Zeitschrift: Helvetica Physica Acta

Band: 53 (1980)

Heft: 1

Artikel: On the existence of the wave operator in relativistic quantum scattering theory

Autor: Horwitz, L.P. / Soffer, A.

DOI: https://doi.org/10.5169/seals-115112

Helvetica Physica Acta, Vol. 53 (1980), Birkhäuser Verlag, Basel

On the existence of the wave operator in relativistic quantum scattering theory[1])

by **L. P. Horwitz** and **A. Soffer**

Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel

(18. I. 1980, rev. 13. III. 1980)

Abstract. We discuss scattering theory in a covariant relativistic quantum mechanics. The equation of evolution, with respect to an invariant parameter τ, is of Schrödinger form, and potential scattering theory is formally similar to that of non-relativistic potential scattering. Although an invariant potential $V(x^2)$ cannot be in $L^2(\mathbf{R}^4)$ we prove that $\int_T^\infty \|Ve^{-iK_0\tau}\varphi\|\,d\tau$ exists for V bounded and decreasing fast enough as $|x^2| \to \infty$ for every φ in a set dense in $L^2(\mathbf{R}^4)$, and hence the corresponding (relativistic) wave operator exists.

The dynamical variables needed to describe a spin zero particle are $(\vec{x}, t, \vec{p}, E)$ [1]. They have Heisenberg equations of the form

$$\frac{d}{d\tau} x_\mu(\tau) = i[K, x_\mu] \tag{1}$$

where K is the generator of motion in τ, an evolution parameter (historical time) needed to order the events along the path taken by the particle in space-time, and x_μ and K are viewed as self-adjoint operators in the Hilbert space $L^2(\mathbf{R}^4)$. The corresponding Schrödinger equation is [2]

$$i\frac{\partial \psi_\tau}{\partial \tau} = K\psi_\tau, \tag{2}$$

where $\psi_\tau \in L^2(\mathbf{R}^4)$. We shall assume that

$$K = K_0 + V(x^2) \tag{3}$$

with the Stueckelberg free evolution operator

$$K_0 = \frac{p_\mu p^\mu}{2M} = \frac{\vec{p}^2 - p_0^2}{2M}$$

and the invariant potential $V(x^2)$ is a function of $x^2 = \vec{x}^2 - x^{02}$ only ($\hbar = c = 1$). In the presence of an external electromagnetic field, $2MK$ would have the form $(p^\mu - eA^\mu(x))(p_\mu - eA_\mu(x))$. In this paper, we shall restrict ourselves to the potential model (3).

---

[1]) This research was supported in part by the U.S.–Israel Binational Science Foundation, Jerusalem, Israel.

The Heisenberg relations (1), with the evolution operator (2), imply that ($[x^\mu, p^\nu] = ig^{\mu\nu}$)

$$\frac{dx^\mu}{d\tau}(\tau) = \frac{p^\mu}{M} \tag{4}$$

$$\frac{dp^\mu}{d\tau}(\tau) = -\frac{\partial V}{\partial x_\mu} \tag{5}$$

The central motion of a scattering wave packet is described by the expectation value of (4). If the motion, as a function of τ, brings x^μ to a region for which V and its derivatives are small, $\langle p^\mu \rangle$ becomes approximately constant. Further evolution of $\langle t \rangle$ is therefore proportional (since $dt/d\tau = E/M$) to τ. Since $\tau \to \pm\infty$ implies $t \to \pm\infty$ (for $E > 0$) if these limits coincide with regions where the potential vanishes, the usual asymptotic condition on scattering, involving limits in t, can be replaced by limits in τ.

This qualitative equivalence is closely analogous to the heuristic observation that the asymptotic condition in t in the non-relativistic theory is qualitatively equivalent to an asymptotic condition on the form of the wave function at large distances.

