# Projective representations of the Schrödinger group

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# Projective representations of the Schrödinger group

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*Abstract.* A classification of the projective unitary irreducible representations of the Schrödinger group $\mathscr{S}(3)$ is given and a representative of each class is explicitly constructed by the method of induced representations. The connection between some of these representatives and the realizations found by U. Niederer on spaces of wave functions is established. The physical interpretation of these representations is very similar to the case of the Galilei group; however the usefulness of generalizing the concept of elementary systems to these representations is not very clear in view of the appearance of an infinite number of degrees of internal freedom.

## I. Introduction

In his two first articles devoted to the Schrödinger group, U. Niederer [1, 2] constructed a projective unitary irreducible representation of this group in two different realizations. The first one was constructed on the Hilbert space of the solutions of the Schrödinger equation for a free particle and the second one on the Hilbert space of the solutions of the Schrödinger equation for a harmonic oscillator. In these two examples, a natural realization of the space of the states for which the position observables are 'diagonal' is given by $L^2(\mathbb{R}^3)$; this space inherits a representation of the group induced by the ones defined on the spaces of wave functions. Actually it is the choice of a law of evolution which permits, starting from this representation to obtain the various realizations on the wave functions. Any localizable three dimensional system with spin 0 has $L^2(\mathbb{R}^3)$ as a space of states (as long as its Hamiltonian is defined on a dense domain of it) and therefore it admits the Schrödinger group as an invariance group. To each choice of a Hamiltonian there corresponds a particular realization on the corresponding space of wave functions. The third article of U. Niederer [3] devoted to Hamiltonians with arbitrary potentials does not contradict this fact: it is the 'kinematical requirement', characterized by a specified law of transformation for the wave function, which restricts the invariance group. The operators of the representation always permute the solutions, but not necessarily via 'kinematical transformations'.

In the two examples mentioned above, the Hamiltonian is the generator of a one parameter subgroup of the Schrödinger group and the idea of interpreting the

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generators of all one parameter subgroups as possible Hamiltonians was explored by G. Burdet and M. Perrin [4]. The knowledge of the representations of the Schrödinger group may be useful here: they explicitly give the evolution operator and therefore the propagator of the corresponding Schrödinger equation. The interest in the representations of this group is not limited to the cases where the group of evolution is a subgroup; as a subgroup of the canonical group, the Schrödinger group through its representations can serve as a spectrum-generating group for Hamiltonians, the symmetry group of which does not remove all degeneracies.

At a more fundamental level it is of interest to compare the projective irreducible representations of the Schrödinger group with those of the Galilei group; the former containing the latter. The possibility of usefully generalizing the notion of elementary Galilean systems to some projective irreducible representations of the Schrödinger group arises. However the answer, after a somewhat superficial analysis, seems to be negative.

The first three sections are devoted to a review of some useful information on the Schrödinger group, its central extensions and the induction procedure for their representations. In Sections 5, 6, 7, 8 we classify and construct the projective irreducible representations of the group. In the last section we examine the consequences of a dynamical postulate for a class of these representations and establish the link between the representations and the various realizations on certain spaces of wave functions given by U. Niederer.

## II. The Schrödinger group and its central extensions

Within the group of all unitary transformations of the Hilbert space of the solutions of the Schrödinger equation for a free particle

$$i\partial_t \psi(\bar{x}, t) + \frac{\hbar}{2m} \Delta\psi(\bar{x}, t) = 0,$$

U. Niederer [1] considered the subgroup of transformations of the form

$$U(g)\psi = (f_g \cdot \psi) \circ g^{-1}$$

generated by the maximal local Lie group of the transformations

$$g: E_3 \times \mathbb{R} \to E_3 \times \mathbb{R}$$

of space-time. The associated Lie group, called *the maximal kinematical invariance group of the free Schrödinger equation* and denoted $\mathscr{S}(3)$, admits a faithful matrix realization in the group of $5 \times 5$ real matrices

$$\mathscr{S}(3) = \left\{ \begin{pmatrix} R & T \\ 0 & S \end{pmatrix} \in M_{5\times 5}(\mathbb{R}) \mid R \in SO(3), \quad S \in SL(2, \mathbb{R}), \quad T \in M_{3\times 2}(\mathbb{R}) \right\}. \tag{2.1}$$

The local group of transformations of space-time is deduced from the linear action of $\mathscr{S}(3)$ on $E_3 \times \mathbb{R} \times \mathbb{R}$:

$$\begin{aligned} \mathbf{x} &\mapsto \frac{R\mathbf{x} + t\mathbf{v} + \mathbf{a}}{ct + d} \\ t &\mapsto \frac{at + b}{ct + d} \end{aligned} \tag{2.2}$$

where $R \in SO(3)$, $[\mathbf{v}, \mathbf{a}] \in M_{3\times 2}(\mathbb{R})$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$.

By restricting $S$ to the elements of the subgroup

$$N = \left\{ \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}; \quad \tau \in \mathbb{R} \right\}$$

of $SL(2, \mathbb{R})$, we obtain *the full Galilei group*, while the restrictions to the subgroups

$$K = \left\{ \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}; \quad \tau \in [0, 2\pi[ \right\}$$

or

$$L = \left\{ \begin{pmatrix} \cosh\tau & \sinh\tau \\ \sinh\tau & \cosh\tau \end{pmatrix}; \quad \tau \in \mathbb{R} \right\}$$

furnish the two Newton's groups which are the contractions (with respect to the speed of light) of the two de Sitter groups $SO(4, 1)$ or $SO(3, 2)$ [5].

A basis for the Lie algebra $\mathscr{s}(3)$ of the group is easily obtained from the matrix realization (2.1). Setting

$$\begin{aligned}
&J_1 = -E_{23} + E_{32}, \quad J_2 = E_{13} - E_{31}, \quad J_3 = -E_{12} + E_{21} \quad \text{for } so(3),\\
&G_+ = E_{45}, \quad G_0 = E_{44} - E_{55}, \quad G_- = E_{54} \quad \text{for } sl(2, \mathbb{R}),\\
&K_i = E_{i4}, \quad P_i = E_{i5}, \quad i = 1, 2, 3 \quad \text{for } T_6,
\end{aligned}$$

we get the following commutation relations

$$\begin{aligned}
&[J_i, J_j] = \varepsilon_{ijk} J_k\\
&[J_i, K_j] = \varepsilon_{ijk} K_k\\
&[J_i, P_j] = \varepsilon_{ijk} P_k\\
&[J_i, G_+] = 0, \quad [J_i, G_0] = 0, \quad [J_i, G_-] = 0\\
&[G_0, G_\pm] = \pm 2G_\pm, \quad [G_+, G_-] = G_0\\
&[G_+, K_i] = -P_i, \quad [G_0, K_i] = -K_i, \quad [G_-, K_i] = 0\\
&[G_+, P_i] = 0, \quad [G_0, P_i] = P_i, \quad [G_-, P_i] = -K_i\\
&[K_i, K_j] = 0, \quad [K_i, P_j] = 0, \quad [P_i, P_j] = 0.
\end{aligned} \tag{2.3}$$

(2.3)

The group $\mathscr{S}(3)$ admits the Levi–Malcev decomposition

$$\mathscr{S}(3) = T_6 \,\Box\, (SO(3) \times SL(2, \mathbb{R}))$$

(where $T_6$ denotes the additive Abelian group $\mathbb{R}^6$). We note that this is a connected but not a simply connected group; its universal covering group $(\mathscr{S}(3)^*; \Pi)$ is

$$\mathscr{S}(3)^* = T_6 \,\Box\, (SU(2) \times SL(2, \mathbb{R})^*)$$

with $\Pi: \mathscr{S}(3)^* \to \mathscr{S}(3)$, the canonical projection.

Setting $g^* = (\mathbf{a}, \mathbf{v}, s, \sigma)$ for an element of $\mathscr{S}(3)^*$, the multiplication law becomes

$$(\mathbf{a}, \mathbf{v}, s, \sigma)(\mathbf{a}', \mathbf{v}', s', \sigma') = (R_s\mathbf{a}' + b'\mathbf{v} + d'\mathbf{a}, R_s\mathbf{v}' + a'\mathbf{v} + c'\mathbf{a}, ss', \sigma\sigma') \tag{2.4}$$

where $R_s$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are the $\Pi$ projections into $SO(3)$ and $SL(2, \mathbb{R})$ of the elements $s \in SU(2)$ and $\sigma' \in SL(2, \mathbb{R})^*$.

Referring to the fundamental article of *V.* Bargmann [6] on the projective unitary representations (P.U.R.) of the Lie groups, we make the following assertions.

If

$$U: \mathscr{S}(3)^* \to \mathscr{U}(\mathscr{H})$$

is a P.U.R. of $\mathscr{S}(3)^*$, $U$ will also be a P.U.R. of $\mathscr{S}(3)$ if

$$U(g^*) = e^{i\alpha} id_{\mathscr{H}}, \qquad \alpha \in \mathbb{R}, \qquad \forall g^* \in \operatorname{Ker} \Pi.$$

Moreover each P.U.R. of $\mathscr{S}(3)^*$ is obtained from a unitary representation (U.R.) of a central extension of $\mathscr{S}(3)^*$ by a one-parameter Lie group

$$\tilde{\mathscr{S}}(3) = \mathbb{R} \,\square\, \mathscr{S}(3)^*.$$

More precisely, all P.U.R. of $\mathscr{S}(3)^*$ are either given by

$$U(g^*) = V(0, g^*),$$

where $V$ is a U.R. of $\tilde{\mathscr{S}}(3)$, or they are projectively equivalent to such a representation.

