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Extension of the final state theorem and the analysis of accurate low energy pion–nucleon scattering and pion photoproduction experiments

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Abstract. The final state theorem, originally formulated by Watson, is studied for the case of $m + n$ coupled two-body channels, m of which are coupled together 'strongly' with 'weak' perturbations, while the other n channels are coupled only 'weakly' to the 'strong' channels. The results are applied to expose the difficulties in a useful analysis of new, more accurate experiments at low energies on pion–nucleon scattering and pion photoproduction from nucleons, which will be performed at SIN, TRIUMF and LAMPF and at several medium energy electron accelerators.

1. Introduction

In a paper reviewing and comparing work on electromagnetic effects in pion–nucleon scattering [1] we discussed our earlier work on the analysis of $\pi^- p$ experiments [2], and pointed out that, once electromagnetic effects on pion–nucleon scattering are being considered consistently to first order in the fine structure constant α, it is essential to take account of the competing two-body photon–nucleon channel. Thus, for coupled channels with total charge zero, we need to consider $\pi^- p$, $\pi^0 n$ and both electric and magnetic multipoles for γn. For each (JP) with $J \geqslant \frac{3}{2}$, this gives a 4×4 unitary, symmetric S-matrix. Similarly, for coupled channels with total charge $+e$, we need to consider $\pi^0 p$, $\pi^+ n$ and both electric and magnetic multipoles for γp, again giving a 4×4 unitary, symmetric S-matrix for each (JP) with $J \geqslant \frac{3}{2}$. In this paper we propose to study these coupled channel situations carefully and to consider the analysis of experiments on pion–nucleon scattering and pion photoproduction from nucleons at low and medium energies.

To this end, since two of the channels may be characterized as 'strong' and two as 'weak', it is clear that we need to consider the well known final state theorem first formulated in zeroth order by Watson [3] in connection with this very problem. Roughly put, Watson showed that suitably chosen multipole amplitudes for photoproduction of pions from nucleons have phases which are just the phase shifts for pion–nucleon scattering for a definite set of values of J, P and total isospin I. Once Coulomb and mass difference effects are taken into account in analysing experiments

on pion–nucleon scattering, complications arise. There is the question of which phase shifts are to be used in the analysis of photoproduction experiments, strictly nuclear phases or phases modified by the addition of electromagnetic corrections of order α. One also needs to ask whether the electromagnetic correction to the charge independent mixing angle needs to be taken into account. We shall answer these questions in Section 2, in the course of a study of the final state theorem for the case of m 'strong' and n 'weak' two-body channels which are coupled together. In this study we shall investigate the consequences of the approximation that the S-matrix for a particular (JP), $\mathbf{S}_{m+n}$, be unitary only to first order in α:

$$\bar{\mathbf{S}}_{m+n}\mathbf{S}_{m+n} - \mathbf{1}_{m+n} = \mathrm{O}(\alpha^2). \tag{1}$$

In writing equation (1) time-reversal invariance has been assumed to hold, so that $\mathbf{S}_{m+n}$ is symmetric. In the usual zeroth order form of the final state theorem, the right side of equation (1) is of order α only. Our study is therefore the extension of the final state theorem to the next highest order in α. The results of Section 2 will then be applied in Section 3, to show that it is difficult to make a consistent analysis of the new generation of more accurate experiments at low and medium energies on pion–nucleon scattering and pion–photoproduction from nucleons, which takes proper account of electromagnetic effects.

2. The Final State Theorem

The matrix $\mathbf{S}_{m+n}$ of equation (1) is partitioned as follows:

$$\mathbf{S}_{m+n} = \begin{pmatrix} \mathbf{S}_m & \mathbf{S}_{mn} \\ \mathbf{S}_{mn}^t & \mathbf{S}_n \end{pmatrix}. \tag{2}$$

In view of the application we have in mind, we think of the relative strengths of the 'strong' and 'weak' channels as being measured by a number $\alpha \ll 1$ (in our application the fine structure constant). More specifically, using the usual mathematical notation, we assume that

$$\mathbf{S}_m - \mathbf{1}_m = \mathrm{O}(1), \qquad \mathbf{S}_{mn} = \mathrm{O}(\alpha^{\frac{1}{2}}), \qquad \mathbf{S}_n - \mathbf{1}_n = \mathrm{O}(\alpha). \tag{3}$$

We need however to look more closely at $\mathbf{S}_m$. If the 'weak' channels were absent altogether, $\mathbf{S}_m$ would be a symmetric unitary matrix. It is therefore convenient to decompose $\mathbf{S}_m$ in the following way:

$$\mathbf{S}_m = \mathbf{S}_m^{(0)} + (\mathbf{S}_m - \mathbf{S}_m^{(0)}), \tag{4}$$

with

$$\mathbf{S}_m^{(0)t} = \mathbf{S}_m^{(0)}, \qquad \overline{\mathbf{S}_m^{(0)}}\mathbf{S}_m^{(0)} = \mathbf{1}_m, \tag{5}$$

$$\mathbf{S}_m^{(0)} - \mathbf{1}_m = \mathrm{O}(1), \qquad \mathbf{S}_m - \mathbf{S}_m^{(0)} = \mathrm{O}(\alpha). \tag{6}$$

This decomposition of $\mathbf{S}_m$ is not unique, because $\mathbf{S}_m^{(0)}$ may include 'weak' effects of order α other than those which arise from the presence of the 'weak' channels. We shall return to this point later. Finally we write

$$\mathbf{S}_n = \mathbf{1}_n + 2i\boldsymbol{\varepsilon}_n, \tag{7}$$

where

$$\boldsymbol{\varepsilon}_n = \mathrm{O}(\alpha), \tag{8}$$

by (3).