We shall require that

$$\|e^{-iK\tau}\psi^{(\pm)} - e^{-iK_0\tau}\phi\| \to 0 \tag{6}$$

for $\tau \to \mp\infty$. Hence,

$$\psi^{(\pm)} = \Omega_\pm \phi \tag{7}$$

where, if the limits exist,

$$\Omega_\pm = \lim_{\tau \to \pm\infty} e^{iK\tau} e^{-iK_0\tau} \tag{8}$$

are the wave operators for relativistic scattering theory. We shall formulate the scattering problem in the following, and then prove that the wave operators exist for conditions on V which are essentially the same as the well-known sufficient conditions for the non-relativistic scattering problem.

To formulate the scattering problem [3] [4], let us assume that asymptotically, the average motion of the wave packet is that of a free particle, i.e., for the ith wave packet in the incoming beam,

$$\begin{aligned} \langle \vec{x}_i(\tau) \rangle &= \langle x_i(0) \rangle + \frac{\vec{p}^*}{M}\tau \\ \langle t_i(\tau) \rangle &= \langle t_i(0) \rangle + \frac{E^*}{M}\tau \end{aligned} \tag{9}$$

The probability for a particle to be scattered into a small volume d^4p of momentum space after a collision is given by

$$\omega(d^4p \leftarrow \phi_i) = d^4p\, |\psi_i^{(+)\text{scatt.}}(p)|^2 \tag{10}$$

The incoming beam has average momentum $p^{\mu *}$, and the wave packets are initially (at $\tau = 0$) at a given large distance (in the beam direction) from the target. Summing over directions transverse to the beam and displacements in time [5], we find the number of particles scattering into d^4p to be

$$N_{sc}(d^4p) = \sum_i \omega(d^4p \leftarrow \phi_i) \cong \int d^2\rho \int dT n_{inc.} \omega(d^4p \leftarrow \phi_{\rho T}) \tag{11}$$

where

$$\phi_{\rho T} = e^{-i(\vec{\rho}\cdot\vec{p} - TE)}\phi(p)$$

and $n_{inc.}$ is the number of packets per unit area and time. This integral can effectively 'cover' a potential $V(x^2)$ of the type considered if it falls off sufficiently fast as a function of $x^\mu x_\mu$, since such functions decrease rapidly in all directions in space-time except in a diminishingly small region near the light cone (the overlap of a wave packet with this tail of the potential decreases in the $L^2(\mathbf{R}^4)$ norm). Assuming $n_{inc.}$ to be constant in a large enough region, the differential cross-section ($p = \sqrt{\vec{p}^2}$) is

$$\frac{d\sigma}{d\Omega\, dE}(d\Omega, dE \leftarrow \phi) = \int d^2\rho \int dT \int dp p^2 \, |\psi_{\rho T}^{(+)\text{scatt.}}(\vec{p}, E)|^2. \tag{12}$$

Defining a 'T matrix' in the usual way [6] from the S matrix associated with the wave operator (8), one obtains [3] [4] [7]

$$\frac{d\sigma}{d\Omega\, dE}(d\Omega, dE \leftarrow \phi) = (2\pi)^5 M^2 \frac{p}{|\vec{p}^*|} |T(p \leftarrow p^*)|^2, \tag{13}$$

where $p = \sqrt{p_\mu^* p^{\mu *} + E^2}$ ($p^\mu p_\mu$ is conserved). We remark that $d\sigma$ has the dimensions of area$\times$time (for example, the electromagnetic scattering of a charged particle on a heavy charged target yields, in lowest approximation, the usual Rutherford scattering cross-section times what may be interpreted as an interaction time [4]).

We now turn to the proof of the existence of the wave operator.