The problem of constructing the irreducible P.U.R. (P.I.U.R.) of $\mathscr{S}(3)$ thus reduces of constructing the irreducible U.R. (I.U.R.) of the central extensions $\tilde{\mathscr{S}}(3)$ and then if necessary to choose those which give the P.I.U.R. of $\tilde{\mathscr{S}}(3)$.

The easiest way of finding the central extensions of $\mathscr{S}(3)^*$ is given by the Bargmann procedure starting from those of the Lie algebra

$$\tilde{\mathfrak{s}}(3) = \mathbb{R} \times \mathfrak{s}(3).$$

It is not difficult to convince oneself, in view of the appearance of the subalgebras $su(2) \times sl(2, \mathbb{R})$ and $\mathbb{R}^3 \times su(2)$ which admit only trivial extensions, that $\mathfrak{s}(3)$ has only *one class of non trivial extensions.* A representative of this latter is characterized by the bilinear from $B$ appearing in the commutation relations

$$[\theta, h), (\theta', h')] = (B(h, h'), [h, h'])$$

of $\tilde{\mathfrak{s}}(3)$. It can be chosen to be zero on all couples of elements of the basis (2.3) except for the couples $\{K_i, P_i\}$ for which we can put

$$B(K_i, P_i) = 1, \quad i = 1, 2, 3.$$

Identifying $\mathfrak{s}(3)$ with its image in $\tilde{\mathfrak{s}}(3)$ $((0, h) \equiv h)$ and setting $M = (1, 0)$, we obtain for $\tilde{\mathfrak{s}}(3)$ the same commutation relations as in (2.3) except for $[K_i, P_j] = 0$ which must be replaced by

$$[K_i, P_j] = \delta_{ij} M.$$

The generator $M$ of course commutes with the entire algebra.

The corresponding extension of the group can be found either by the Bargmann method or by formal exponentiation of the Lie algebra. Denoting by $\tilde{g} = (\xi, g^*) = (\xi, \mathbf{a}, \mathbf{v}, s, \sigma)$ an element of $\tilde{\mathscr{S}}(3)$, we can write the law of composition as

$$(\xi, g^*)(\xi', g'^*) = (\xi + \xi' + \Xi(g^*, g'^*), g^* g'^*) \tag{2.5}$$

where $g^* g'^*$ is defined by (2.4) and

$$\Xi(g^*, g'^*) = \tfrac{1}{2}(a'b'|\mathbf{v}|^2 + c'd'|\mathbf{a}|^2 + 2b'c'\mathbf{v}\cdot\mathbf{a} + 2(a'\mathbf{v} + c'\mathbf{a})\cdot R_s \mathbf{a}').$$

(Recall that every analytical function on $\mathscr{S}(3)^*$ allows us to pass to an equivalent extension by putting

$$\Xi'(g^*, g'^*) = \Xi(g^*, g'^*) + \Lambda(g^*) + \Lambda(g'^*) - \Lambda(g^*g'^*).)$$

From now on we shall call $\tilde{\mathscr{S}}(3)$ *the extended Schrödinger group*; the following sections will be devoted to the construction of the I.U.R. of this group. Indeed it is sufficient to limit oneself to these latter, since the P.I.U.R. of $\mathscr{S}(3)$ arising from the I.U.R. of the trivial extension $\mathbb{R} \times \mathscr{S}(3)^*$ will be already given by those I.U.R. of $\tilde{\mathscr{S}}(3)$ whose kernel contains the group of extension.

## III. The induction procedure

The extended group $\tilde{\mathscr{S}}(3)$ admits the following semidirect decomposition

$$\tilde{\mathscr{S}}(3) = \tilde{G}(3) \,\square\, SL(2, \mathbb{R})^*$$

where

$$\tilde{G}'(3) = W_3 \,\square\, SU(2)$$

is the derived group of the extended Galilei group (the isochronous group of J.-M. Lévy-Leblond [7]); $W_3 = \mathbb{R} \,\square\, T_6$ denotes the Heisenberg group of dimension 7. $\tilde{G}'(3)$ is a type I group and its I.U.R. are easy to find; we are therefore in a favourable position to construct the I.U.R. of $\tilde{\mathscr{S}}(3)$ by an induction method.

The normal subgroup $\tilde{G}'(3)$ is not Abelian as in the case of the Euclidean or Poincaré groups; the Wigner method of induction [8] must be modified. We restrict ourselves to giving the individual steps of the recipe we have followed in the inducing procedure. It is an application of the general method due to G. Mackey [9, 10].

(1) Find all the equivalence classes of I.U.R. of $\tilde{G}'(3)$; we call this set the dual $\mathring{\tilde{G}}'(3)$ of the group.

(2) Classify the orbits of $\tilde{\mathscr{S}}(3)$ in $\mathring{\tilde{G}}'(3)$; the $\tilde{\mathscr{S}}(3)$ action is defined as follows: for $[L] \in \mathring{\tilde{G}}'(3)$ and $g \in \tilde{\mathscr{S}}(3)$, $[L] \mapsto [L^g]$ is given by $L^g(h) = L(ghg^{-1})$, $\forall h \in \tilde{G}'(3)$. Choose a class $[L]_0$ in each orbit and find its stabilizer $\Gamma_0 = \tilde{G}'(3) \,\square\, K$. ($K \subseteq SL(2, \mathbb{R})^*$ is called the little group of $\Gamma_0$.)

(3) 'Extend' a representative $L \in [L]_0$ to a unitary representation $L^0$ (projective if necessary) of $\Gamma_0$ and lift an I.U.R. $\mathscr{D}$ (projective if $L^0$ is projective) of $K$ to a representation $\mathscr{D}^0$ of $\Gamma_0$ ($\mathscr{D}^0(g) = \mathscr{D}(g\tilde{G}'(3))$) in order to obtain a representation $\mathscr{L} = L^0 \otimes \mathscr{D}^0$ of $\Gamma_0$ acting on a Hilbert space $\mathscr{H}$.

(4) If the orbit is a point ($\Gamma_0 = \tilde{\mathscr{S}}(3)$), the induced representation is simply given by $\mathscr{L}$. If the orbit is not just a point, find an invariant measure $\mu$ on the orbit ($\tilde{\mathscr{S}}(3)$ and its semidirect decomposition are 'good ones' for this) and construct a section $\Lambda: \mathcal{O} \to \tilde{\mathscr{S}}(3)$ ($\mathcal{O} \simeq \tilde{\mathscr{S}}(3)/\Gamma_0$) such that $\Delta_{[L^g]}: [L]_0 \mapsto [L^g]$.

(5) Induce starting from $\mathscr{L}$ and $\Lambda$: the unitary induced representation of $\tilde{\mathscr{S}}(3)$ which acts on the Hilbert space $L^2_\mu(0; \mathscr{H})$ is given by

$$(U(g)F)(x) = \mathscr{L}(\Lambda_x^{-1} g \Lambda_{g^{-1}x}) F(g^{-1}x).$$

The equivalence class $[U]$ is characterized by the class $[L]_0$ of $\tilde{G}'(3)$ and by a class $[\mathscr{D}]$ of the little group $K$.

*Remarks.* (i) When the normal subgroup is an Abelian one, this recipe reduces to that of Wigner.

(ii) The stabilizers $\Gamma_0$ are of type I for each case under consideration and the rather involved measure theoretical conditions on the dual $\mathring{\tilde{G}}'(3)$ are satisfied so that $\mathscr{S}(3)$ is a type I group.

## IV. The I.U.R. of $\tilde{G}'(3)$ and the orbits of $\mathring{\tilde{G}}'(3)$

Denoting by $g = (\xi, \mathbf{a}, \mathbf{v}, s)$ an element of $\tilde{G}'(3)$, the law of composition can be written as

$$(\xi, \mathbf{a}, \mathbf{v}, s)(\xi', \mathbf{a}', \mathbf{v}', s') = (\xi + \xi' + \mathbf{v}\cdot R_s\mathbf{a}', \mathbf{a} + R_s\mathbf{a}', \mathbf{v} + R_s\mathbf{v}', ss'). \tag{4.1}$$

The I.U.R. of this group are the most easily found by the induction method starting from the semidirect decomposition

$$\tilde{G}'(3) = K \square E(3)^*$$

where $K = \{(\xi, \mathbf{a}, \mathbf{0}, 1)\}$ is an Abelian normal subgroup and $E(3)^* = \{(0, \mathbf{0}, \mathbf{v}, s)\}$ is the universal covering of the Euclidean group $E(3)$. Because of the Abelian character of $K$, the method is standard and we limit ourselves to giving some outlines.