To proceed further we note that since $\mathbf{S}_m^{(0)}$ is unitary and symmetric it may be written as

$$\mathbf{S}_m^{(0)} = \mathbf{O}_m \exp(2i\boldsymbol{\Theta})\mathbf{O}_m^t, \tag{9}$$

where $\mathbf{O}_m$ is a real orthogonal matrix,

$$\mathbf{O}_m^t \mathbf{O}_m = \mathbf{1}_m, \tag{10}$$

and $\boldsymbol{\Theta}$ is a diagonal matrix with real elements θ_i,

$$\boldsymbol{\Theta}_{ij} = \theta_i \delta_{ij}, \quad -\pi/2 < \theta_i \leqslant \pi/2. \tag{11}$$

We now write the matrix $\mathbf{S}_{m+n}$ of equation (2) as

$$\mathbf{S}_{m+n} = \begin{pmatrix} \mathbf{O}_m \exp(i\boldsymbol{\Theta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_n \end{pmatrix} \mathbf{S}'_{m+n} \begin{pmatrix} \exp(i\boldsymbol{\Theta})\mathbf{O}_m^t & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_n \end{pmatrix}, \tag{12}$$

with $\mathbf{S}'_{m+n}$ having the form

$$\mathbf{S}'_{m+n} = \mathbf{1}_{m+n} + 2i \begin{pmatrix} \boldsymbol{\sigma}_m & \boldsymbol{\lambda}_{mn} \\ \boldsymbol{\lambda}_{mn}^t & \boldsymbol{\varepsilon}_n \end{pmatrix}. \tag{13}$$

Then, from equations (2), (4) and (9), we have

$$\mathbf{S}_m - \mathbf{S}_m^{(0)} = 2i\mathbf{O}_m \exp(i\boldsymbol{\Theta})\boldsymbol{\sigma}_m \exp(i\boldsymbol{\Theta})\mathbf{O}_m^t, \tag{14}$$

$$\mathbf{S}_{mn} = 2i\mathbf{O}_m \exp(i\boldsymbol{\Theta})\boldsymbol{\lambda}_{mn}, \tag{15}$$

while (12) and (13) incorporate (7). From (3) and (6),

$$\boldsymbol{\sigma}_m = \mathrm{O}(\alpha), \qquad \boldsymbol{\lambda}_{mn} = \mathrm{O}(\alpha^{1/2}). \tag{16}$$

One checks easily that $\mathbf{S}'_{m+n}$ is symmetric and unitary; furthermore equation (1) requires

$$\bar{\mathbf{S}}'_{m+n}\mathbf{S}'_{m+n} - \mathbf{1}_{m+n} = \mathrm{O}(\alpha^2). \tag{17}$$

Now from equation (13) we have

$$\bar{\mathbf{S}}'_{m+n}\mathbf{S}'_{m+n} - \mathbf{1}_{m+n} = 4 \begin{pmatrix} -\mathrm{Im}\,\boldsymbol{\sigma}_m + \bar{\boldsymbol{\sigma}}_m\boldsymbol{\sigma}_m + \bar{\boldsymbol{\lambda}}_{mn}\boldsymbol{\lambda}_{mn}^t & -\mathrm{Im}\,\boldsymbol{\lambda}_{mn} + \bar{\boldsymbol{\sigma}}_m\boldsymbol{\lambda}_{mn} + \bar{\boldsymbol{\lambda}}_{mn}\boldsymbol{\varepsilon}_n \\ -\mathrm{Im}\,\boldsymbol{\lambda}_{mn}^t + \bar{\boldsymbol{\lambda}}_{mn}^t\boldsymbol{\sigma}_m + \bar{\boldsymbol{\varepsilon}}_n\boldsymbol{\lambda}_{mn}^t & -\mathrm{Im}\,\boldsymbol{\varepsilon}_n + \bar{\boldsymbol{\varepsilon}}_n\boldsymbol{\varepsilon}_n + \bar{\boldsymbol{\lambda}}_{mn}^t\boldsymbol{\lambda}_{mn} \end{pmatrix}. \tag{18}$$

Thus we see that for the off-diagonal submatrices in (18) to vanish to lowest order ($\alpha^{1/2}$ by (16)) we must have

$$\mathrm{Im}\,\boldsymbol{\lambda}_{mn}^{(0)} = \mathbf{0}, \qquad \boldsymbol{\lambda}_{mn}^{(0)} \text{ real}. \tag{19}$$

We use the superscript (0) on $\boldsymbol{\lambda}_{mn}$ to denote that this is the lowest order non-vanishing approximation to $\boldsymbol{\lambda}_{mn}$. Equation (19) is in fact the generalization to our $(m + n)$ channel situation of the usual statement of the final state theorem. Taken together with (9) and (15), it says that the mn complex S-matrix elements connecting 'strong' and 'weak' channels can be expressed in terms of the mn real elements of the matrix

$\boldsymbol{\lambda}_{mn}^{(0)}$, once the approximate unitary symmetric matrix $\mathbf{S}_m^{(0)}$ for the 'strong' channels is known.