A sufficient condition for the existence of the wave operator (8) is that

$$\int_T^\infty \|Ve^{-iK_0\tau}\varphi\| \, d\tau < \infty \tag{14}$$

for some finite T and for φ any element of some dense set in the Hilbert space. We start with

Theorem 1. *The operator $p^\mu p_\mu$, defined as a multiplication operator in momentum space, is essentially self-adjoint on $\mathcal{S}$-space functions contained in $L^2(\mathbf{R}^4)$.*

Proof. Given the set of $\mathcal{S}$ space functions with support in some open domain of $\mathbf{R}^4$, consider the algebra of multiplication operators $O^M(\mathbf{R}^4)$ which leaves $\mathcal{S}$ invariant. This algebra consists of those $\mathcal{S}_\infty$ functions which, together with all their derivatives, are polynomially bounded on all of $\mathbf{R}^4$. $\mathcal{S}$ is then an ideal of O^M.

Consider the linear functional l on $\mathcal{S}$ *only* defined by $l(\varphi) = \int \varphi \, d^4p$ for all $\varphi \in \mathcal{S}$. Then, $l(\alpha\varphi_1 + \varphi_2) = \alpha l(\varphi_1) + l(\varphi_2)$; $l(\varphi^*\varphi) \geq 0$, and the equality holds if and only if $\varphi = 0$. If we define the scalar product on $\mathcal{S}$ by

$$(\varphi, \psi)_l = l(\varphi^*\psi) \, \forall \varphi, \psi \in \mathcal{S}, \tag{15}$$

the completion of $\mathcal{S}$ under this product is $L^2(\Omega)$, where Ω is the support of $\mathcal{S}$.

For each element a of O^M, define the operator A on $\mathcal{S} \subset L^2$ by $A\tilde{\varphi} = j(a\varphi) \in \mathcal{S} \subset L^2$, where $\tilde{\varphi}$ is $j(\varphi)$ and j is the identity map of $\mathcal{S} \subset O^M$ onto $\mathcal{S} \subset L^2$. Then, A is densely defined on L^2, leaving $\mathcal{S}$ invariant. If a is real, then A is symmetric. One may verify, moreover, that if $a \in O^M$, then a^*a, $(1+a^*a)^{-1}$ and $e^{i\alpha a}$ (for every real α) are also in O^M. Suppose $\tilde{\psi}$ is in $\mathcal{S} \subset L^2$. We shall show that $\varphi_\pm \in \mathcal{S} \subset O^M$ exist, for which $(A \pm i)\tilde{\varphi}_\pm = \tilde{\psi}$. In fact, if $\tilde{\psi} = (A \pm i)\tilde{\varphi}$, then $\psi = (a \pm i)\varphi_\pm$ or

$$(a^* \mp i)\psi = (a^* \mp i)(a \pm i)\varphi_\pm = (a^*a+1)\varphi_\pm$$

and hence

$$\varphi_\pm = (a^*a+1)^{-1}(a^* \mp i)\psi.$$

Since ψ belongs to the ideal $\mathcal{S}$ of O^M, it follows that $\varphi_\pm$ are in $\mathcal{S}$, and hence $\tilde{\varphi}_\pm$ are in $\mathcal{S} \subset L^2$. Taking a to be a real polynomial function of the $p^\mu \in \mathbf{R}^4$, this result implies that the corresponding operator is essentially self-adjoint. ■

Lemma 1.[2]

$$(e^{-iK_0\tau}\varphi)(x) = \frac{1}{(4\pi i\tau)^2} \int e^{-i(x-y)^2/4\tau}\varphi(y)\, d^4y \tag{16}$$

where $x^2 = x \cdot x = \vec{x}^2 - x^{02}$ *in the Minkowski metric.*

Proof. $e^{-iK_0\tau}$ is defined by Theorem 1 and Stone's theorem; the result follows by application of the Fourier transform

$$\varphi(x) = \frac{1}{(2\pi)^2} \int e^{ip\cdot x}\hat{\varphi}(p)\, d^4p. \quad ■ \tag{17}$$

Theorem 2.

$$(e^{-iK_0\tau}\varphi)(x) \xrightarrow[L^2]{} \frac{1}{(2i\tau)^2} e^{i(x^2/4\tau)}\hat{\varphi}\left(\frac{x}{2\tau}\right),$$

i.e., in the L^2 *sense, as* $\tau \to \infty$.