The action of $\tilde{G}'(3)$ on $K$ follows from (4.1):

$$\mathscr{A}(\xi, \mathbf{a}, \mathbf{v}, s): (\xi', \mathbf{a}') \mapsto (\xi' + \mathbf{v}\cdot R_s\mathbf{a}', R_s\mathbf{a}')$$

$$\mathscr{A}(\xi, \mathbf{a}, \mathbf{v}, s)^{-1}: (\xi', \mathbf{a}') \mapsto (\xi' - \mathbf{v}\cdot\mathbf{a}', R_s^{-1}\mathbf{a}').$$

If $\mathring{K} = \{(m, \mathbf{p}) \mid m \in \mathbb{R}, \mathbf{p} \in \mathbb{R}^3\}$ denotes the dual of $K$, the I.U.R. (one-dimensional) of $K$ are given by

$$\langle (m, \mathbf{p}), (\xi', \mathbf{a}')\rangle = \exp i(m\xi' + \mathbf{p}\cdot\mathbf{a}')$$

and the action of $\tilde{G}'(3)$ on $\mathring{K}$, which is defined as

$$\langle \mathscr{A}^*(\xi, \mathbf{a}, \mathbf{v}, s)(m, \mathbf{p}), (\xi', \mathbf{a}')\rangle = \langle (m, \mathbf{p}), \mathscr{A}(\xi, \mathbf{a}, \mathbf{v}, s)^{-1}(\xi', \mathbf{a}')\rangle$$

gives

$$\mathscr{A}^*(\xi, \mathbf{a}, \mathbf{v}, s): (m, \mathbf{p}) \mapsto (m, R_s\mathbf{p} - m\mathbf{v}). \tag{4.2}$$

The orbits of $\tilde{G}'(3)$ in $\mathring{K}$ are thus of 3 types:

(1) $O_m = \{(m, \mathbf{p}) \mid \mathbf{p} \in \mathbb{R}^3\} \simeq \mathbb{R}^3$, $m \neq 0$; the stabilizer of the point $(m, \mathbf{0}) \in O_m$ is $\Gamma_m = K \square SU(2)$.

(2) $O_p = \{(0, \mathbf{p}) \mid |\mathbf{p}|^2 = p^2\} \simeq S^2(p)$, $p > 0$; the stabilizer of the point $(0, (0, 0, p)) \in O_p$ is $\Gamma_p = K \square (T_3 \square SU(1))$, where

$$T_3 \square SU(1) = \left\{\left[\mathbf{v}, \begin{pmatrix} \exp i\psi/2 & 0 \\ 0 & \exp -i\psi/2 \end{pmatrix}\right] \,\middle|\, \mathbf{v} \in \mathbb{R}^3,\ 0 \leqslant \psi < 4\pi\right\}$$

(3) $O_0 = \{(0, \mathbf{0})\}$, point orbit; the stabilizer is $\Gamma_0 = K \square E(3)^*$.

The next step consists of constructing the I.U.R. of the little groups of the stabilizers; but these latter are either well known ($SU(2)$, $E(3)^*$) or they are easy to compute by induction ($T_3 \square SU(1)$). Finally we induce, starting from a section $\Lambda: O \to \tilde{G}'(3)$. The results for the three types of orbits are the following.

(1) The U.R. of $\Gamma_m$ associated with the stabilized point $(m, \mathbf{0})$ are:

$$D_{m,j}(\xi, \mathbf{a}, \mathbf{0}, s) = e^{-im\xi} D_j(s).$$

where $D_j(s): \mathbb{C}^{2j+1} \to \mathbb{C}^{2j+1}$ denotes the U.R. of $SU(2)$, $2j = 0, 1, 2, \ldots$. The action of $\tilde{G}'(3)$ on $O_m$ is given by (4.2) and a choice of section is $\Lambda_{\mathbf{p}} = (0, \mathbf{0}, -1/m\mathbf{p}, 1)$ $(\Lambda_{\mathbf{p}}: (m, \mathbf{0}) \mapsto (m, \mathbf{p}))$; then

$$D_{m,j}(\Lambda_{\mathbf{p}}^{-1} g \Lambda_{g^{-1}\mathbf{p}}) = e^{i(m\xi + \mathbf{p}\cdot\mathbf{a})} D_j(s)$$

and the induced I.U.R. of $\tilde{G}'(3)$ is

$$(L_{m,j}(\xi, \mathbf{a}, \mathbf{v}, s)f)(\mathbf{p}) = e^{i(m\xi + \mathbf{p}\cdot\mathbf{a})} D_j(s) f(R_s^{-1}(\mathbf{p} + m\mathbf{v})), \quad m \neq 0, \quad 2j \in \mathbb{N},$$
$$f \in L^2(\mathbb{R}^3; \mathbb{C}^{2j+1}; d^3p). \quad (4.3)$$

(2) The U.R. of $\Gamma_p$ associated to the point $(0, (0, 0, p))$ are here of two types:

(a) $(D_{\mathbf{p},\mathbf{q}}(\xi, \mathbf{a}, \mathbf{v}, s(\psi))f)(\theta) = e^{i(\mathbf{p}\cdot\mathbf{a} + R(\theta)\mathbf{q}\cdot\mathbf{v})} f(\theta - \psi)$
$\mathbf{p} = (0, 0, p), \quad p > 0; \quad \mathbf{q} = (0, q, r), \quad q > 0, \quad r \in \mathbb{R}$
$f \in L^2([0, 2\pi]; d\theta)$

$$s(\psi) = \begin{pmatrix} \exp i\psi/2 & 0 \\ 0 & \exp -i\psi/2 \end{pmatrix}; \quad R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) $D_{\mathbf{p},\mathbf{q},\nu}(\xi, \mathbf{a}, \mathbf{v}, s(\psi)) = e^{i(\mathbf{p}\cdot\mathbf{a} + \mathbf{q}\cdot\mathbf{v} + \nu\psi)}$
$\mathbf{p} = (0, 0, p), \quad p > 0; \quad \mathbf{q} = (0, 0, q), \quad q \in \mathbb{R}; \quad 2\nu \in \mathbb{Z}.$

The action of $\tilde{G}'(3)$ on $O_p \simeq S^2(p) \simeq pS^2$ is given by (4.2) with $m = 0$ and a section can be chosen as $\Lambda_{\mathbf{z}} = (0, \mathbf{0}, \mathbf{0}, s_{\mathbf{z}})$ $(\Lambda_{\mathbf{z}}: (0, 0, p) \to R_{s\mathbf{z}}(0, 0, p) = p\mathbf{z}$, (cf. [11]). For the case (a), the induced representations act on the space $L^2(S^2; L^2([0, 2\pi]; d\theta); d\Omega)$; thanks to the Euler angle parametrization, this space is isometric to $L^2(SO(3); dR)$; thus

(a) $(L_{\mathbf{p},\mathbf{q}}(\xi, \mathbf{a}, \mathbf{v}, s)f)(R) = e^{i(R\mathbf{p}\cdot\mathbf{a} + R\mathbf{q}\cdot\mathbf{v})} f(R_s^{-1}R)$
$\mathbf{p} = (0, 0, p), \quad p > 0; \quad \mathbf{q} = (0, q, r), \quad q > 0, \quad r \in \mathbb{R}$ (4.4a)
$f \in L^2(SO(3); dR)$.

(b) $(L_{p,q,\nu}(\xi, \mathbf{a}, \mathbf{v}, s)f)(\mathbf{z}) = e^{i(\mathbf{z}\cdot(p\mathbf{a} + q\mathbf{v}) + \nu\theta(s, \mathbf{z}))} f(R_s^{-1}\mathbf{z})$
$p > 0, \quad q \in \mathbb{R}, \quad 2\nu \in \mathbb{Z}$ (4.4b)
$f \in L^2(S^2, d\Omega)$ (4.4b)
(with $s(\theta(s, \mathbf{z})) = s_{\mathbf{z}}^{-1} \cdot s \cdot s_{R_s^{-1}\mathbf{z}}$, [11]).

(3) The U.R. of the stabilizer of $(0, \mathbf{0})$ are those of group $E(3)^*$; since the orbit is a point, the induction is trivial and we obtain the two types of representations [11]:

(a) $(L_{q,\nu}(\xi, \mathbf{a}, \mathbf{v}, s)f)(\mathbf{z}) = e^{i(q\mathbf{z}\cdot\mathbf{v} + \nu\theta(s, \mathbf{z}))} f(R_s^{-1}\mathbf{z})$
$q > 0, \quad 2\nu \in \mathbb{Z}$ (4.5a)
$f \in L^2(S^2; d\Omega)$

(b) $L_j(\xi, \mathbf{a}, \mathbf{v}, s) = D_j(s)$
$2j \in \mathbb{N}$. (4.5b)

The dual $\mathring{\tilde{G}}'(3)$ of $\tilde{G}'(3)$, i.e. the set of all its equivalence classes of I.U.R., is described by the set of indices appearing in the representations listed above. The action of $\tilde{\mathscr{S}}(3)$ on $\mathring{\tilde{G}}'(3)$ was defined in the Section III by

$$g: [L] \mapsto [L^g], \quad \text{where} \quad L^g(h) = L(ghg^{-1}), \quad \forall h \in \tilde{G}'(3).$$

If $g \in \tilde{G}'(3)$, $L^g(h) = L(g)L(h)L(g)^{-1}$, then of course $[L^g] = [L]$. Since each element $g = (\xi, \mathbf{a}, \mathbf{v}, s, \sigma)$ of $\tilde{\mathscr{S}}(3)$ decomposes uniquely under the form $g = \text{kh}$ with $k = (0, \mathbf{0}, \mathbf{0}, 1, \sigma) \equiv \sigma \in SL(2, \mathbb{R})^*$ and $h = (\xi, \mathbf{a}, \mathbf{v}, s, 1) \in \tilde{G}'(3)$, we see that the action of $\tilde{\mathscr{S}}(3)$ on the dual is uniquely determined by that of the elements $\sigma$ of $SL(2, \mathbb{R})^*$.