However, if it is possible to perform experiments on 'strong' channel $\leftrightarrow$ 'weak' channel processes with an accuracy of around α or better, one will need for the purposes of analysis of these experiments to take into account the terms of order $\alpha^{3/2}$ in the off-diagonal submatrices in (18). That is why we prescribed the condition (1) on $\mathbf{S}_{m+n}$, from which condition (17) on $\mathbf{S}'_{m+n}$ follows. This condition in effect means that in the matrix on the right side of (18) the off-diagonal submatrices are to be of order $\alpha^{5/2}$ and the diagonal submatrices of order α^2. To this order $\boldsymbol{\lambda}_{mn}$ acquires an imaginary part and we have, from (18),

$$\operatorname{Im} \boldsymbol{\lambda}_{mn}^{(1)} = \operatorname{Re} \boldsymbol{\sigma}_m^{(0)} \boldsymbol{\lambda}_{mn}^{(0)} + \boldsymbol{\lambda}_{mn}^{(0)} \operatorname{Re} \boldsymbol{\varepsilon}_n^{(0)}. \tag{20}$$

Equation (20) is true because, in order that the diagonal submatrices in (18) be of order α^2, we must have

$$\operatorname{Im} \boldsymbol{\sigma}_m^{(0)} = \boldsymbol{\lambda}_{mn}^{(0)} \boldsymbol{\lambda}_{mn}^{(0)t}, \qquad \operatorname{Im} \boldsymbol{\varepsilon}_n^{(0)} = \boldsymbol{\lambda}_{mn}^{(0)t} \boldsymbol{\lambda}_{mn}^{(0)}, \tag{21}$$

giving

$$-\operatorname{Im} \boldsymbol{\sigma}_m^{(0)} \boldsymbol{\lambda}_{mn}^{(0)} + \boldsymbol{\lambda}_{mn}^{(0)} \operatorname{Im} \boldsymbol{\varepsilon}_n^{(0)} = \mathbf{0}.$$

In equation (20) we have used the superscript (1) to denote the approximation to $\boldsymbol{\lambda}_{mn}$ correct to order $\alpha^{3/2}$. Thus, to analyse the results of sufficiently accurate experiments on 'strong' channel $\leftrightarrow$ 'weak' channel processes it is necessary to include the non-zero imaginary part of $\boldsymbol{\lambda}_{mn}$. From (15) and (20) the expression for $\mathbf{S}_{mn}$, correct to order $\alpha^{3/2}$, is

$$\mathbf{S}_{mn} = 2i\mathbf{O}_m \exp(i\boldsymbol{\Theta}) [\operatorname{Re} \boldsymbol{\lambda}_{mn} + i(\operatorname{Re} \boldsymbol{\sigma}_m \operatorname{Re} \boldsymbol{\lambda}_{mn} + \operatorname{Re} \boldsymbol{\lambda}_{mn} \operatorname{Re} \boldsymbol{\varepsilon}_n)]. \tag{22}$$

In (22) we have written $\operatorname{Re} \boldsymbol{\lambda}_{mn}$ in place of $\boldsymbol{\lambda}_{mn}^{(0)}$ in giving $\operatorname{Im} \boldsymbol{\lambda}_{mn}^{(1)}$, since the difference does not affect the result to order $\alpha^{3/2}$. Similarly, $\operatorname{Re} \boldsymbol{\sigma}_m$ and $\operatorname{Re} \boldsymbol{\varepsilon}_n$ in (22) are required just to lowest order (α by (8) and (16)). If these latter quantities are known to this order, then analysis of the experiments will yield the elements of $\operatorname{Re} \boldsymbol{\lambda}_{mn}$, provided $\mathbf{S}_m^{(0)}$ is also known.

At this point we can answer the questions posed in Section 1. We shall show that it does not matter whether one uses the 'strictly strong' S-matrix or a modified S-matrix for the strong channels which already includes corrections of order α for any 'weak' (in our case electromagnetic) effects which can be calculated in some approximate theory (in our case Coulomb and mass difference effects) which neglects the presence of the 'weak' channels. To see this, suppose that two different unitary symmetric matrices $\mathbf{S}_m^{(0)}$ are given, which differ by a matrix whose elements are of order α. We require at the same time that the matrix $\boldsymbol{\sigma}_m$ is to be modified in such a way that $\mathbf{S}_m$ given by (9) and (14) is unaltered to first order in α. Dropping the subscript m for convenience, given $\Delta\mathbf{O}$ and $\Delta\boldsymbol{\Theta}$ we wish to determine $\Delta\boldsymbol{\sigma}$ so that

$$\Delta[\mathbf{O} \exp(i\boldsymbol{\Theta}) (\mathbf{1} + 2i\boldsymbol{\sigma}) \exp(i\boldsymbol{\Theta}) \mathbf{O}^t] = \mathbf{0}.$$

To first order in α this gives

$$\Delta\mathbf{O} \exp(2i\boldsymbol{\Theta}) \mathbf{O}^t + \mathbf{O} \exp(2i\boldsymbol{\Theta}) \Delta\mathbf{O}^t$$
$$+ 2i\mathbf{O} \exp(2i\boldsymbol{\Theta}) \Delta\boldsymbol{\Theta} \mathbf{O}^t + 2i\mathbf{O} \exp(i\boldsymbol{\Theta}) \Delta\boldsymbol{\sigma} \exp(i\boldsymbol{\Theta}) \mathbf{O}^t = \mathbf{0}.$$

Further, $\mathbf{O}$ is assumed to remain orthogonal so that, again to first order in α,

$$(\Delta\mathbf{O}^t)\mathbf{O} + \mathbf{O}^t\Delta\mathbf{O} = \mathbf{0}. \tag{23}$$

