tag{A.5}$$

one can see the asymptotic s behaviour of $A_n(s, t)$ is dominated by the singularity of $\tilde{A}_n(\tau, t)$ at $\tau = 1$, so that

$$\tilde{A}_n(\tau, t) \underset{\tau=-1}{\sim} i(1 + \tau)^{-2}K_n(t) \tag{A.6}$$

and

$$A_n(s, t) \underset{s\to\infty}{\sim} \frac{\log s}{s}K_n(t) \tag{A.7}$$

with

$$K_n(t) = \int_0^\infty d\beta(\beta + \lambda)^{1-n}\exp\left[-i\left(\beta\mu^2 + \frac{\lambda^2}{\beta + \lambda}t\right)\right] \tag{A.8}$$

In the same way one can compute the asymptotic behaviour of $B_n(s, t)$. This time, due to the presence of the extra factor γ, we have a simple pole at $\tau = -1$, coming from δ^τ, and from (A.5) we have

$$B_n(s,t) \underset{s\to\infty}{\sim} i\frac{1}{s}G_n(t) \tag{A.9}$$

with

$$G_n(t) = \int_0^\infty d\gamma\, d\beta(\beta+\lambda)^{1-n}\exp[-iJ(\delta=0,\gamma,t)] \tag{A.10}$$

Appendix B. The non-leading contribution

We shall compute in this Appendix the contribution to the first order correction given by $M'_L(s,t) + M'_R(s,t)$ which has not been considered in Section 3. We shall get factorization using a new permutation scheme, not used in Section 3 and in Ref. (I). Also the results from Appendix A will be used to study the high energy behaviour of this contribution. In fact, we shall find it will be non-leading and negligible with respect to $M_L(s,t) + M_R(s,t)$.

Let us compute first the amplitude $M'_L(s,t)$. From the definition (3.8) we have for $i < l-1$

$$\begin{aligned}(M_L^{(1)}(\lambda))_{iln} = {} & \left(-2im\frac{g^2}{16\pi^2}\right)^n (2m)^{n-2}\int_0^\infty \prod_j \left\{d\alpha_{2j}\, d\alpha_{2j-1}\, d\beta_j \frac{e^{-iB_j}}{(\beta_j+\lambda)^2}\right\} \\ & \times \left\{A(t)\left(\sum_{K=1}^{i}\frac{1}{\beta_K+\lambda}\right) + B\left(\sum_{K=1}^{i}\frac{\gamma_K}{\beta_K+\lambda}\right)\right. \\ & \left. + C(s)\left(\sum_{K=1}^{i}\frac{\delta_K}{\beta_K+\lambda}\right)\right\}\bar{u}_{\lambda'_1}(p'_1)u_{\lambda_1}(p_1)\bar{u}_{\lambda'_2}(p'_2)u_{\lambda_2}(p_2)\end{aligned} \tag{B.1}$$

where $A(t) = -2\lambda p_2 q$, $B = -2m^2$ and $C(s) = 2p_1p_2$, and we have used the identity $(\not{p}_2 + m)^h = (2m)^{h-1}(\not{p}_2 + m)$ for $h > 1$, and

$$\bar{u}_{\lambda'_2}(p'_2)(\not{p}_2+m)\bar{K}_i u_{\lambda_2}(p_2) = 2p_2\bar{K}_i\bar{u}_{\lambda'_2}(p'_2)u_{\lambda_2}(p_2)$$

We must sum over permutations π_1, to consider the different topologies, and π_2, π'_2, π_4 to get factorization, and divide by the number of times one has counted each diagram. As in the scalar case

$$\sum_{\pi_1}\int_0^\infty \prod_{j\neq 1}\{d\alpha_{2j-1}\} = \int_0^\infty \prod_{j\neq 1} d\,\delta_j$$

and the same thing happens with π_4-permutations because they do not affect the spinor factor.