By virtue of the law of composition (2.4) and (2.5):

$$\sigma \cdot (\xi, \mathbf{a}, \mathbf{v}, s) \cdot \sigma^{-1} = (\xi - \tfrac{1}{2}(ac|\mathbf{a}|^2 + bd|\mathbf{v}|^2 - 2bc\mathbf{a} \cdot \mathbf{v}, a\mathbf{a} - b\mathbf{v}, d\mathbf{v} - c\mathbf{a}, s) \tag{4.6}$$

with $\Pi: \sigma \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$.

On the representatives (4.3)–(4.5) of $[L]$, we obtain respectively

$$(1) \quad (L_{m,j}(\xi, \mathbf{a}, \mathbf{v}, s)f)(\mathbf{p}) = \exp i\left(m\xi - \frac{m}{2}(ac|\mathbf{a}|^2 + bd|\mathbf{v}|^2 - 2bc\mathbf{a} \cdot \mathbf{v}) + \mathbf{p} \cdot (a\mathbf{a} - b\mathbf{v})\right) D_j(s) f(R_s^{-1}(\mathbf{p} + m(d\mathbf{v} - c\mathbf{a}))) \tag{4.7}$$

$L^\sigma_{m,j}$ is always in the class $(m, j)$; the orbits are here reduced to points.

$$(2a) \quad (L^\sigma_{\mathbf{p},\mathbf{q}}(\xi, \mathbf{a}, \mathbf{v}, s)f)(R) = \exp i(R(a\mathbf{p} - c\mathbf{q}) \cdot \mathbf{a} + R(d\mathbf{q} - b\mathbf{p}) \cdot \mathbf{v}) \cdot f(R_s^{-1}R). \tag{4.8}$$

Recall that $(\mathbf{p}, \mathbf{q}) = ((0, 0, p), (0, q, r))$ with $p > 0, q > 0$; in general the two vectors $(a\mathbf{p} - c\mathbf{q}, d\mathbf{q} - b\mathbf{p})$ are not of this form, but it is easy to verify that there always exists a rotation $s \in \tilde{G}'(3)$ such that $[L^\sigma_{\mathbf{p},\mathbf{q}}] = [L^{\sigma s}_{\mathbf{p},\mathbf{q}}] = L_{\mathbf{p}',\mathbf{q}'}]$ with $(\mathbf{p}', \mathbf{q}')$ of the required form and $\mathbf{p}' = R_s(a\mathbf{p} - c\mathbf{q})$, $\mathbf{q}' = R_s(d\mathbf{q} - b\mathbf{p})$. We remark that the action of $\sigma$ on these classes:

$$(\mathbf{p}, \mathbf{q}) \mapsto (a\mathbf{p} - c\mathbf{q}, d\mathbf{q} - b\mathbf{p}) \text{ (modulo a rotation)}$$

leaves invariant the cross product $\mathbf{p} \times \mathbf{q} = (-pq, 0, 0)$; it follows from this that the orbits contain all the classes $(\mathbf{p}, \mathbf{q})$ for which $|\mathbf{p} \times \mathbf{q}|$ is fixed $\neq 0$.

(2b) and (3a) (We set $L_{q,v} = L_{0,q,v}$.)

$$(L^\sigma_{p,q,v}(\xi, \mathbf{a}, \mathbf{v}, s)f)(\mathbf{z}) = \exp i((ap - cq)\mathbf{z} \cdot \mathbf{a} + (dq - bp)\mathbf{z} \cdot \mathbf{v} + v\theta(s, \mathbf{z})) \cdot f(R_s^{-1}\mathbf{z}). \tag{4.9}$$

The classes $(p, q, v)$ satisfy the positivity conditions $p > 0$, or $q > 0$ when $p = 0$. These conditions are in general not satisfied by the transformed quantities $ap - cq$ and $dq - bp$. However, because of the property $\theta(s, -\mathbf{z}) = -\theta(s, \mathbf{z})$, it is easily seen that $L_{p,q,v} \sim L_{-p,-q,-v}$. This together with the transitivity of the action of $SL(2, \mathbb{R})$ on the couples $(p, q)$ implies that the orbits contain all the classes $(p, q, v)$ with $|v|$ fixed.

$$(3b) \quad L^\sigma_j = L_j. \tag{4.10}$$

The action of $\sigma$ in this case is trivial and the orbits are points.

To summarize, there exist 4 classes of orbits in $\mathring{\tilde{G}}'(3)$:

*Class 1.* $O_{m,j} = \{(m, j)\}, \quad m \in \mathbb{R}, \quad m \neq 0, \quad 2j \in \mathbb{N}.$

*Class 2.* $O_\mu = \{((0, 0, p), (0, q, r)) \mid p > 0, q > 0, r \in \mathbb{R}, pq = \mu\}, \quad \mu > 0.$

*Class 3.* $O_\nu = \{(p, q, \pm\nu) \mid p > 0, q \in \mathbb{R} \quad \text{or} \quad p = 0, q > 0\}, \quad 2\nu \in \mathbb{N}.$

*Class 4.* $O_j = \{(j)\}, \qquad 2j \in \mathbb{N}.$

To each class of orbits corresponds a type of I.U.R. of $\tilde{\mathscr{S}}(3)$. The following sections are devoted to the construction of representatives for each type.

## V. Class 1

Since the orbits $O_{m,j}$ reduce to a point, the stabilizer is $\tilde{\mathscr{S}}(3) = \tilde{G}'(3) \square SL(2, \mathbb{R})^*$, its little group being $SL(2, \mathbb{R})^*$. Following the procedure exposed in Section III, the first step consists of 'extending' a representation $L_{m,j}$ of $\tilde{G}'(3)$ to the stabilizer $\tilde{\mathscr{S}}(3)$. This is essentially the very problem to solve in this case since the I.U.R. of the little group $SL(2, \mathbb{R})^*$ have already been determined [12, 13] and the question of choosing a section does not arise since the orbits are points.

The problem of 'extending' $L_{m,j}$ reduces to that of constructing a unitary representation $\sigma \mapsto T(\sigma)$ of $SL(2, \mathbb{R})^*$ (possibly a projective one), such that

$$T(\sigma)L_{m,j}(h)T(\sigma)^{-1} = L^\sigma_{m,j}(h), \qquad \forall h \in \tilde{G}'(3). \tag{5.1}$$

Actually, writing the composition law (2.4) and (2.5) in the form

$$(h, \sigma)(h', \sigma') = (\sigma'^{-1}h\sigma'h', \sigma\sigma'), \ h, h' \in \tilde{G}'(3); \sigma, \sigma' \in SL(2, \mathbb{R})^*$$

we see at once that

$$L^0_{m,j}: (h, \sigma) \mapsto T(\sigma)L_{m,j}(h) \tag{5.2}$$

is a representation of the stabilizer $\tilde{\mathscr{S}}(3)$ which extends $L_{m,j}$ (in the sense that $L^0_{m,j} \downarrow_{\tilde{G}'(3)} = L_{m,j}$). Due to the irreducibility of $L_{m,j}$, the representation $T$ so defined is unique up to a phase factor.

When constructing the representation $T$, it is advantageous to use a parametrization of $SL(2, \mathbb{R})^*$ related to the Iwasawa decomposition of $SL(2, \mathbb{R})$:

$$\Pi: \sigma(\theta, u, v) \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \tag{5.3}$$

$$\theta \in \mathbb{R}, \quad u \in \mathbb{R}^+, \quad v \in \mathbb{R}.$$

This decomposition of $SL(2, \mathbb{R})^*$ into three one parameter subgroups makes it possible to express the intertwining operators $T(\sigma)$ as a product of 3 unitary operators

$$T(\sigma(\theta, u, v)) = T_1(\theta)T_2(u)T_3(v).$$

The operators $T_2(u)$ and $T_3(v)$ are easy to find; by restricting the expression (4.7) for $L^\sigma_{m,j}$ to $\sigma = \sigma(0, u, 0)$ and to $\sigma = \sigma(0, 1, v)$, we can verify directly that

$$(T_2(u)f)(\mathbf{p}) = |u|^{3/2}f(u\mathbf{p}) \tag{5.4}$$

and

$$(T_3(v)f)(\mathbf{p}) = e^{(iv/2m)|\mathbf{p}|^2}f(\mathbf{p}) \tag{5.5}$$

satisfy the intertwining condition (5.1) and define a unitary representation for the subgroups $\{\sigma(0, u, 0)\}$ and $\{\sigma(0, 1, v)\}$ of $SL(2, \mathbb{R})^*$. The construction of $T_3(\theta)$ is more awkward. Let us first remark that when restricting $L^\sigma_{m,j}$ to $\sigma = \sigma(2k\pi, 1, 0)$, the corresponding operator $T_3(2k\pi)$ must be a multiple of the identity (from this follows that Ker $\Pi$ will be represented in $L^0_{m,j}$ by phase factors). Secondly, when restricting $L^\sigma_{m,j}$ to $\sigma = \sigma(\pi/2 + 2k\pi, 1, 0)$, we obtain

$$(L^\sigma_{m,j}(\xi, \mathbf{a}, \mathbf{v}, s)f)(\mathbf{p}) = e^{i(m\xi + \mathbf{v}\cdot(\mathbf{p} - m\mathbf{a}))} D_j(s) f(R_s^{-1}(\mathbf{p} - m\mathbf{a})).$$