Combining these expressions, we have

$$\Delta\boldsymbol{\sigma} = -\Delta\boldsymbol{\Theta} - \mathrm{Im}\,[\exp(-i\boldsymbol{\Theta})\mathbf{O}^t\Delta\mathbf{O}\exp(i\boldsymbol{\Theta})], \tag{24}$$

a real matrix as required, since Im $\boldsymbol{\sigma}$ is fixed to first order in α by (21). Turning to (22), we now show that it is possible to modify Re $\boldsymbol{\lambda}_{mn}$ in such a way that $\mathbf{S}_{mn}$ is unaltered to order $\alpha^{3/2}$ by the changes in $\mathbf{O}_m$, $\boldsymbol{\Theta}$ and $\boldsymbol{\sigma}_m$. We need not worry about the final term in (22), which is of order $\alpha^{3/2}$ and is not altered to that order by the changes in $\mathbf{O}_m$, $\boldsymbol{\Theta}$ and Re $\boldsymbol{\lambda}_{mn}$, provided Re $\boldsymbol{\varepsilon}_n$ is known to order α. Thus, again dropping subscripts, we want

$$\begin{aligned}\mathbf{0} &= \Delta[\mathbf{O}\exp(i\boldsymbol{\Theta})(\mathbf{1} + i\,\mathrm{Re}\,\boldsymbol{\sigma})\,\mathrm{Re}\,\boldsymbol{\lambda}]\\ &= \Delta\mathbf{O}\exp(i\boldsymbol{\Theta})\,\mathrm{Re}\,\boldsymbol{\lambda} + i\mathbf{O}\exp(i\boldsymbol{\Theta})\,\Delta\boldsymbol{\Theta}\,\mathrm{Re}\,\boldsymbol{\lambda}\\ &\quad + i\mathbf{O}\exp(i\boldsymbol{\Theta})\,\Delta(\mathrm{Re}\,\boldsymbol{\sigma})\,\mathrm{Re}\,\boldsymbol{\lambda} + \mathbf{O}\exp(i\boldsymbol{\Theta})\,\Delta(\mathrm{Re}\,\boldsymbol{\lambda});\end{aligned}$$

on substituting $\Delta\boldsymbol{\sigma}$ from (24), this gives

$$\Delta(\mathrm{Re}\,\boldsymbol{\lambda}) = -\mathrm{Re}\,[\exp(-i\boldsymbol{\Theta})\mathbf{O}^t\Delta\mathbf{O}\exp(i\boldsymbol{\Theta})]\,\mathrm{Re}\,\boldsymbol{\lambda},$$

which is real as required. Thus which 'strong' matrix $\mathbf{S}_m^{(0)}$ is used in this approximation depends only on practical considerations.

In fact there is a practical consideration which leads to a particular choice. We want expressions for $\mathbf{S}_m$, $\mathbf{S}_{mn}$ which include electromagnetic corrections to charge independence correct to order α, $\alpha^{3/2}$ respectively, and which can be conveniently used for the analysis of sufficiently accurate experiments. We have, from (9) and (14)

$$\mathbf{S}_m = \mathbf{O}_m\exp(i\boldsymbol{\Theta})(\mathbf{1}_m - 2\,\mathrm{Im}\,\boldsymbol{\sigma}_m + 2i\,\mathrm{Re}\,\boldsymbol{\sigma}_m)\exp(i\boldsymbol{\Theta})\mathbf{O}_m^t. \tag{25}$$

In this expression we have $\frac{1}{2}m(m+1)$ real quantities of zeroth order needed to specify $\mathbf{O}_m$ and $\boldsymbol{\Theta}$, and $\frac{1}{2}m(m+1)$ real quantities of first order in α needed to specify Re $\boldsymbol{\sigma}_m$. Im $\boldsymbol{\sigma}_m$ is of course given by (21). Is it possible to write an expression for $\mathbf{S}_m$ in terms of just $\frac{1}{2}m(m+1)$ real quantities which include first order corrections to the zeroth order quantities which specify $\mathbf{O}_m$ and $\boldsymbol{\Theta}$? By giving a twist to the argument of the previous paragraph, we can show that the answer is yes. For suppose that $\mathbf{S}_m$ is expressed as in (25) in terms of $\mathbf{O}_m$, $\boldsymbol{\Theta}$ and Re $\boldsymbol{\sigma}_m$. Then, using (24), we can choose $\Delta\mathbf{O}_m$ and $\Delta\boldsymbol{\Theta}$ so that

$$\Delta\boldsymbol{\sigma}_m = -\mathrm{Re}\,\boldsymbol{\sigma}_m. \tag{26}$$

This is done in the following way. Noting from (23) that $\mathbf{O}_m^t\Delta\mathbf{O}_m$ is antisymmetric, it follows from (24) and (26) that, provided all the $\theta_i (i = 1, \ldots, m)$ are different (which we take to be the case),

$$\Delta\theta_i = (\mathrm{Re}\,\boldsymbol{\sigma}_m)_{ii}, \quad i = 1, \ldots, m, \tag{27}$$

$$(\mathbf{O}_m^t\Delta\mathbf{O}_m)_{ii} = 0, \quad i = 1, \ldots, m, \tag{28}$$

$$(\mathbf{O}_m^t\Delta\mathbf{O}_m)_{ij} = -(\mathrm{Re}\,\boldsymbol{\sigma}_m)_{ij}[\sin(\theta_i - \theta_j)]^{-1}, \quad i \neq j. \tag{29}$$

