Generating functions of canonical maps

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Abstract. There are many possible descriptions of a canonical map of a symplectic manifold (phase space) by means of generating functions. These possibilities form a continuous set which we investigate in a very general way. The generalized form of the Hamilton–Jacobi equation for generating functions of time dependent canonical maps is established. The composition law of generating functions is studied. Important sub-groups of canonical maps are treated as illustrations.

1. Introduction

Generating functions of canonical maps play a well known role in classical mechanics as solutions of the Hamilton–Jacobi equation. They appear also in quantum mechanics in the WKB method and more generally as exponents in the matrix elements of time evolution operators. It is not an exaggeration to say that generating functions play a key role as a link between classical and quantum mechanics.

As a consequence, it is important to try to understand geometrically the meaning of the different kinds of generating functions. At first glance, the different possibilities for choosing the variables are rather confusing.

The present paper makes use of a remark due to Abraham [1] that the graph of a canonical map is a Lagrangian manifold in the product space of the phase space with itself. The sometimes strange choice of variables in the theory is necessitated by the parametrization of this Lagrangian manifold. We give a general method for obtaining all possible generating functions of a given canonical map. We study more carefully a new possibility proposed by Marinov. This kind of generating function also appears as an exponent in the Wigner-function of unitary operators, but the present paper deals only with classical properties.

2. Generating functions of canonical maps

Let E be a $2n$ dimensional real symplectic manifold, l its nondegenerate closed 2-form, and let Φ be a canonical map of E

$$\Phi : E \ni x \to \bar{x} \in E. \tag{2.1}$$

The condition of canonicity, requiring the invariance of the action integral, can also be written

$$\int l(\Phi(x) \mid \Phi'(x)\, dx_1, \Phi'(x)\, dx_2) = \int l(x \mid dx_1, dx_2). \tag{2.2}$$

The graph $\mathscr{V}$ of Φ,

$$\mathscr{V} = \{u \mid x, x' = \Phi(x) \in E\} \subset E \times E, \tag{2.3}$$

in the product space

$$E \times E = \{u \mid u = (x, x'); x, x' \in E\} \tag{2.4}$$

is a $2n$-dimensional submanifold of $E \times E$ having a characteristic geometry. Introducing into $E \times E$ a symplectic structure by means of the 2-form

$$\mathscr{L}(u \mid du_1, du_2) = l(x \mid dx_1, dx_2) - l(x' \mid dx_1', dx_2') \tag{2.5}$$

which, like l, is nondegenerate and closed

$$\mathscr{L}(u \mid du_1, du_2) = 0 \;\forall\; du_2 \Rightarrow du_1 = 0, \tag{2.6}$$

$$d\mathscr{L} = 0, \tag{2.7}$$

we see that the condition (2.2) can be written

$$\int_{\mathscr{D}} \mathscr{L} = 0, \qquad \mathscr{D} \subset \mathscr{V}, \tag{2.8}$$

for any simply connected compact 2-dimensional domain $\mathscr{D}$ of $\mathscr{V}$. In other words, $\mathscr{V}$ is a *Lagrangian sub-manifold* of the symplectic manifold

$$\mathscr{E} = (E \times E; \mathscr{L}). \tag{2.9}$$

Conversely, if a Lagrangian sub-manifold of $\mathscr{E}$ is the graph of a map, it defines a canonical map of E. To give explicitly a canonical map of E is thus essentially a problem of finding a convenient parametrization of a Lagrangian sub-manifold of $\mathscr{E}$.

The property (2.7) implies that $\mathscr{L}$ is locally the skew derivative of a 1-form $\mathscr{A}$,

$$\mathscr{L} = d\mathscr{A}. \tag{2.10}$$

Further, the condition (2.8) means that the restriction $\mathscr{L}_{\mathscr{V}}$ of $\mathscr{L}$ to the manifold $\mathscr{V}$ is a vanishing 2-form,

$$0 = \mathscr{L}_{\mathscr{V}} = d\mathscr{A}_{\mathscr{V}}. \tag{2.11}$$

In consequence, the restriction $\mathscr{A}_{\mathscr{V}}$ is locally the total derivative

$$\mathscr{A}_{\mathscr{V}} = dA \tag{2.12}$$

of a differentiable function A defined on $\mathscr{V}$.