Proof. The proof is almost the same as in the three-dimensional case [8], where now the three dimensional Fourier transform is replaced by the four dimensional Minkowski Fourier transform. We reproduce the proof here since we will utilize some of the intermediate steps later. For τ fixed the map

$$V_\tau : \varphi \to \frac{1}{(2i\tau)^2} e^{ix^2/4\tau}\hat{\varphi}\left(\frac{x}{2\tau}\right) \tag{19}$$

is unitary. It is therefore sufficient to demonstrate the result for $\varphi \in \mathcal{S}$, for example. We assume that $\varphi \in \mathcal{S}$ in the remainder of the proof. Since

$$e^{i(x-y)^2/4\tau} = e^{ix^2/4\tau}e^{-ix\cdot y/2\tau}e^{iy^2/4\tau},$$

we obtain

$$(e^{-iK_0\tau}\varphi)(x) - (V_\tau\varphi)(x) = \frac{1}{(2i\tau)^2} e^{ix^2/4\tau}\hat{G}_\tau\left(\frac{x}{2\tau}\right), \tag{20}$$

---

[2]) We take $2M = 1$ in the following.

where the function $G_\tau(y)$ is defined as

$$G_\tau(y)=(e^{iy^2/4\tau}-1)\varphi(y). \tag{21}$$

Then, for $\varphi\in\mathscr{S}(\|\cdot\|$ is the $L^2(\mathbf{R}^4)$ norm),

$$\|e^{-iK_0\tau}\varphi-V_\tau\varphi\|=\frac{1}{(2\tau)^2}\left\|\hat{G}_\tau\left(\frac{\cdot}{2\tau}\right)\right\|=\|\hat{G}_\tau\|=\|G_\tau\|\leqslant\frac{1}{4\tau}\int|y^2\varphi(y)|^2\,d^4y\to 0$$

as $\tau\to\infty$.

At this stage, it is possible to estimate the effect of the tails of the wave function off the classical trajectories using stationary phase approximation [9] [10].[3] In the following, we use direct calculation.

Theorem 3. *$\int_T^\infty\|Ve^{-iK_0\tau}\varphi\|\,d\tau$ (T finite) exists for V bounded and decreasing fast enough as $x^2\to\infty$ on a dense set in $\{\varphi\in L^2(\mathbf{R}^4)\mid\hat{\varphi}\in L^2_\chi(\mathbf{R}^4)\}$, where $L^2_\chi(\mathbf{R}^4)$ is the Hilbert space of square integrable functions with support in the positive cone.*

Proof. Consider the set of states φ with support of $\hat{\varphi}$ in the forward cone and with $p^2\geqslant\epsilon>0$ (no zero-mass states in φ). Now, calling $e^{-iK_0\tau}\varphi=\varphi_\tau$, we have

$$\|V\varphi_\tau\|=\|V(\varphi_\tau-V_\tau\varphi)+V_\tau\varphi\|\leqslant\|V(\varphi_\tau-V_\tau\varphi)\|+\|VV_\tau\varphi\| \tag{22}$$

Consider the last term of (22):

$$\begin{aligned}\|VV_\tau\varphi\|^2&=\frac{1}{(2\tau)^4}\int(V(x^2))^2\left|\hat{\varphi}\left(\frac{x}{2\tau}\right)\right|^2 d^4x\\&=\int(V(4\tau^2y^2))^2|\hat{\varphi}(y)|^2\,d^4y\\&\leqslant\max_{y^2\geqslant\epsilon>0}(V(4\tau^2y^2))^2=\max_{x^2\geqslant4\tau^2\epsilon}(V(x^2))^2.\end{aligned} \tag{23}$$