If then we write

$$\mathbf{O}'_m = \mathbf{O}_m + \Delta\mathbf{O}_m, \qquad \boldsymbol{\Theta}' = \boldsymbol{\Theta} + \Delta\boldsymbol{\Theta},$$

with $\Delta\mathbf{O}_m$, $\Delta\boldsymbol{\Theta}$ given by (27)–(29), it follows that $\mathbf{O}'_m$ is orthogonal and that

$$\mathbf{S}_m = \mathbf{O}'_m\exp(i\boldsymbol{\Theta}')(\mathbf{1}_m - 2\boldsymbol{\lambda}_{mn}^{(0)}\boldsymbol{\lambda}_{mn}^{(0)t})\exp(i\boldsymbol{\Theta}')\mathbf{O}_m'^t, \tag{30}$$

since Re $\boldsymbol{\sigma}'_m$ = Re $\boldsymbol{\sigma}_m$ + $\Delta\boldsymbol{\sigma}_m$ = $\mathbf{0}$, by (26). In (30) we have used (21) to give Im $\boldsymbol{\sigma}_m^{(0)}$ explicitly. Further, from the argument in the previous paragraph, Re $\boldsymbol{\lambda}_{mn}$ may be modified in such a way that $\mathbf{S}_{mn}$ is not altered to order $\alpha^{3/2}$ by the changes in $\mathbf{O}_m$, $\boldsymbol{\Theta}$ and Re $\boldsymbol{\sigma}_m$. Thus, by (22), $\mathbf{S}_{mn}$ may be written in the convenient form

$$\mathbf{S}_{mn} = 2i\mathbf{O}'_m \exp(i\boldsymbol{\Theta}') \text{ Re } \boldsymbol{\lambda}_{mn}(\mathbf{1}_n + \text{Re } \boldsymbol{\varepsilon}_n). \tag{31}$$

Equations (30) and (31) are the expressions we need for the analysis of experiments.

3. The analysis of experiments on pion–nucleon scattering and pion photoproduction

In this section we discuss in detail the analysis of experiments on pion–nucleon scattering and pion photoproduction from nucleons at low and medium energies, emphasizing the difficulties arising from the increased accuracy of recent experiments. Much of the background to our discussion is contained in Reference [1] and equations from that paper are distinguished by the prefix 1-. Until recently the analysis of these experiments was carried out at what we might call 'zeroth order' (in α) level. For the analysis of differential cross-sections and polarizations for pion–nucleon scattering the full amplitudes $\mathscr{F}$ and $\mathscr{G}$ of (1-1) are used. Even a zeroth order analysis included the full Coulomb amplitudes, computed in the way described in Section 2 of [1]. Truncated partial wave expansions of the 'modified nuclear' parts of $\mathscr{F}$ and $\mathscr{G}$ are used. The partial wave amplitudes $\mathscr{F}_+(l, J; q)$ for $\pi^+p \to \pi^+p$ are written in (1-3) and the amplitudes $\mathscr{F}_-(l, J; q)$, $\mathscr{F}_{0-}(l, J; q)$ for $\pi^-p \to \pi^-p$, $\pi^-p \to \pi^0n$ respectively are written in (1-18). In the zeroth order analysis, kinematical effects of mass differences (resulting in different channel momenta) were included, as were modifying Coulomb phases, again calculated as discussed in Section 2 of [1]. However, at this level of accuracy the electromagnetic corrections c_+ in (1-3) and c_1, c_3, C_{31} in (1-18) and (1-19) were neglected completely. The analysis then yielded, for each partial wave included, approximate nuclear phase shifts δ_1 and δ_3, no distinction being made between strictly nuclear quantities and quantities modified by the presence of electromagnetic corrections. At energies where pion production becomes important, inelasticity parameters η_1 and η_3 were included in the analysis, but no account was taken of the effect of the competing radiative capture channel γn on the analysis of π^-p experiments at low energies. This sort of zeroth order analysis worked pretty well up to about 2GeV pion laboratory kinetic energy for experiments performed up to about 1971; a comprehensive analysis of this type is that of Almehed and Lovelace [4].

Equipped with these approximate nuclear phase shifts for the lower partial waves, the next step was a 'zeroth order' analysis of experiments on pion photoproduction from nucleons. There are four processes, namely $\gamma p \to \pi^0 p$, $\gamma p \to \pi^+ n$, $\gamma n \to \pi^0 n$ and $\gamma n \to \pi^- p$. Data on the last two processes is extracted from experiments with deuterons, and is thus less accurate than that for the first two processes. The zeroth order photoproduction analysis uses the final state theorem in the simple form given in Equation (19). By about 1970 the photoproduction data was of sufficient accuracy for energy independent analyses to be performed across the first resonance, from threshold to about 450 MeV lab photon energy; the main analyses are listed in [5–8]. These analyses rely on the predictions of dispersion relations for partial waves with $l > 1$ and determine the multipole amplitudes $E_{0+}(J, P = \frac{1}{2}, -1)$, $M_{1-}(J, P = \frac{1}{2}, +1)$, and M_{1+} and $E_{1+}(J, P = \frac{3}{2}, +1)$. The analysis of Berends and Donnachie [8], which uses phase shifts from Carter et al. [9], has difficulties with the choice of

resonant phase. For the analysis of the neutron data they use the resonant phase δ_{3-} obtained in [9] from the analysis of $\pi^- p \to \pi^- p$, $\pi^0 n$, though they remark that the neutron data is not sufficiently accurate to test the reliability of this phase. For the analysis of the proton data they use the so-called nuclear resonant phase of [9], but need to modify it in the neighbourhood of the resonance position in order to obtain the best possible fit to the data. The analyses in [5–7] use approximate nuclear phase shifts from zeroth order pion–nucleon phase shift analyses. All the analyses use a strictly charge independent mixing angle in calculating $\mathbf{O}_2$ in (15). For the analysis of $\gamma p \to \pi^0 p$, $\pi^+ n$ the data is good enough for eight real parameters to be determined, one for each of the four multipole amplitudes given above and for each of the two processes. However, the data for $\gamma n \to \pi^0 n$, $\pi^- p$ can determine only four real parameters, the other four being taken from the analysis of proton experiments, by making assumptions about SU(2) invariance. If $\mathbf{A}$ denotes a multipole amplitude, then, denoting the relevant process by its final state, we have