The potentials $\mathscr{A}$ and A are not unique. If $\mathscr{F}$ is any differentiable function on $\mathscr{E}$, then

$$\hat{\mathscr{A}} = \mathscr{A} + d\mathscr{F} \tag{2.13}$$

is another 'potential'; $\mathscr{L} = d\hat{\mathscr{A}}$. Correspondingly, one has for the restrictions

$$\hat{\mathscr{A}}_{\mathscr{V}} = \mathscr{A}_{\mathscr{V}} + d\mathscr{F}_{\mathscr{V}} = d\hat{A} \tag{2.14}$$

$$\hat{A} = A + \mathscr{F}_{\mathscr{V}}. \tag{2.15}$$

The potentials $\hat{\mathscr{A}}$ and $\hat{A}$ are moreover only defined up to an additive constant.

In order to make tractable the fundamental equation (2.12) of $\mathscr{V}$, the three following tools have to be conveniently chosen. (1) A chart of $\mathscr{E}$, (2) a 'potential' $\mathscr{A}$, (3) a pair $\mathcal{M}, \mathcal{M}^c$ of reference Lagrangian sub-manifolds of $\mathscr{E}$, complementary to each other: $\mathscr{E} = (\mathcal{M} \times \mathcal{M}^c\,; \mathscr{L})$. Then by adapting the chart to this new reduction of $\mathscr{E}$ into a product

$$\mathscr{E} = (E \times E\,; \mathscr{L}) \ni (x, x') \leftrightarrow (\alpha, \beta) \in (\mathcal{M} \times \mathcal{M}^c\,; \mathscr{L}) = \mathscr{E}$$

in a way that $(\alpha, 0) \in \mathcal{M}$, $(0, \beta) \in \mathcal{M}^c$, the equation (2.12) will lead to the simple result (see Fig. 1)

$$\mathscr{V} \ni (\alpha, \beta) \leftrightarrow \beta(\alpha) = \nabla \hat{A}(\alpha).$$

For the first tool, it is obviously convenient to choose a canonical chart of E. The coefficients of l are then constants,

$$l(x \mid dx_1, dx_2) = dx_1 \cdot L\, dx_2. \tag{2.16}$$

The constant matrix L will only be supposed to have the properties

$$\tilde{L} = -L, \qquad L^2 = -\mathbf{1}_{2n}, \qquad \det L = 1, \tag{2.17}$$

its precise form being irrelevant. This choice yields a canonical chart of $\mathscr{E}$, in which $\mathscr{L}$ reads

$$\mathscr{L}(u \mid du_1, du_2) = du_1 \cdot \mathscr{L}\, du_2, \qquad du_i = (dx_i, dx_i'), \tag{2.18}$$

where

$$\mathscr{L} = \begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix}. \tag{2.19}$$

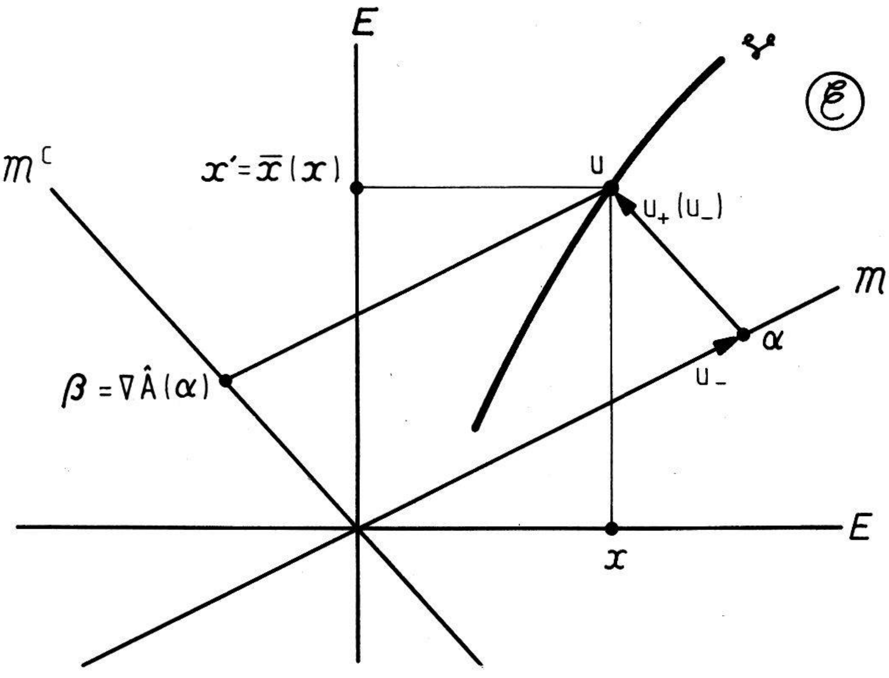

Figure 1
Schematic view in $\mathscr{E}$ of the graph $\mathscr{V}$ of a canonical map. The points $u \in \mathscr{V}$ are located relative to the reference manifold $\mathcal{M}$.