Next, we consider the first term of (22):

$$V(\varphi_\tau-V_\tau\varphi)(x)=i^{-2}e^{ix^2/4\tau}(f_1(x)+f_2(x))$$

where

$$\begin{aligned}f_1(x)&=\frac{1}{4\tau^2}V(x^2)\int d^4y\varphi(y)\left(e^{iy^2/4\tau}-1-\frac{iy^2}{4\tau}\right)e^{-iy\cdot x/2\tau}\\f_2(x)&=\frac{1}{4\tau^2}V(x^2)\int d^4y\varphi(y)\frac{iy^2}{4\tau}e^{-iy\cdot x/2\tau},\end{aligned} \tag{24}$$

and we now choose φ such that $y^2\varphi(y)\in L^2$. Hence

$$\|V(\varphi_\tau-V_\tau\varphi)\|\leqslant\|f_1\|+\|f_2\|. \tag{25}$$

Consider the second term of (25):

$$\begin{aligned}\|f_2\|^2&=\frac{1}{(2\tau)^4}\int d^4x(V(x^2))^2\left|\int d^4y\varphi(y)e^{-ix\cdot y/2\tau}\frac{y^2}{4\tau}\right|^2\\&=\int d^4z(V(4z^2\tau^2))^2\left|\int d^4y\varphi(y)e^{-iz\cdot y}\frac{y^2}{4\tau}\right|^2.\end{aligned}$$

[3]) We wish to thank B. Simon for pointing out this possibility to us.

Suppose now that the support of

$$\psi(z)=\frac{1}{(2\pi)^2}\int d^4y e^{-iz\cdot y}y^2\varphi(y)$$

is outside of the set $z^2<\epsilon$. Then,

$$\|f_2\|^2\leq\frac{(2\pi)^4}{16\tau^2}\max_{z^2\geq\epsilon}(V(4z^2\tau^2))^2\int d^4z\,|\psi(z)|^2$$
$$=\frac{(2\pi)^4}{16\tau^2}\max_{z^2\geq\epsilon}(V(4z^2\tau^2))^2\int d^4y\,|y^2\varphi(y)|^2. \tag{26}$$

For the first term of (25),

$$\|f_1\|^2=\frac{1}{(2\tau)^4}\int d^4x(V(x^2))^2\left|\int d^4y e^{-ix\cdot y/2\tau}\left(e^{iy^2/4\tau}-1-\frac{iy^2}{4\tau}\right)\varphi(y)\right|^2$$
$$=\int d^4z(V(4\tau^2z^2))^2\left|\int d^4y e^{-iz\cdot y}\left(e^{iy^2/4\tau}-1-\frac{iy^2}{4\tau}\right)\varphi(y)\right|^2$$
$$\leq\max_{x\in\mathbf{R}^4}(V(x^2))^2\int d^4z\left|\int d^4y e^{-iz\cdot y}\left(e^{iy^2/4\tau}-1-\frac{iy^2}{4\tau}\right)\varphi(y)\right|^2$$
$$=(2\pi)^4\max_{x\in\mathbf{R}^4}(V(x^2))^2\int d^4y\left|\left(e^{iy^2/4\tau}-1-\frac{iy^2}{4\tau}\right)\varphi(y)\right|^2 \tag{27}$$

We now break the integral up into two regions, the first for $|y^2/4\tau|<1$ and the second for $|y^2/4\tau|\geq 1$. Using the bound $|e^{ix}-1-ix|\leq x^2/(1-x/2)$ for $|x|<2$, we obtain for the first part

$$\int_{|y^2/4\tau|<1}d^4y\left|\left(e^{iy^2/4\tau}-1-\frac{iy^2}{4\tau}\right)\varphi(y)\right|^2\leq\frac{4}{(4\tau)^4}\int d^4y(y^2)^4|\varphi(y)|^2=\frac{\alpha}{\tau^4} \tag{28}$$

for φ such that $(y^2)^2\varphi(y)\in L^2$. Using the bound $|e^{ix}-1-ix|\leq 1+\sqrt{1+x^2}\leq 3|x|^2$ for $|x|\geq 1$, we obtain for the second part