$$\begin{aligned} &\mathbf{A}(\pi^0 p) = {}_p\mathbf{A}^{1/2} + \tfrac{2}{3}{}_p\mathbf{A}^{3/2}, && \mathbf{A}(\pi^+ n) = \sqrt{2}\,({}_p\mathbf{A}^{1/2} - \tfrac{1}{3}{}_p\mathbf{A}^{3/2}), \\ &\mathbf{A}(\pi^0 n) = -{}_n\mathbf{A}^{1/2} + \tfrac{2}{3}{}_n\mathbf{A}^{3/2}, && \mathbf{A}(\pi^- p) = \sqrt{2}\,({}_n\mathbf{A}^{1/2} + \tfrac{1}{3}{}_n\mathbf{A}^{3/2}), \\ &{}_p\mathbf{A}^{3/2} = {}_n\mathbf{A}^{3/2}. \end{aligned} \tag{32}$$

The amplitude ${}_N\mathbf{A}^I (I = \frac{1}{2}, \frac{3}{2};\ N = p, n)$ is for γN going to a pion–nucleon state with total isospin I, and the crucial assumption is the last line of (32). In a 'first order' photoproduction analysis one could no longer assume that ${}_p\mathbf{A}^{3/2} = {}_n\mathbf{A}^{3/2}$, so that analysis of the neutron data would also require the determination of eight real parameters.

Since 1970 data for $\pi^+ p$ and $\pi^- p$ scattering experiments has been obtained for which a zeroth order analysis fails. The first attempt to go beyond such an analysis was that of Carter et al. [9], who analysed the accurate data taken across the first resonance by their group. Their analysis is discussed in detail in [1]. They include the correction $-2\boldsymbol{\lambda}^{(0)}\boldsymbol{\lambda}^{(0)t}$ in the central factor on the right side of (30) in analysing $\pi^- p$ experiments, employing for this purpose the results of zeroth order photoproduction analyses; this is clearly essential in performing a proper first order analysis. Focussing on the resonant wave, where the failure of the zeroth order analysis shows up, we find an awkward 'in between' situation. Analysis of the $\pi^+ p$ experiments yields a modified nuclear phase which differs from the strictly nuclear $I = \frac{3}{2}$ phase by a correction of order α. A proper first order analysis of experiments on $\pi^- p \to \pi^- p$, $\pi^0 n$ should use the expression (30). However, as we pointed out already in [2], there are *three* parameters to be obtained for each partial wave, two eigenphases θ_i which differ from the strictly nuclear $I = \frac{1}{2}, \frac{3}{2}$ phases by corrections of order α, and a mixing angle which differs from the charge independent one by a correction of order α. Borrowing the notation of (1-18) and (1-19) and writing $\mathbf{O}_2$ correct to order α, we have, using θ_1 and θ_3 for the eigenphases,

$$\begin{aligned} &\theta_1 = \delta_1 + c_1, && \theta_3 = \delta_3 + c_3 \\ &\mathbf{O}_2 = \begin{pmatrix} \sqrt{\tfrac{2}{3}}\,(1 - C_{31}/\sqrt{2}) & \sqrt{\tfrac{1}{3}}\,(1 + \sqrt{2}\,C_{31}) \\ -\sqrt{\tfrac{1}{3}}\,(1 + \sqrt{2}\,C_{31}) & \sqrt{\tfrac{2}{3}}\,(1 - C_{31}/\sqrt{2}) \end{pmatrix}. \end{aligned} \tag{33}$$

It is clear from the analysis of Carter et al. [9] that at this stage only two parameters can be found; it is possible to look at their analysis in the following way. Their potential theory model for the electromagnetic corrections is used to fix C_{31}; the

$\pi^- p$ experiments are then able to determine the two eigenphases θ_1 and θ_3. Their model also gives the corrections c_1 and c_3 and the correction c_+ for $\pi^+ p$ scattering, but the corrections are such that the values of the strictly nuclear phase δ_3, obtained separately from the $\pi^+ p$ and the $\pi^- p$ experiments, do not agree.

The fact that a zeroth order analysis fails, but an analysis of $\pi^- p$ experiments can determine the two eigenphases θ_1 and θ_3 after C_{31} has been fixed from a model, poses great difficulties for a proper first order analysis of $\pi^- p$ experiments. In particular, we see from the discussion in Sect. 2 that, in addition to the electromagnetic effects (Coulomb and mass difference) which one might hope to calculate from some approximate potential theory model, there are in principle *further* first order corrections (the components of Re $\boldsymbol{\sigma}_2$) which arise from the presence of the γn channel; it is not clear how these could be calculated in any model theory.[1]) The analysis of Zimmerman [11], which was discussed in [1], shows that it is possible to assume that these corrections due to the γn channel are zero and still compute Coulomb and mass difference corrections from a 'charge independent' potential theory model which enable the present experimental data to be acceptably fitted. This is probably all that can be done until experiments on $\pi^- p \to \pi^- p, \pi^0 n$ are performed which are of sufficient accuracy to determine θ_1, θ_3 and C_{31} reliably for the resonant wave.