To choose the second tool, we remark that a 1-form $\mathscr{A}$ can always be written

$$\mathscr{A}(u \mid du) = a(u) \cdot du = \mathscr{L}(a^*(u) \mid du); \tag{2.20}$$

the dual a^* of the field a is well defined because $\mathscr{L}$ is non-degenerate (2.6). In a given canonical chart of $\mathscr{E}$, the simplest choice for a^* is a linear function. With

$$a^*(u) = \tfrac{1}{2}u \tag{2.21}$$

one has

$$\mathscr{A}(u \mid du) = \tfrac{1}{2}u \cdot \mathscr{L}\, du = \tfrac{1}{2}(x \cdot L\, dx - x' \cdot L\, dx') \tag{2.22}$$

and the correct form (2.18) for $\mathscr{L}$.

This expression is invariant under symplectic changes of chart. In order to preserve the linearity in u, subsequent changes of chart should obviously be limited to linear ones. For the third tool, we remark that in a given canonical chart, the simplest Lagrangian sub-manifolds are linear. They are given using an arbitrary anticanonical involution,

$$\tilde{I}\mathscr{L}I = -\mathscr{L}, \qquad I^2 = -\mathbf{1}_{4n}. \tag{2.23}$$

The polarization projectors

$$\mathbf{P}_\pm = \tfrac{1}{2}(\mathbf{1} \pm I), \tag{2.24}$$

having the properties

$$\mathbf{P}_\pm^2 = \mathbf{P}_\pm, \qquad \mathbf{P}_\pm \mathbf{P}_\mp = 0, \qquad \tilde{\mathbf{P}}_\pm \mathscr{L} \mathbf{P}_\pm = 0, \qquad \tilde{\mathbf{P}}_\pm \mathscr{L} = \mathscr{L}\mathbf{P}_\mp, \tag{2.25}$$

decompose the tangent spaces into the sum of two isotropic subspaces,

$$du = du_+ + du_-, \tag{2.26}$$

$$du_\pm = \mathbf{P}_\pm\, du, \tag{2.27}$$

$$a^*(u) = \mathbf{P}_+ a^*(u) + \mathbf{P}_- a^*(u) = \tfrac{1}{2}u_+ + \tfrac{1}{2}u_-. \tag{2.28}$$

The two manifolds

$$\mathscr{M} = \{u \mid u_- = 0\} \tag{2.29}$$

and

$$\mathscr{M}^c = \{u \mid u_+ = 0\} \tag{2.30}$$

are complementary and Lagrangian. (Note that $\mathscr{M}$ does not define I completely because $\mathscr{L}$ is skew symmetric. Knowledge of $\mathscr{M}^c$ is required to fix I.) If, in addition, $\mathscr{M}^c$ is transverse to $\mathscr{V}$, the points of $\mathscr{V}$ can be located relative to the reference manifold $\mathscr{M}$ as shown in Fig. 1. The 'potential' $\mathscr{A}$ is decomposed into a sum

$$\mathscr{A}(u \mid du) = \hat{\mathscr{A}}(u \mid du) - \tfrac{1}{4}d(u \cdot \mathscr{L}Iu) \tag{2.31}$$

in which the gauge transformed

$$\hat{\mathscr{A}}(u \mid du) = (\mathbf{P}_- u) \cdot \mathscr{L}\mathbf{P}_+\, du = u_- \cdot \mathscr{L}\, du_+ \tag{2.32}$$

have vanishing restrictions to both $\mathscr{M}$ and $\mathscr{M}^c$. Eventually, by adapting the coordinates to $\mathcal{M}$ and $\mathcal{M}^c$ with a linear change of chart

$$u=(x,x')\mapsto\psi=(\alpha,\beta)=U^{-1}u, \tag{2.33}$$

such that

$$U^{-1}IU=T=\begin{pmatrix}\mathbf{1}_{2n} & 0\\ 0 & -\mathbf{1}_{2n}\end{pmatrix} \tag{2.34}$$

$$\tilde{U}\mathcal{L}U=\Gamma=\begin{pmatrix}0 & -\mathbf{1}_{2n}\\ \mathbf{1}_{2n} & 0\end{pmatrix} \tag{2.35}$$

one has $u_-=U\begin{pmatrix}\alpha\\0\end{pmatrix}\in\mathcal{M}$, $u_+=U\begin{pmatrix}0\\\beta\end{pmatrix}\in\mathcal{M}^c$. This leads to

$$\hat{\mathcal{A}}(u\mid du)=\beta\cdot d\alpha \tag{2.36}$$

$$\mathcal{A}(u\mid du)=\beta\cdot d\alpha-\tfrac{1}{2}d\,(\beta\cdot\alpha)=\tfrac{1}{2}(\beta\cdot d\alpha-\alpha\cdot d\beta). \tag{2.37}$$