Comparing this expression with the one for $L_{m,j}$ (4.3), we easily see that the corresponding operator $T_3(\pi/2 + 2k\pi)$ must be, up to a phase factor, a Fourier transform $\mathscr{F}_m$:

$$\left(T_3\left(\frac{\pi}{2} + 2k\pi\right)f\right)(\mathbf{p}) = e^{i\Lambda(k)}(\mathscr{F}_m f)(\mathbf{p}) = \frac{e^{i\Lambda(k)}}{|2\pi m|^{3/2}} \int_{\mathbb{R}^3} d^3q\, e^{-i/(m)\mathbf{p}\cdot\mathbf{q}} f(\mathbf{q}). \quad (5.6)$$

Finally, recalling that any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $d \neq 0$ of $SL(2, \mathbb{R})$ can be decomposed as

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

it is possible, once

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is written in this form, to construct the operator $T_3(\theta)$, for $\theta \neq k\pi$, as a product of operators (5.4), (5.5) and (5.6)

$$\left(\Pi\colon \sigma\left(\frac{\pi}{2} + 2k\pi, 1, 0\right) \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right).$$

We thus obtain

$$(T_3(\theta)f)(\mathbf{p}) = \frac{e^{i\Lambda(\theta)}}{|2\pi m \sin\theta|^{3/2}} \int_{\mathbb{R}^3} d^3q \exp\left[\frac{i}{2m\sin\theta}(\cos\theta(|\mathbf{p}|^2 + |\mathbf{q}|^2) - 2\mathbf{p}\cdot\mathbf{q})\right] f(\mathbf{q}). \quad (5.7)$$

One checks directly that $T_3(\theta)$ satisfies the intertwining condition (5.1) for any factor $\Lambda(\theta)$; it remains to examine if, for a judicious choice of this factor, $\theta \mapsto T_3(\theta)$ defines a (continuous) representation of the subgroup $\{\sigma(\theta, 1, 0) \mid \theta \in \mathbb{R}\}$. This requires

(a) $T_3(\theta_1)T_3(\theta_2) = T_3(\theta_1 + \theta_2)$

(b) $\lim_{\theta \to k\pi} T_3(\theta) = T_3(k\pi)$ exists (strongly).

Putting the expression (5.7) in the condition (a), we obtain the equation

$$\Lambda(\theta_1) + \Lambda(\theta_2) - \Lambda(\theta_1 + \theta_2) \equiv -\frac{3\pi}{4} \operatorname{sig}(m \sin\theta_1 \sin\theta_2 \sin(\theta_1 + \theta_2)) \pmod{2\pi}.$$

By inspection, one assures oneself that a solution is given by

$$\Lambda(\theta) = \frac{\pi}{2}\operatorname{sig}(m)\left(\operatorname{sig}\left(\sin\frac{\theta}{2}\right) + \frac{3}{2}\operatorname{sig}(\sin\theta)\right). \tag{5.8}$$

Using this factor and the limit

$$f(\mathbf{p}) = \lim_{\varepsilon\to 0}\frac{e^{i(3\pi/4)\operatorname{sig}(\varepsilon)}}{|2\pi\varepsilon|^{3/2}}\int_{\mathbb{R}^3} d^3q\, e^{-(i/2\varepsilon)|\mathbf{q}-\mathbf{p}|^2} f(\mathbf{q})$$

(more precisely the strong limit $id = \lim_{\varepsilon\to 0}\mathscr{F}T_2(\varepsilon)\mathscr{F}^{-1}$) in condition (b), we obtain

$$(T_3(k\pi)f)(\mathbf{p}) = e^{i(k\pi/2)\operatorname{sig}(m)} f((-1)^k\mathbf{p}). \tag{5.9}$$

Thus $T_3$ turns out to be a faithful representation of the double covering of $SO(2)$ and a 'two valued representation' of $SO(2)$; we shall see in Section IX that the $4\pi$-periodicity of the evolution operator for a harmonic oscillator is a consequence of this representation. Let us mention that the solution (5.8) is not unique; however one can show that no factor $\Lambda(\theta)$ exists making it possible to obtain a 'single valued representation' of $SO(2)$.

Collecting the partial results (5.4), (5.5), (5.7) and (5.8), we can write down the 'extended' representation $L^0_{m,j}$ (5.2). In order to deduce from it the expressions for the infinitesimal generators (2.3) in this representation, we make the identification

$$\Pi\colon \sigma(\theta, u, v) \mapsto \begin{pmatrix} u\cos\theta & uv\cos\theta - u^{-1}\sin\theta \\ u\sin\theta & uv\sin\theta + u^{-1}\cos\theta \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$(L^0_{m,j}(\xi, \mathbf{a}, \mathbf{v}, s, \sigma)f)(\mathbf{p}) = \frac{e^{i(\Lambda(\sigma)+m\xi)}}{|2\pi mc|^{3/2}}$$

$$\times \int_{\mathbb{R}^3} d^3q \exp\left[\frac{i}{2mc}(a|\mathbf{p}|^2 - 2\mathbf{p}\cdot\mathbf{q} + d|\mathbf{q}|^2) + i\mathbf{q}\cdot\mathbf{a}\right] D_j(s) f(R_s^{-1}(\mathbf{q} + m\mathbf{v})). \tag{5.9}$$

This integral operator must be taken in its generalized sense ([14] page 494); for $\sigma = \sigma(k\pi, u, v)$ it becomes

$$(L^0_{m,j}(\xi, \mathbf{a}, \mathbf{v}, s, \sigma)f)(\mathbf{p}) = \exp\left[i\left(\frac{k\pi}{2} + m\xi + \frac{u^2v}{2m}|\mathbf{p}|^2 + (-1)^k u\mathbf{p}\cdot\mathbf{a}\right)\right]$$
$$\times D_j(s) f(R_s^{-1}((-1)^k\mathbf{p} + m\mathbf{v})). \tag{5.9'}$$

The I.U.R. of the little group $SL(2, \mathbb{R})^*$ of the stabilizer are known: they were classified and constructed by L. Pukansky [12]. Let us simply mention here that their classes are completely characterized by a couple $(q, h)$, where $q$ is the value of the Casimir operator of $\mathfrak{sl}(2, \mathbb{R})$ and $h$ is some real number closely related to the spectrum of one of its infinitesimal generators, similarly as in the $SL(2, \mathbb{R})$ case [13].

Since the orbits are here reduced to a point, the induced representations of $\tilde{\mathscr{S}}(3)$ are simply given by

$$U_{m,j,q,h} = L^0_{m,j} \otimes \mathscr{D}^0_{q,h} \tag{5.10}$$

where $\mathscr{D}^0_{q,h}$ denotes the lifting to $\tilde{\mathscr{S}}(3)$ of a I.U.R. $\mathscr{D}_{q,h}$ of $SL(2, \mathbb{R})^*$.

It can be verified directly from the expression (5.9) for $L^0_{m,j}$ that these induced representations define projective representations of $\mathscr{S}(3)$ and that there exists a one-to-one correspondence between the classes $(m, j, q, h)$ of I.U.R. of $\tilde{\mathscr{S}}(3)$ and the corresponding projectively equivalent classes of P.I.U.R. of $\mathscr{S}(3)$.

We shall see that these representations exhaust the list of the P.I.U.R. of $\mathscr{S}(3)$ for which the extension group is not trivially represented. Because of their physical importance, we return to them in Section IX.

We finish this section by giving the list of the infinitesimal generators (2.3) in these representations; more exactly we give the associated self-adjoint generators $-i\ell$, $\ell \in \mathfrak{Z}(3)$.

$$\begin{aligned}
J_1 &= i(p_2\partial_3 - p_3\partial_2) + \Sigma_1 \\
J_2 &= i(p_3\partial_1 - p_1\partial_3) + \Sigma_2 \\
J_3 &= i(p_1\partial_2 - p_2\partial_1) + \Sigma_3 \\
G_+ &= \frac{1}{2m}|\mathbf{p}|^2 + g_+ \\
G_0 &= -i\left(p^i\partial_i + \frac{3}{2}\right) + g_0 \\
G_- &= \frac{m}{2}\Delta + g_- \\
\left.\begin{aligned} K_k &= -im\partial_k \\ P_k &= p_k \end{aligned}\right\} &\quad k = 1, 2, 3 \\
M &= m.
\end{aligned} \tag{5.11}$$

($\{\Sigma_i\}$ is a basis in the $D_j$ representation and $\{g_+, g_0, g_-\}$ is a basis in the $\mathscr{D}_{g,h}$ representation.)

The Lie algebra $\mathfrak{Z}(3)$ has 3 fundamental invariant operators:

$$M$$

$$S^2 = \left(\mathbf{J} + \frac{1}{m}\mathbf{P} \times \mathbf{K}\right)^2 = \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2$$

$$\begin{aligned}
G^2 &= \left(G_0 - \frac{1}{m}\left(\mathbf{P}\cdot\mathbf{K} - \frac{3i}{2}M\right)\right)^2 + 2\left\{\left(G_+ - \frac{1}{2m}\mathbf{P}^2\right), \left(G_- + \frac{1}{2m}\mathbf{K}^2\right)\right\}_+ \\
&= g_0^2 + 2\{g_+, g_-\}_+,
\end{aligned}$$

the eigenvalues of which, in the representations $(m, j, q, h)$ are given by $-m$, $j(j+1)$ and $q$.