We consider now the possibilities for a first order analysis of pion photoproduction experiments across the first resonance. As a general remark, we note that the multipole amplitudes with $l > 1$ will still need to be fed into the analysis from calculations with partial wave dispersion relations. The proton and neutron experiments need to be considered separately, since the difficulties encountered are quite different in the two cases. For the processes $\gamma p \to \pi^+ n, \pi^0 p$ we note first that they should not be analysed without analysing data on $\gamma p \to \gamma p$ at the same time. This is the conclusion of Pfeil et al. [12], who point out that the imaginary parts of the Compton amplitudes, given by (21) in terms of the results of a zeroth order photoproduction analysis, may very easily give too large contributions to the differential cross-sections for $\gamma p \to \gamma p$, thus leaving no room for the real parts required by partial wave dispersion relations. It follows that it is essential to analyse simultaneously data on $\gamma p \to \pi^+ n, \pi^0 p, \gamma p$, as was done in [12]. For a proper first order analysis, equation (31) has to be used for the photoproduction data, with $m = 2$, $n = 1$ for each of the multipole amplitudes with $J = \frac{1}{2}$ and $n = 2$ for the amplitudes with $J \geqslant \frac{3}{2}$. The unitary connection of (21) will be used, with ${}_p\lambda_{2n}^{(0)}$ replaced by Re ${}_p\lambda_{2n}$ obtained from the photoproduction analysis without affecting Im $p\boldsymbol{\varepsilon}_n$ to lowest order.

The great difficulty in carrying out the analysis just outlined is in the choice of the matrix ${}_p\mathbf{O}_2'$ and of the eigenphases ${}_p\theta_1$ and ${}_p\theta_3$, the problem being that there is no data for $\pi^0 p \to \pi^0 p$, $\pi^+ n \to \pi^+ n$ and $\pi^0 p \leftrightarrow \pi^+ n$. All one can do therefore is to use the strictly nuclear phases obtained from an analysis like that of Zimmerman [11], and to modify these phases and the charge independent mixing angle by calculating mass difference corrections using the nuclear potentials employed in the analysis of $\pi^{\pm}$p experiments. There are no Coulomb corrections in this case, and it would be necessary to assume that corrections due to the presence of the γp channel can be neglected. This strategy might run into trouble for two reasons. As noted by Berends and Donnachie [8], the analysis of the photoproduction data near the resonance position is very sensitive to the eigenphase ${}_p\theta_3$ for the resonant wave

[1]) The same situation occurs in threshold $\pi^- p$ scattering and was commented on at the beginning of Section 4 of [10].

($J = \frac{3}{2}$, $P = +1$). The value of ${}_p\theta_3$ for the resonant wave, obtained from a potential theory model as just described, might therefore not give the best fit to the photoproduction data. In other words, this data could be used to give a value of ${}_p\theta_3$ in the neighbourhood of the resonance, but only after the mixing angle has been fixed from a model. This is the same awkward situation as we noted earlier for the analysis of $\pi^- p$ experiments. The second possible trouble is noted by Pfeil et al. [12]; the real parts of the $\gamma p \to \gamma p$ multipole amplitudes f_{MM} and f_{ME}, for $J = \frac{3}{2}$, $P = +1$, obtained from the analysis may differ substantially from the real parts calculated from the imaginary parts by means of partial wave dispersion relations.

Ideally one would like to be able to calculate ${}_p\mathbf{O}'_2$ and ${}_p\theta_1$, ${}_p\theta_3$ as described in the last paragraph for $J = \frac{1}{2}$, $P = \pm 1$ and for $J = \frac{3}{2}$, $P = +1$, and to fit all the data on $\gamma p \to \pi^+ n$, $\pi^0 p$, γp across the first resonance with eight real quantities at each energy, namely

$$\text{Re } {}_p\lambda_{21}(J = \tfrac{1}{2}, P = -1), \qquad \text{Re } {}_p\lambda_{21}(J = \tfrac{1}{2}, P = +1),$$
$$\text{Re } {}_p\lambda_{22}(J = \tfrac{3}{2}, P = +1).$$

One would hope that the Compton scattering data could be fitted using (21) to obtain the imaginary parts of the relevant multipole amplitudes, and partial wave dispersion relations to obtain the real parts. If this programme broke down, as it well might for the reasons just described, it would be necessary to explore whether a modification of ${}_p\theta_3$ for $J = \frac{3}{2}$, $P = +1$ was sufficient to give a good fit to the photoproduction data, with the Compton scattering data being satisfactorily fitted by means of multipole amplitudes calculated from (21) and partial wave dispersion relations as before.

For reasons which will appear below, we conclude this discussion of the analysis of γp experiments by writing the 2×2 matrix of amplitudes for $\gamma p \to \pi^+ n$, $\pi^0 p$, with $J = \frac{3}{2}$, $P = +1$, in two forms:

$$\begin{aligned}
&\begin{pmatrix} {}_pS_{+1} & {}_pS_{+2} \\ {}_pS_{01} & {}_pS_{02} \end{pmatrix} \\
&= 2i \begin{pmatrix} \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} 1 & {}_pC_{31} \\ -{}_pC_{31} & 1 \end{pmatrix} \begin{pmatrix} \exp(i\,{}_p\theta_1) & 0 \\ 0 & \exp(i\,{}_p\theta_3) \end{pmatrix} \text{Re } {}_p\lambda_{22}(\mathbf{1}_2 + \\
&\quad + i \text{ Re } {}_p\boldsymbol{\varepsilon}_2) \\
&= i\sqrt{2q_\gamma} \begin{pmatrix} \sqrt{q_+} & 0 \\ 0 & \sqrt{{}_pq_0} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} {}_pE_{1+}^{1/2} & {}_pM_{1+}^{1/2} \\ {}_pE_{1+}^{3/2} & {}_pM_{1+}^{3/2} \end{pmatrix} \times \\
&\quad \begin{pmatrix} -3 & \sqrt{3} \\ -1 & -\sqrt{3} \end{pmatrix}.
\end{aligned} \tag{34}$$