To factorize the part corresponding to $K_1, \ldots, K_{l-1}$ we perform the following transformation of variables

$$\{\alpha_2, \alpha_4, \ldots, \alpha_{2(l-1)}\} \to \{\gamma'_1, \gamma'_2, \ldots, \gamma'_i, \gamma_{i+1}, \ldots, \gamma_{l-1}\}$$

defined by

$$\begin{aligned}\gamma'_K &= \sum_{j=K}^{i}\alpha_{2j} \quad (K \leqslant i) \\ \gamma_K &= \sum_{j=K}^{l-1}\alpha_{2j} \quad (K > i)\end{aligned} \tag{B.2}$$

and the integration domain $\gamma'_1 \geqslant \cdots \geqslant \gamma'_i;\ \gamma_{i+1} \geqslant \cdots \geqslant \gamma_{l-1}$.

From (B.2) we have

$$\gamma_K = \gamma'_K + \gamma_{i+1} \quad (K \leqslant i) \tag{B.3}$$

and where γ_K, $K \leqslant i$, are the old variables we handled in Section 3. As it is indicated in Figure 2, π_2 are permutations over $(K_1, \ldots, K_i)$ while π'_2 are permutations over $(K_{i+1}, \cdots, K_{l-1})$. It is simple to see that

$$b_j(\gamma'_j + \gamma_{i+1}) \xrightarrow{\pi_2} b_j(\gamma'_{\pi_2}(j) + \gamma_{i+1}), \qquad j = 1, \ldots, i \tag{B.4}$$

Let us write

$$\begin{aligned}
&\int_0^\infty \prod_{j=1}^{l-1} \left\{ d\alpha_{2j}\, d\alpha_{2j-1}\, d\beta_j \frac{e^{-iB_j}}{(\beta_j + \lambda)^2} \right\} \left\{ A(t) \left(\sum_{K=1}^{i} \frac{1}{\beta_K + \lambda} \right) + B \left(\sum_{K=1}^{i} \frac{\gamma_K}{\beta_K + \lambda} \right) \right. \\
&\left. + C(s) \left(\sum_{K=1}^{i} \frac{\delta_K}{\beta_K + \lambda} \right) \right\} = \int_{\gamma_{i+1} \geqslant \cdots \geqslant \gamma_{l-1}}^{\infty} \prod_{j=i+1}^{l-1} \left\{ d\gamma_j\, d\delta_j\, d\beta_j \frac{e^{-iB_j(\gamma_j)}}{(\beta_j + \lambda)^2} \right\} \\
&\int_{\gamma'_1 \geqslant \cdots \geqslant \gamma'_i}^{\infty} \prod_{j=1}^{i} \left\{ d\gamma'_j\, d\delta_j\, d\beta_j \frac{e^{-iB_j(\gamma'_j + \gamma_{i+1})}}{(\beta + \lambda)^2} \right\} \left\{ A(t) \left(\sum_{K=1}^{i} \frac{1}{\beta_K + \lambda} \right) \right. \\
&\left. + B \sum_{K=1}^{i} \frac{\gamma'_K + \gamma_{i+1}}{\beta_K + \lambda} \right) + C(s) \left(\sum_{K=1}^{i} \frac{\delta_K}{\beta_K + \lambda} \right) \Bigg\}
\end{aligned} \tag{B.5}$$

Under π_2-permutations, B_j changes as $\gamma'_j \to \gamma'_{\pi_2(j)}$ while the spinor factor changes as $\gamma'_j \to \gamma'_{\pi_2\pi_2(j)}$, $\beta_j \to \beta_{\pi_2(j)}$ and $\delta_j \to \delta_{\pi_2(j)}$ and