## VI. Class 2

In the orbit $O_\mu$ we choose the point $(\mathbf{p}, \mathbf{q}) = ((0, 0, \mu), (0, \mu, 0))$ and, for this class, the representative $L_{\mathbf{p},\mathbf{q}} \equiv L_\mu$ given by (4.4a)

$$(L_\mu(\xi, \mathbf{a}, \mathbf{v}, s)f)(R) = e^{i(R\mathbf{p}\cdot\mathbf{a} + R\mathbf{q}\cdot\mathbf{v})}f(R_s^{-1}R). \tag{6.1}$$

Since the classes $[L_{\mathbf{p},\mathbf{q}}]$ are defined by the couples $(\mathbf{p}, \mathbf{q})$ modulo a rotation ($L_{R\mathbf{p},R\mathbf{q}} \sim L_{\mathbf{p},\mathbf{q}}$), the stabilizer $\Gamma_\mu$ of $[L_\mu]$ is

$$\Gamma_\mu = \tilde{G}'(3) \,\square\, K$$

where the little group $K = \{\sigma(\theta, 1, 0) \equiv \sigma(\theta) \mid \theta \in \mathbb{R}\}$ is the subgroup of $SL(2, \mathbb{R})^*$ introduced by (5.3).

By restricting $\sigma$ to $\sigma(\theta, 1, 0)$ and $(\mathbf{p}, \mathbf{q})$ to $((0, 0, \mu), (0, \mu, 0))$ in (4.8) we obtain

$$(L_\mu^\sigma(\xi, \mathbf{a}, \mathbf{v}, s)f)(R) = e^{i(RR(\theta)\mathbf{p}\cdot\mathbf{a} + RR(\theta)\mathbf{q}\cdot\mathbf{v})}f(R_s^{-1}R) \tag{6.2}$$

where

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 1 \end{pmatrix}.$$

One verifies easily that the unitary operators

$$(T(\theta)f)(R) = f(RR(\theta)) \tag{6.3}$$

intertwine the representations (6.1) and (6.2) and define a representation of $K$. An extension of $L_\mu$ to $\Gamma_\mu$ is thus given by

$$L_\mu^0: (h, \sigma(\theta)) \mapsto T(\theta)L_\mu(h), \quad h \in \tilde{G}'(3). \tag{6.4}$$

The I.U.R. of $K$ are one dimensional and labelled by a real number $\alpha$; $\sigma(\theta) \mapsto e^{i\alpha\theta}$. The representation $\mathscr{L}_{\mu,\alpha}$ of $\Gamma_\mu$ associated to the stabilized point $((0, 0, \mu), (0, \mu, 0))$ is thus

$$(\mathscr{L}_{\mu,\alpha}(\xi, \mathbf{a}, \mathbf{v}, s, \sigma(\theta))f)(R) = e^{i\alpha\theta}\, e^{i(RR(\theta)\mathbf{p}\cdot\mathbf{a} + RR(\theta)\mathbf{q}\cdot\mathbf{v})}f(R_s^{-1}RR(\theta)). \tag{6.5}$$

The orbit $O_\mu \simeq \tilde{\mathscr{S}}(3)/\Gamma_\mu \simeq SL(2, \mathbb{R})/SO(2)$ can be realized by a sheet of the hyperboloid:

$$H_+^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z \geqslant 1, -x^2 - y^2 + z^2 = 1\}.$$

The group $\tilde{\mathscr{S}}(3)$ acts on $H_+^2$ via the $SO(2, 1)$ transformations:

$$T(\sigma): X \mapsto \Pi(\sigma)X\Pi(\sigma)^{-1}$$

where $\Pi: (0, \mathbf{0}, \mathbf{0}, 1, \sigma) \mapsto \Pi(\sigma) \in SL(2, \mathbb{R})$ and

$$(x, y, z) \mapsto X = \begin{pmatrix} -y & x + z \\ x - z & y \end{pmatrix}.$$

The invariant measure on the orbit is therefore given by $d\mu = z^{-1}\,dx\,dy$ and a section $\Lambda: H_+^2 \to \tilde{\mathscr{S}}(3)$, such that $T(\Lambda_\mathbf{x}): (0, 0, 1) \mapsto \mathbf{x}$ is for example

$$\Lambda_\mathbf{x} = \left(0, \mathbf{0}, \mathbf{0}, 1, \sigma\left(\frac{\pi}{2}, \frac{1}{\sqrt{x + z}}, -y\right)\right)$$

with respect to the decomposition (5.3) of $SL(2, \mathbb{R})^*$.

The representation of $\tilde{\mathscr{S}}(3)$ induced by $\mathscr{L}_{\mu,\alpha}$ and $\Lambda_\mathbf{x}$ is thus

$$\begin{aligned}(U_{\mu,\alpha}(\xi, \mathbf{a}, \mathbf{v}, s, \sigma)F)(R, \mathbf{x}) = \mathscr{L}_{\mu,\alpha}&(\Lambda_\mathbf{x}^{-1}\cdot(\xi, \mathbf{a}, \mathbf{v}, s, \sigma)\cdot\Lambda_{T(\sigma)^{-1}\mathbf{x}}) \\ &\times F(R_s^{-1}RR(\theta(\sigma, \mathbf{x})), T(\sigma)^{-1}\mathbf{x})\end{aligned} \tag{6.6}$$

where $F \in L^2(SO(3) \times H_+^2; dR\,d\mu)$ and $\theta(\sigma, \mathbf{x})$ is defined by

$$\Lambda_\mathbf{x}^{-1}\cdot\sigma\cdot\Lambda_{T(\sigma)^{-1}\mathbf{x}} = \sigma(\theta(\sigma, \mathbf{x}), 1, 0).$$

The kernel of this representation $U_{\mu,\alpha}$ contains the extension subgroup $\{(\xi, \mathbf{0}, \mathbf{0}, 1, 1)\}$; it is therefore an I.U.R. of $\mathscr{S}(3)^*$ which gives, as is easy to check, a P.I.U.R. of $\mathscr{S}(3)$ (a true I.U.R. if $\alpha = 2k\pi$).

The pairs $(\mu, \alpha)$, $\mu > 0$, $\alpha \in \mathbb{R}$, characterize the P.I.U.R. of these Class 2 representations of $\mathscr{S}(3)$.

When trying to give a physical interpretation to this class of P.I.U.R., we are faced with the same difficulties as in the case of the true I.R. of the Galilei group. The localizability conditions of A. S. Wightman [15] are not satisfied and we could present the same arguments here as given by E. Inonu and E. P. Wigner [16] for the Galilei group. We shall not go into this any further here; let us simply mention that by reducing these representations with respect to the Galilei group, only true representations appear in the decomposition. The same is true for the two following classes.

## VII. Class 3

In the orbit $O_\nu$ we choose the point $(p, q, \nu) = (0, 1, \nu)$ and for this class the representative $L_\mu$ is given by (4.4b):

$$(L_\nu(\xi, \mathbf{a}, \mathbf{v}, s)f)(\mathbf{z}) = e^{i(\mathbf{z}\cdot\mathbf{v} + \nu\theta(s, \mathbf{z}))} f(R_s^{-1}\mathbf{z}). \tag{7.1}$$

The stabilizer of this class $(0, 1, \nu)$ is

$$\Gamma_\nu = \tilde{G}'(3) \,\square\, N_\nu$$

with the little group

$$N_\nu = \{\sigma(2k\pi, 1, \tau), \quad \tau \in \mathbb{R}\} \quad \text{if } \nu \neq 0$$

$$N_0 = \{\sigma(k\pi, 1, \tau), \quad \tau \in \mathbb{R}\} \quad \text{if } \nu = 0.$$

(The difference between the case $\nu = 0$ and the cases $\nu \neq 0$ is due to the fact that $L_{p,q,\nu} \nsim L_{-p,-q,\nu}$ if $\nu \neq 0$, while $L_{p,q,0} \sim L_{-p,-q,0}$.)

By restricting $\sigma$ in (4.9) to the elements of $N_\nu$ and $(p, q, \nu)$ to $(0, 1, \nu)$ we see that $L_\nu^\sigma = L_\nu$ if $\nu \neq 0$, while $L_0^\sigma$ differs from $L_0$ by the exponent which changes its sign. The intertwining operators can then be chosen as the identity operator if $\nu \neq 0$ and as

$$(T(\sigma(k\pi, 1, \tau))f)(\mathbf{z}) = f((-1)^k\mathbf{z})$$

if $\nu = 0$. The I.U.R. of $N_\nu$ are one dimensional and characterized by two real numbers $\alpha$ and $\beta$.

$$\sigma(n\pi, 1, \tau) \mapsto e^{i\alpha n\pi} e^{i\beta\tau}$$

$$(n = 2k \quad \text{if } \nu \neq 0, \qquad n = k \quad \text{if } \nu = 0).$$

The representation of the stabilizer $\Gamma_\nu$ of the point $(0, 1, \nu)$ is thus

$$\mathscr{L}_{\nu,\alpha,\beta}(\xi, \mathbf{a}, \mathbf{v}, s, \sigma(n\pi, 1, \tau))f)(\mathbf{z}) = e^{i(\alpha n\pi + \beta\tau + (-1)^n\mathbf{z}\cdot\mathbf{v} + \nu\theta(s,\mathbf{z}))} f((-1)^n R_s^{-1}\mathbf{z}). \tag{7.2}$$

The orbit $O_\nu = \tilde{\mathscr{S}}(3)/\Gamma_\nu \simeq SL(2, \mathbb{R})/N'_\nu$ where

$$N'_\nu = \left\{\begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \quad n = 2k \quad \text{if } \nu \neq 0, \qquad n = k \quad \text{if } \nu = 0\right\}$$

is different for $\nu \neq 0$ or $\nu = 0$.