The amplitude ${}_pS_{ij}$ refers to the process in which γp in the state $j (j = 1, 2)$ goes to the pion–nucleon state i, where $i = +, 0$ denotes $\pi^+ n$, $\pi^0 p$ respectively. The photon–nucleon states 1, 2 are defined in Reference [2] and the results in (34) come from Appendix B of [2] (with an extra factor i included, as noted in footnote [2]) of Reference [10]), and from (31), (32) and the equivalent of (33) for $\pi^+ n$, $\pi^0 p$.

For the analysis of data on $\gamma n \to \pi^- p$, $\pi^0 n$, though a full first order analysis cannot be performed at present, it is desirable to use the framework of such an analysis by employing (31) for each J, P. This means that the modified nuclear phases and mixing angle obtained from an analysis of $\pi^- p \to \pi^- p$, $\pi^0 n$ like that of Zimmerman [11] should be used. Further the final factor ($\mathbf{1} + i$ Re ${}_n\boldsymbol{\varepsilon}$) should be included.

To compute Re ${}_n\varepsilon$, one will first use (21) to obtain Im ${}_n\varepsilon$, correct to lowest order, from a zeroth order analysis of pion photoproduction from neutrons. Partial wave dispersion relations will then be used to obtain Re ${}_n\varepsilon$, though in this case there is no data on Compton scattering from neutrons to provide a check on the results. However, after all this, there are still eight real parameters to be obtained at each energy from experiments on $\gamma n \to \pi^- p$, $\pi^0 n$ and the data at present cannot determine them all with reasonable accuracy.

To see what to do, we write the equations analogous to (34) for $J = \frac{3}{2}$, $P = +1$, the notation being obvious:

$$\begin{pmatrix} {}_nS_{-1} & {}_nS_{-2} \\ {}_nS_{01} & {}_nS_{02} \end{pmatrix}$$

$$= 2i \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} 1 & {}_nC_{31} \\ -{}_nC_{31} & 1 \end{pmatrix} \begin{pmatrix} \exp(i_n\theta_1) & 0 \\ 0 & \exp(i_n\theta_3) \end{pmatrix} \mathrm{Re}\ {}_n\lambda_{22}(\mathbf{1}_2 + i\,\mathrm{Re}\ {}_n\varepsilon_2)$$

$$= i\sqrt{2q_\gamma} \begin{pmatrix} \sqrt{q_-} & 0 \\ 0 & \sqrt{{}_nq_0} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} {}_nE_{1+}^{1/2} & {}_nM_{1+}^{1/2} \\ {}_nE_{1+}^{3/2} & {}_nM_{1+}^{3/2} \end{pmatrix} \times$$

$$\begin{pmatrix} -3 & \sqrt{3} \\ -1 & -\sqrt{3} \end{pmatrix}. \tag{35}$$

Now the usual expression of charge independence, which is a zeroth order statement for pion photoproduction from nucleons is, from (32),

$${}_pE_{1+}^{3/2} = {}_nE_{1+}^{3/2}, \qquad {}_pM_{1+}^{3/2} = {}_nM_{1+}^{3/2}.$$

From (34) and (35) we see that, keeping always to zeroth order, this may be replaced by the statement that the second rows of the matrices Re ${}_p\lambda_{22}$ and Re ${}_n\lambda_{22}$ are the same. Similarly, for the partial waves with $J = \frac{1}{2}$, $P = \pm 1$, we have equality of the second elements of Re ${}_p\lambda_{21}$ and Re ${}_n\lambda_{21}$ in each case, to zeroth order. This zeroth order approximation, at this stage of experimental accuracy, gives values of four of the eight real quantities to be determined which are more accurate than those given by the full analysis. Analysis of the data for photoproduction from neutrons will then give the other four real quantities with reasonable accuracy.

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REFERENCES

[1] G. Rasche and W. S. Woolcock, Helv. Phys. Acta *49*, 435 (1976).
[2] G. Rasche and W. S. Woolcock, Helv. phys. Acta *45*, 642 (1972).
[3] K. M. Watson, Phys. Rev. *95*, 228 (1954).
[4] S. Almehed and C. Lovelace, Nucl. Phys. *B40*, 157 (1972).
[5] P. Noelle, W. Pfeil and D. Schwela, Nucl. Phys. *B26*, 461 (1971).
[6] F. A. Berends and D. L. Weaver, Nucl. Phys. *B30*, 575 (1971).
[7] W. Pfeil and D. Schwela, Nucl. Phys. *B45*, 379 (1972).
[8] F. A. Berends and A. Donnachie, Nucl. Phys. *B84*, 342 (1975).
[9] J. R. Carter, D. V. Bugg and A. A. Carter, Nucl. Phys. *B58*, 378 (1973).
[10] G. Rasche and W. S. Woolcock, Helv. phys. Acta *49*, 557 (1976).
[11] H. Zimmerman, Helv. phys. Acta *48*, 191 (1975).
[12] W. Pfeil, H. Rollnik and S. Stankowski, Nucl. Phys. *B73*, 166 (1974).