$$\begin{aligned}
&\frac{1}{i!} \sum_{\pi_2} \int_{\gamma'_1 \geqslant \cdots \geqslant \gamma'_i}^{\infty} \prod_{j=1}^{i} d\gamma'_j\, d\delta_j\, d\beta_j \frac{e^{-iB_j(\gamma'_{\pi_2}(j) + \gamma_{i+1})}}{(\beta_j + \lambda)^2} \left\{ A(t) \left(\sum_{K=1}^{i} \frac{1}{\beta_{\pi_2(K)} + \lambda} \right) \right. \\
&\left. + B \left(\sum_{K=1}^{i} \frac{\gamma'_{\pi_2\pi_2(K)} + \gamma_{i+1}}{\beta_{\pi_2(K)} + \lambda} \right) + C(s) \left(\sum_{K=1}^{i} \frac{\delta_{\pi_2(K)}}{\beta_{\pi_2(K)} + \lambda} \right) \right\} = \frac{1}{(i-1)!} \\
&\times \{ A(t)\omega'_1(\gamma_{i+1}) + BZ'_1(\gamma_{i+1}) + C(s) Y'_1(\gamma_{i+1}) \} u'_1(\gamma_{i+1}) \equiv G(\gamma_{i+1})
\end{aligned} \tag{B.6}$$

and the functions

$$\begin{aligned}
u'_1(\gamma) &= \int_0^\infty d\rho'(s, t, \gamma) \qquad \omega'_1(\gamma) = \int_0^\infty \frac{1}{\beta' + \lambda} d\rho'(s, t, \gamma) \\
Z'_1(\gamma) &= \int_0^\infty \frac{\gamma' + \gamma}{\beta' + \lambda} d\rho'(s, t, \gamma) \qquad Y'_1(\gamma) = \int_0^\infty \frac{\delta'}{\beta' + \lambda} d\rho'(s, t, \gamma)
\end{aligned} \tag{B.7}$$

and

$$d\rho'(s, t, \gamma) = \frac{1}{(\beta' + \lambda)^2} d\gamma'\, d\delta'\, d\beta' \exp[-iB(s, t, \gamma' + \gamma)] \tag{B.8}$$

The factorization of π'_2-permutations is accomplished in the following way

$$\begin{aligned}
&\frac{1}{(1-i-1)!} \sum_{\pi'_2} \int_{\gamma_{i+1} \geqslant \cdots \geqslant \gamma_{l-1}}^{\infty} \prod_{j=i+1}^{l-1} \left\{ d\gamma_j\, d\delta_j\, d\beta_j \frac{e^{-iB_j(\gamma_{\pi'_2(j)})}}{(\beta_j + \lambda)^2} \right\} G(\gamma_{i+1}) \\
&= \frac{1}{(1-i-2)!} \int_0^\infty d\gamma\, d\beta\, d\delta \frac{e^{-iB(s,t,\gamma)}}{(\beta + \lambda)^2} G(\gamma) (v'_1(\gamma))^{l-i-2}
\end{aligned}$$

with

$$v_1'(\gamma) = \int_0^\gamma d\gamma' \int_0^\infty d\delta' \, d\beta' \, \frac{e^{-iB(s,t,\gamma')}}{(\beta' + \lambda)^2} \tag{B.10}$$

The sum over i, l and n is now straightforward, and the result

$$M_L'(\lambda) = \frac{1}{(2m)^2} \exp[-u^2] \int_0^\infty d\rho(s, t)\{A(t)\omega_1(\gamma) + BZ_1(\gamma) + C(s)Y_1(\gamma)\}$$
$$\times \exp[-(u_1(\gamma) + v_1(\gamma)) \, \bar{u}_{\lambda_1'}(p_1')u_{\lambda_1}(p_1)\bar{u}_{\lambda_2'}(p_2')u_{\lambda_2}(p_2) \tag{B.11}$$

where the functions ω_1, Z_1, u_1, Y_1 and v_1 are given by (B.7), but with the measure of integration changed to

$$d\rho(s, t, \gamma) = i\left(\frac{mg}{2\pi}\right)^2 d\rho'(s, t, \gamma) \tag{B.12}$$