For $\nu \neq 0$, the orbit is realized by the punctured plane $\mathbb{R}^2_*$ with Lebesgue measure as invariant measure. Then $\tilde{\mathscr{S}}(3)$ acts through the transformations $\Pi(\sigma)$ of $SL(2, \mathbb{R})$; a choice of section $\Lambda: \mathbb{R}^2_* \to \tilde{\mathscr{S}}(3)$ is for example

$$\Lambda_{\mathbf{x}} = (0, \mathbf{0}, \mathbf{0}, 1, \sigma(\theta, r, 0))$$

where $(r, \theta)$ are the polar coordinates of $\mathbf{x}$ (the stabilized point is realized in $\mathbb{R}^2_*$ by $(1, 0)$). The representation of $\tilde{\mathscr{S}}(3)$ induced by $\mathscr{L}_{\nu,\alpha,\beta}$ and $\Lambda_{\mathbf{x}}$ is thus

$$(U_{\nu,\beta}(\beta, \mathbf{a}, \mathbf{v}, s, \sigma)F)(\mathbf{z}, \mathbf{x}) = \mathscr{L}_{\nu,0,\beta}(\Lambda_{\mathbf{x}}^{-1}\cdot(\xi, \mathbf{a}, \mathbf{v}, s, \sigma) \times \Lambda_{\Pi(\sigma)^{-1}\mathbf{x}})F(R_s^{-1}\mathbf{z}, \Pi(\sigma)^{-1}\mathbf{x}) \quad (7.3)$$

where $F \in L^2(S^2 \times \mathbb{R}^2_*; d\Omega\, d\mathbf{x})$.

The index $\alpha$ does not appear in the induced representation because the section was chosen so that

$$\Lambda_{\mathbf{x}}^{-1}\cdot\sigma\cdot\Lambda_{\Pi(\sigma)^{-1}\mathbf{x}} = \sigma(0, 1, \tau(\sigma, \mathbf{x})).$$

For $\nu = 0$, the orbit is realized by $O_0 \simeq \mathbb{R}^2_*/\varepsilon$, where $\varepsilon$ is the equivalence relation $\mathbf{x}\varepsilon - \mathbf{x}$; the invariant measure and the action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2_*$ pass to the quotient $([\mathbf{x}] \mapsto [\Pi(\sigma)\mathbf{x}])$ and a section $\Lambda\colon [\mathbf{x}] \mapsto \tilde{\mathscr{S}}(3)$ is for example given by

$$\Lambda_{[\mathbf{x}]} = (0, \mathbf{0}, \mathbf{0}, 1, \sigma(\theta, r, 0))$$

where $r = |\mathbf{x}|$ and $\theta$ is the polar angle, $0 \leqslant \theta < \pi$ of one of the elements $\mathbf{x}$ or $-\mathbf{x}$ of the class $[\mathbf{x}]$. The induced representation of $\tilde{\mathscr{S}}(3)$ associated with $\mathscr{L}_{0,\alpha,\beta}$ and $\Lambda_{[\mathbf{x}]}$ is thus

$$(U_{\alpha,\beta}(\xi, \mathbf{a}, \mathbf{v}, s, \sigma)F)(\mathbf{z}, [\mathbf{x}]) = \mathscr{L}_{0,\alpha,\beta}(\Lambda_{[\mathbf{x}]}^{-1}\cdot(\xi, \mathbf{a}, \mathbf{v}, s, \sigma) \times \Lambda_{[\Pi(\sigma)^{-1}\mathbf{x}]})F((-1)^{\varepsilon}R_s^{-1}\mathbf{z}, [\Pi(\sigma)^{-1}\mathbf{x}])$$

where $\varepsilon = \varepsilon(\sigma, [\mathbf{x}]) = 0$ or $1$ is defined by

$$\Lambda_{[\mathbf{x}]}^{-1}\cdot\sigma\cdot\Lambda_{[\Pi(\sigma)^{-1}\mathbf{x}]} = \sigma(\varepsilon(\sigma, [\mathbf{x}])\pi, 1, \tau(\sigma, \mathbf{x})).$$

Contrary to the case where $\nu \neq 0$, it is not possible to choose here a section which makes the index $\beta$ superfluous.

The I.U.R. of $\mathscr{S}(3)^*$ and the P.I.U.R. of $\mathscr{S}(3)$ of this Class 3 are thus characterized by the couples $(\nu, \beta)$, $\nu = \frac{1}{2}, 1, \ldots, \beta \in \mathbb{R}$ or, if $\nu = 0$ by $(\alpha, \beta)$, $\alpha, \beta \in \mathbb{R}$.

## VIII. Class 4

Here the orbits are reduced to a point and the action of $\tilde{\mathscr{S}}(3)$ is trivial; we obtain immediately the induced representation of $\tilde{\mathscr{S}}(3)$ (or of $\mathscr{S}(3)^*$).

$$U_{j,q,h} = D_j \otimes \mathscr{D}_{q,h} \quad (8.1)$$

that is the tensor product of an I.U.R. of $SU(2)$ and an I.U.R. of $SL(2, \mathbb{R})^*$.

## IX. The wave functions

By reducing the Class 1 P.I.U.R. (5.10) with respect to the Euclidean group, it is easy to show that the criteria of localizability [15] are satisfied. Actually the situation is analogous to the one for the $(m, j, U)$ representations of the Galilei group [7]: the position operators associated with each $(m, j, q, h)$ representation are given by

$$Q_k = \frac{1}{m}K_k = -i\,\partial_k, \quad k = 1, 2, 3. \quad (9.1)$$

From this follows at once the standard interpretation of the generators $P_k$ and $J_k$ as the linear momentum and angular momentum of the system and the identification of the characteristic number $j$ with the spin of the system. The interpretation of the other generators and of the remaining characteristic numbers of the representation depend on a dynamical postulate for the system, i.e. the choice of a law of evolution $t \mapsto U(t)$, where $U(t)$ is a one parameter group of unitary transformations. By the Stone theorem, we know that this is equivalent to giving a Hamiltonian operator $H$. The choice for such an operator is *a priori* arbitrary and does not depend on the considered group. The choice of $H$ as the generator of the temporal translations in the case of the $(m, j, U)$ representations of the Galilei group, for example, is not imposed at all by considerations of relativisitc consistencies: the Galilei group acts correctly and irreducibly on the Hilbert space of the system independently of the law of evolution; it is only a postulate and the definition of a free (or isolated) elementary system. From this follows the interpretation of $m$ and $U$ as the mass and the internal energy for such a system. By choosing this same generator in the $\mathscr{S}(3)$ case, i.e. by setting

$$H = G_+$$

where $G_+$ is given in (5.11), we obtain

$$e^{itH} P_k e^{-itH} = P_k$$

$$e^{itH} Q_k e^{-itH} = Q_k - \frac{t}{m} P_k,$$

i.e. the dynamics of a free three dimensional system with mass $m$. The two other generators $G_0$ and $G_-$ of the subgroup $SL(2, \mathbb{R})$ give by virtue of (5.9)

$$e^{i\mu G_0} P_k e^{-i\mu G_0} = e^{\mu} P_k$$

$$e^{i\mu G_0} Q_k e^{-i\mu G_0} = e^{-\mu} Q_k$$

$$e^{i\lambda G_-} P_k e^{-i\lambda G_-} = P_k - \lambda m Q_k$$

$$e^{i\lambda G_-} Q_k e^{-i\lambda G_-} = Q_k.$$

These two subgroups appear as scale and gauge groups. The Hamiltonian is not invariant under their action; they define conjugacy classes of Hamiltonians distinct from the usual one generated by the pure Galilei transformations

$$e^{iv^k K_k} H e^{-iv^k K_k} = \frac{1}{2m} |\mathbf{p} + m\mathbf{v}|^2 + g_+.$$

Except for the appearance of scale and gauge transformations, the situation seems very analogous to that for the Galilei group. However there exists a serious difficulty of interpreting the representations $(m, j, q, h)$ in the general case as an elementary system. Let us recall that the Hilbert space of these representations is the space of the normalizable functions

$$f: \mathbb{R}^3 \to \mathbb{C}^{2j+1} \otimes \mathscr{H},$$

where $\mathscr{H}$ is the space of the $\mathscr{D}_{q,h}$ representation of $SL(2, \mathbb{R})^*$. With the exception of the trivial representation $\mathscr{D}_{0,0}$, this space is never of finite dimension. For such a representation, the system would possess, in addition to the spin, an infinite number of degrees of internal freedom. Moreover the interpretation of the numbers $q$ and $h$ together with the parts $g_+$, $g_0$ and $g_-$ of the generators of $SL(2, \mathbb{R})$ would be difficult; we do not try to adapt a physical model to these representations: let us simply mention that in reducing them with respect to the Galilei group, the 'internal' space $\mathscr{H}$ decomposes into a direct integral of one dimensional spaces, labelled by the internal energy.