$$\int_{|y^2/4\tau|\geq 1}d^4y\left|\left(e^{iy^2/4\tau}-1-\frac{iy^2}{4\tau}\right)\varphi(y)\right|^2\leq\frac{9}{(4\tau)^4}\int d^4y(y^2)^4|\varphi(y)|^2=\frac{\beta}{\tau^4}, \tag{29}$$

for φ such that $(y^2)^2\varphi(y)\in L^2$. Hence, $\|f_1\|$ is bounded by $O(1/\tau^2)$ if $(y^2)^2\varphi(y)\in L^2$ and $V(x^2)$ is bounded. Gathering these results, we see that the requirements of the theorem are met if $\varphi(y)$, $y^2\varphi(y)$ and $(y^2)^2\varphi(y)$ are in L^2, $\hat{\varphi}$ and $\widehat{(y^2\varphi)}$ have support in the positive cone and vanish on an ϵ-neighbourhood of the boundary, and if $V(x^2)$ is bounded and

$$\max_{y^2\geq\epsilon>0}|V(4\tau^2y^2)|$$

is integrable on (T,∞) in τ for some finite T. For V of the form $|x^2|^{-\alpha}$, $\alpha>0$, for $|x^2|$ large, integrability requires that $\alpha\geq\frac{1}{2}+\delta$, for some $\delta>0$. ■

Corollary. *The wave operator exists for $V(x^2)$ bounded and decreasing fast enough $x^2\to\pm\infty$.*

This result can be seen by replacing y^2 by $|y^2|$ in the appropriate places (to apply the proof to the space of wave functions with support in momentum space inside the space-like region;[4]) in this more general case, $V(x^2)$ must vanish for $x^2 \to -\infty$ as well.

For the method using stationary phase approximations, one should choose the open set containing the support of the Fourier transform of the wave function to be away from the light cone. One can then apply Lemma A.1 of Ref. [9].

Acknowledgements

We wish to thank Barry Simon and Israel Sigal for helpful discussions.

REFERENCES

[1] L. P. HORWITZ and C. PIRON, Helv. Phys. Acta *46*, 316 (1973). The formulation of the theory for particles with spin is given by L. P. HORWITZ, C. PIRON and F. REUSE, Helv. Phys. Acta *48*, 546 (1975), and C. PIRON and F. REUSE, Helv. Phys. Acta *51*, 146 (1978).
[2] E. C. G. STUECKELBERG, Helv. Physica Acta *14*, 588 (1941); *15*, 23 (1948). See also R. P. FEYNMAN, Phys. Rev. *80*, 440 (1950) (Appendix A), and Ref. [1].
[3] L. P. HORWITZ, Y. LAVIE and A. SOFFER, *Scattering Theory in Relativistic Quantum Mechanics*, Tel Aviv University preprint TAUP 754-79, and in the Proceedings of the VIII International Colloquium on Group Theoretical Methods in Physics. (Kiriat Anavim, Israel, March 25–29, 1979) Vol. 3, Annals of the Israel Physical Society (1980).
[4] L. P. HORWITZ and Y. LAVIE, to be published.
[5] One may think of t as the relative time between the incident particle and the target. In this case, we are studying the relative motion. Since K is quadratic in momenta, the center of mass motion may be separated as in the non-relativistic problem [1] [3].
[6] J. R. TAYLOR, *Scattering Theory*, John Wiley & Sons, Inc., New York, 1972.
[7] This formula was obtained by J. H. COOKE, Phys. Rev. *166*, 1293 (1968) from a somewhat different point of view.
[8] M. REED and B. SIMON, *Methods of Modern Mathematical Physics II*, Aca. Press New York and London, 1975.
[9] L. HÖRMANDER, Math. Z. *146*, 69 (1976).
[10] M. REED and B. SIMON, *Methods of Modern Mathematical Physics III*, Aca. Press, New York and London, 1979.

[4]) Applicable to the two body problem with center-of-mass motion removed [3], [4].