Let us note that $d\rho(s, t, 0) = d\rho(s, t)$ and $u_1(\gamma) + v_1(\gamma) = u_1$, so that

$$-u_2 - u_1(\gamma) - v_1(\gamma) = i\chi(\lambda).$$

The calculation of $M_R'(\lambda)$ follows along the same lines, and the sum $M'(\lambda) = M_L'(\lambda) + M_R'(\lambda)$ can be written as

$$M'(\lambda) = \frac{1}{(2m)^2} \exp[i\chi(\lambda)] \left\{ \int_0^\infty d\rho(s, t)[A(t)\omega_1(\gamma) + BZ_1(\gamma) + C(s)Y_1(\gamma)] \right.$$
$$\left. + \int_0^\infty d\rho(u, t)[A(t)\omega_2(\gamma) + BZ_2(\gamma) + C(u)Y_2(\gamma)] \right\} \tag{B.13}$$
$$\times \bar{u}_{\lambda_1'}(p_1')u_{\lambda_1}(p_1)\bar{u}_{\lambda_2'}(p_2')u_{\lambda_2}(p_2)$$

where the functions $X_2(\gamma)$ are obtained from $X_1(\gamma)$ by the change $s \to u$.

In this way the formula (B.13) is the result we were looking for. As we have said these terms are negligible at high energy. To prove this assertion, let us compute the $s \to \infty$ behaviour of (B.13). We have seen that $\chi(\lambda) \sim 1/s$ in such a way that $e^{i\chi} = 1 + 0(1/s)$. The behaviour of the integral which multiplies this exponential can be computed following the methods described in Appendix A. It is found in the region at t fixed and $s \to \infty$, that

$$\int_0^\infty d\rho(s, t) \begin{Bmatrix} \omega_1(\gamma) \\ Z_1(\gamma) \\ C(s)Y_1(\gamma) \end{Bmatrix} = K_2(t) \frac{\log s}{s^2} \begin{Bmatrix} f(t) \\ g(t) \\ h(t) \end{Bmatrix} \tag{B.14}$$

where $K_2(t)$ is given by (A.8) and

$$\begin{Bmatrix} f(t) \\ g(t) \\ h(t) \end{Bmatrix} = \left(\frac{gm}{2\pi}\right)^4 \int_0^\infty d\gamma \, d\beta \exp[-iJ(\delta = 0, \gamma, \beta, t)] \begin{Bmatrix} \dfrac{1}{\gamma} \dfrac{1}{(\beta + \lambda)^2} \\ \dfrac{1}{(\beta + \lambda)^2} \\ \dfrac{1}{\gamma^2} \dfrac{1}{\beta + \lambda} \end{Bmatrix} \tag{B.15}$$

The behaviour of the other piece in (B.13) is given by (B.14) with the substitution $s \to u$. It is proven, in this way, the statement that the amplitude M' is dominated at high energy by the amplitude M, and

$$M'(\lambda) \underset{s\to\infty}{\sim} \frac{\log s}{s^2} \tag{B.16}$$

REFERENCES

[1] M. Levy and J. Sucher, Phys. Rev. *D2*, 1716 (1970).
[2] D. Leger and M. Levy, Nuovo Cimento *25A*, 53 (1975).
[3] M. Levy and J. Sucher, Phys. Rev. *186*, 1656 (1969).
[4] B. Humpert and M. Quiros, Nucl. Phys. *B59*, 141 (1973) (referred to as I hereafter).
[5] R. J. Eden *et al.*, 'The Analytic S-matrix' (Cambridge University Press, 1966).
[6] A. Erdelyi *et al.*, 'Table of Integral Transforms', Vol. 1 (McGraw Hill, Inc., 1954).
[7] B. Humpert and M. Quiros, Lett. Nuovo Cimento, *7*, 201 (1973).
[8] T. L. Trueman and T. Yao, Phys. Rev. *132*, 2741 (1963).
[9] G. Tiktopoulos and S. B. Treiman, Phys. Rev. *D2*, 805 (1970).
[10] H. Cheng and T. T. Wu *D5*, 3170 (1972).
[11] H. Banerjee and S. Mallik, Phys. Rev. *D9*, 956 (1974).