From now on, we limit ourselves to $(m, j, 0, 0)$ representations which we are going to realize on the space of wave functions $\psi(\mathbf{x}, t)$.

By virtue of (9.1), we see that the realization of these projective representations of $\mathscr{S}(3)$ for which the position operators are 'diagonal' is simply obtained from (5.9) by a Fourier transformation:

$$\left(\tilde{U}_{m,j}\left(\mathbf{a}, \mathbf{v}, R, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f\right)(\mathbf{x}) = \left(\mathscr{F} U_{m,j}\left(\mathbf{a}, \mathbf{v}, R, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\mathscr{F}^{-1} f\right)(\mathbf{x})$$

$$= \frac{e^{i\tilde{\Lambda}(\sigma)}}{|2\pi m^{-1} b|^{3/2}} \int_{\mathbb{R}^3} d^3y \exp\left[-\frac{im}{2b}(a|\mathbf{y}|^2 - 2\mathbf{x}\cdot\mathbf{y} + d|\mathbf{x}|^2 - 2b\mathbf{v}\cdot(\mathbf{y} - \mathbf{a})\right] \times D_j(R) f(R^{-1}(\mathbf{y} - \mathbf{a})). \quad (9.2)$$

Here $\tilde{\Lambda}(\sigma)$ is a factor analogous to (5.8) and $D_j(R)$, when $j$ is a half integer, is a choice of one of the two values of the representation.

The evolution operator

$$U(t) = e^{-itH} = \tilde{U}_{m,j}\left(\mathbf{0}, \mathbf{0}, 1, \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\right)$$

is of course the usual propagator

$$(U(t)f)(\mathbf{x}) = \frac{e^{-i(3\pi/4)\,\mathrm{sig}(mt)}}{|2\pi m^{-1} t|^{3/2}} \int_{\mathbb{R}^3} d^3y\, e^{(im/2t)|\mathbf{x}-\mathbf{y}|^2} f(\mathbf{y}).$$

Let us now recall the construction of the space of wave functions.

Let $\mathscr{H}$ be a Hilbert space for a physical system and $U(t) = e^{-itH}$ be an evolution law. A wave function with initial state $f \in \mathscr{H}$ is defined by $\psi(t) = U(t)f$, for any $t$ in $\mathbb{R}$; thus it is a function $\psi: \mathbb{R} \to \mathscr{H}$ satisfying the Schrödinger equation

$$(i\partial_t \psi)(t) = H\psi(t).$$

Since $U(t)$ is unitary,

$$(\psi(t) \mid \phi(t)) = (\psi(0) \mid \psi(0)) = (f \mid g)$$

and this property makes it possible to provide the set $S_H$ of these functions with a Hilbert space structure by setting

$$(\psi \mid \phi) = (\psi(t) \mid \phi(t)).$$

This Hilbert space is referred to as the space of wave functions for the Hamiltonian $H$.

From this construction follows at once that the operator

$$T: \mathscr{H} \to S_H; \qquad (Tf)(t) = U(t)f \quad (9.3)$$

is an isometry with the inverse

$$T^{-1}: S_H \to \mathscr{H}; \qquad T^{-1}\psi = \psi(0). \quad (9.3')$$

Then, if $U(g)$ is a unitary representation of a group acting on $\mathscr{H}$, the representation

$$\tilde{U}(g) = TU(g)T^{-1} \quad (9.4)$$

is also unitary and acts on $S_H$.

In our case, the Schrödinger equation is

$$i\partial_t \psi(\mathbf{x}, t) = -\frac{\Delta}{2m}\psi(\mathbf{x}, t)$$

since the wave functions are defined as

$$\psi(\mathbf{x}, t) = \left(\tilde{U}_{m,j}\left(\mathbf{0}, \mathbf{0}, 1, \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\right) f\right)(\mathbf{x})$$

$f \in L^2(\mathbb{R}^3, \mathbb{C}^{2j+1})$.

The representation of $\mathscr{S}(3)$ on $S_H$ is, by virtue of (9.3), (9.3′) and (9.4):

$$\left(\hat{U}_{m,j}\left(\mathbf{a}, \mathbf{v}, R, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\psi\right)(\mathbf{x}, t) = \left(\tilde{U}_{m,j}\left(\mathbf{0}, \mathbf{0}, 1, \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\right)\right.$$
$$\left. \times \tilde{U}_{m,j}\left(\mathbf{a}, \mathbf{v}, R, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) T^{-1}\psi\right)(\mathbf{x}).$$

By (2.4) and (2.5) we have:

$$\left(0, \mathbf{0}, \mathbf{0}, 1, \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\right)\left(0, \mathbf{a}, \mathbf{v}, R, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$
$$= \left(\frac{1}{2}\left(\frac{dt - b}{-ct + a}\right)|\mathbf{v}|^2, \mathbf{a} + \left(\frac{dt - b}{-ct + a}\right)\mathbf{v}, \mathbf{v}, R, \begin{pmatrix} -ct + a & 0 \\ c & (-ct + a)^{-1} \end{pmatrix}\right)$$
$$\times \left(0, \mathbf{0}, \mathbf{0}, 1, \begin{pmatrix} 1 & -\dfrac{dt - b}{-ct + a} \\ 0 & 1 \end{pmatrix}\right)$$

and thus

$$\left(\hat{U}_{m,j}\left(\mathbf{a}, \mathbf{v}, R, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\psi\right)(\mathbf{x}, t) = \frac{e^{i\tilde{\Lambda}(\sigma)}}{|-ct + a|^{3/2}} \exp\left\{-\frac{im}{-ct + a}\left(\frac{c}{2}|\tilde{x}|^2\right.\right.$$
$$\left.\left. + \frac{dt - b}{2}|\mathbf{v}|^2 - \mathbf{v}\cdot\mathbf{x} + (-ct + a)\mathbf{a}\cdot\mathbf{v}\right)\right\}$$
$$\times D_j(R)\psi\left(\frac{R^{-1}(\mathbf{x} - (-ct + a)\mathbf{a} - (dt - b)\mathbf{v})}{-ct + a}, \frac{dt - b}{-ct + a}\right). \tag{9.5}$$

For $j = 0$, up to some changes of notation we recognize here the representation found by U. Niederer [1].

All the above holds for the Hamiltonian of an isolated system; we can realize this construction of wave functions for any choice of a one parameter group of evolution. By choosing for example

$$U(t) = \tilde{U}_{m,j}\left(\mathbf{0}, \mathbf{0}, 1, \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}\right), \quad \omega > 0,$$

we get the Hamiltonian for a harmonic oscillator

$$H = \omega(G_+ - G_-) = \frac{1}{2M}(-\Delta + M^2\omega^2|\mathbf{x}|^2)$$

with $M = m/\omega > 0$.

Since, by (9.2)

$$U(t) = \frac{e^{i\Lambda(\omega t)}}{|2\pi m^{-1}\sin\omega t|^{3/2}} \times \int_{\mathbb{R}^3} d^3y \exp\left[\frac{im}{2\sin\omega t}(\cos\omega t(|\mathbf{y}|^2 + |\mathbf{x}|^2) - 2\mathbf{x}\cdot\mathbf{y})\right] f(\mathbf{y})$$

we see, taking into account the expression (5.8) for $\Lambda(\omega t)$, that $U(t) = U(t + 4(k\pi/\omega))$; from this periodicity condition follows the particular spectrum of $H$.

Using the same techniques as for the free Hamiltonian, we obtain the representations of $\mathscr{S}(3)$ on the space of wave functions of the harmonic oscillator. For $j = 0$, they are the representations given by U. Niederer [2].

**Conclusion**

The classification and the physical analysis of the P.I.U.R. of the Schrödinger group is very similar to that for the Galilei group; except for the Class 1 representations for which the factor is not trivial, the other representations have no immediate physical interpretation. However, for the Class 1 representations the situation is not as clear as it is for the Galilei group: the appearance in the general case of an infinite number of internal degrees of freedom makes doubtful the usefulness of defining an elementary system by such representations. Contrary to the Galilei group, the Schrödinger group is not really a kinematical group for space-time: its action on $E_3 \times \mathbb{R}$ is only defined locally. It is of course possible to find homogeneous spaces of dimension 4 for $\mathscr{S}(3)$: they are the spaces of constant curvature, the nonrelativistic contreparts of the de Sitter spaces; but we know that the interpretable kinematical groups for these spaces are the Newton groups [5] which are only subgroups of $\mathscr{S}(3)$.

The situation of $\mathscr{S}(3)$ as opposed to the Galilei group is analogous to the conformal group $O(4, 2)$ as opposed to the Poincaré group: The introduction of these groups is somewhat artificial since it rests on a particular realization of the Hilbert space of the system: it would for instance be very difficult to define them in the '$p$-representation'. What is more, they are only subgroups of the group of canonical transformations of a quantum system. Other subgroups, such as the group $W_3 \square Sp(3)$ generated by the Lie algebra of the polynomials of degree $\leqslant 2$ in $p_i$, $q_i$ and 1 [17] might be considered much more important than $\mathscr{S}(3)$, simply because it is larger but it is still a Lie group!

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