The canonicity condition (2.21) is equivalent to

$$\hat{\mathcal{A}}_{\mathcal{V}}=d\hat{A} \tag{2.38}$$

or, explicitly

$$\beta(\alpha)=\nabla\hat{A}(\alpha)=\nabla(A(\alpha)+\tfrac{1}{2}\alpha\cdot\beta(\alpha)). \tag{2.39}$$

In summary: Given a reference Lagrangian sub-manifold $\mathcal{M}$ of $\mathscr{E}$ and a potential $\hat{A}$, a canonical map Φ of E is defined by the points $u=(x,\bar{x})$ of its graph $\mathcal{V}$ expressed in the following parametric form

$$\begin{aligned}x(\alpha)&=U_{11}\alpha+U_{12}\beta(\alpha)\\ \bar{x}(\alpha)&=U_{21}\alpha+U_{22}\beta(\alpha)\end{aligned} \tag{2.40}$$

The $2n$-dimensional square matrices U_{ik} are sub-matrices of U (2.33),

$$U=\begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}. \tag{2.41}$$

The convenient reference manifolds defined in (2.29–30) form a continuous set.[1]) A limited discrete subset only is exploited in the literature (Section 3). The potential $\hat{A}$ is commonly called the *generating function* of the canonical transformation Φ. It is clear that for each choice I one has to select an appropriate $\hat{A}_I$ in order to describe the same map Φ. The relation between these generating functions is easily established by remarking that the potential A itself does not depend on I. We thus have for two equivalent pairs $(I,\hat{A})$ and $(I',\hat{A}')$

$$A=\hat{A}'(\alpha')-\tfrac{1}{2}\alpha'\cdot\nabla\hat{A}'(\alpha')=\hat{A}(\alpha)-\tfrac{1}{2}\alpha\cdot\nabla\hat{A}(\alpha). \tag{2.42}$$

The coordinates ψ and ψ' labelling the same point of $\mathcal{V}$ in the two descriptions are connected by the linear relation

$$u=U\psi=U'\psi', \tag{2.43}$$

where U and U' put I and I' respectively into the standard form T (2.34).

[1]) The set of all U's (2.35) modulo those which leave I invariant.

However, (2.43) does not give a linear relation between α and α' if β or β' is a non-linear function of respectively α or α'.

The description of a canonical map by means of a pair $(I, \hat{A})$ usually has a local validity only. The picture may break down for two reasons. First, the map Φ may be defined on a subset E' of E only. If E' is made of disconnected open subsets E_i' of E, the graph $\mathscr{V}$ of Φ has as many branches $\mathscr{V}_i$, and, for each of them, a distinct generating function $\hat{A}_i$ is needed. In the case where the subsets E_i' have common boundaries, and an $\mathcal{M}^c$ transverse to the $\mathscr{V}_i$'s exists, the $\hat{A}_i$'s may be equal to the different values of one multiply valued function $\hat{A}$ (see Section 6, Example 2). Secondly, even if Φ is a diffeomorphism of E, a reference manifold $\mathcal{M}$ with a complementary $\mathcal{M}^c$ everywhere transverse to $\mathscr{V}$ does not always exist. As long as $\mathcal{M}^c$ is transverse to $\mathscr{V}$, the relations (2.40) are bijective and the Jacobians

$$\det\left(\frac{\partial x}{\partial \alpha}\right) = \det\left(U_{11} + U_{12}\frac{\partial^2 \hat{A}}{\partial \alpha\, \partial \alpha}\right) = \det\left(U_{21} + U_{22}\frac{\partial^2 \hat{A}}{\partial \alpha\, \partial \alpha}\right) = \det\left(\frac{\partial \bar{x}}{\partial \alpha}\right) \tag{2.44}$$

take a finite, non-zero value. Indeed, tangent vectors to $\mathscr{V}$ and to $\mathcal{M}^c$ can be respectively expressed as

$$du = (dx, \Sigma(x)\, dx) \tag{2.45}$$

and

$$du' = (U_{12}\, d\beta, U_{22}\, d\beta), \tag{2.46}$$

where

$$\Sigma(x) = \left(\frac{\partial \bar{x}}{\partial x}\right) = \left(U_{21} + U_{22}\frac{\partial^2 \hat{A}}{\partial \alpha\, \partial \alpha}\right)\left(U_{11} + U_{12}\frac{\partial^2 \hat{A}}{\partial \alpha\, \partial \alpha}\right)^{-1} \in \mathrm{Sp}\,(TE_x); \tag{2.47}$$

the matrices U_{ik} are sub-matrices of U (2.41). The lack of transversality means that at least one tangent vector of $T\mathscr{V}_u$ is co-linear to a tangent vector of $T\mathcal{M}^c$:

$$du' = \lambda\, du, \qquad \lambda \in \mathbf{R}. \tag{2.48}$$

Eliminating dx from this relation, one obtains

$$(U_{22} - \Sigma(x)U_{12})\, d\beta = 0. \tag{2.49}$$

Therefore, whenever

$$\det\,(U_{22} - \Sigma(x)U_{12}) \neq 0 \tag{2.50}$$

the solution is trivial, $d\beta = 0$, and $\mathcal{M}^c$ is transverse to $\mathscr{V}$ in $(x, \bar{x}(x))$. On the other hand, one deduces from (2.40) and (2.46) the equation

$$(U_{22} - \Sigma(x)U_{12})\Omega = \Sigma(x)U_{11} - U_{21} \tag{2.51}$$

which relates $\Sigma(x(\alpha))$ to the matrix

$$\Omega(\alpha) = \left(\frac{\partial^2 \hat{A}}{\partial \alpha^\mu\, \partial \alpha^\nu}\right) \tag{2.52}$$

of second derivatives of $\hat{A}$. When the transversality condition (2.50) holds, one has

$$\Omega = (U_{22} - \Sigma U_{12})^{-1}(\Sigma U_{11} - U_{21}) \tag{2.53}$$

or

$$\Sigma = (U_{21} + U_{22}\Omega)(U_{11} + U_{12}\Omega)^{-1}. \tag{2.54}$$

Points at which (2.50) fails to hold are singular points of $\hat{A}$.

To conclude this point, we show that there exists no $\mathcal{M}^c$ everywhere transverse to $\mathcal{V}$ if the set

$$S' = \{\Sigma(x) \mid x \in E\} \subset \mathrm{Sp}\,(2n, \mathbf{R}) \tag{2.55}$$

contains a closed path $C = \{\Sigma(x(t)) \mid t \in \mathbf{R}\}$ which is not homotopic to zero. Considering the generic case $\det U_{22} \neq 0$ for simplicity, we rewrite (2.50) as

$$\det(\mathbf{1} + \Sigma_0\Sigma(x(t))) \neq 0. \tag{2.56}$$

By virtue of (2.35), the matrix

$$\Sigma_0 = -U_{12}U_{22}^{-1} \tag{2.57}$$

is symplectic. For any $\Sigma_0 \in \mathrm{Sp}\,(2n, \mathbf{R})$ the path $\gamma' = \Sigma_0\gamma$ is homotopic to γ because $\mathrm{Sp}\,(2n, \mathbf{R})$ is connected. If γ is not homotopic to zero, neither is γ' and at least one eigenvalue $\sigma_\mu(t)$ of $\Sigma_0\Sigma(x(t))$ is equal to -1 for some t_0. Since

$$\det(\mathbf{1} + \Sigma_0\Sigma(x(t))) = \prod_{\mu=1}^{2n} (1 + \sigma_\mu(t))$$

this function vanishes in $x(t_0)$, and (2.56) is not fulfilled.

We remark that the existence of an $\mathcal{M}^c$ transverse to $\mathcal{V}$ is not excluded when Φ is only locally one-to-one. The above discussion remains valid in this case.

A reference manifold $\mathcal{M}$ may or may not be appropriate for the discussion of canonical maps lying in the vicinity of the map identity $\Phi_e : x \mapsto x$. The graph $\mathcal{V}_e = \{u \mid x = \bar{x}\}$ of Φ_e is given, using an arbitrary $\mathcal{M}$, by a generating function $\hat{A}_e$ which, according to (2.40), fulfills the relation

$$(U_{11} - U_{21})\alpha = (U_{22} - U_{12})\nabla\hat{A}_e(\alpha). \tag{2.58}$$

The most convenient choice is naturally $\mathcal{M} = \mathcal{V}_e$; in this case, $\hat{A}_e = 0$ and $U_{11} = U_{21}$ (case (e), Section 3). Otherwise, $\hat{A}_e$ is a quadratic function

$$\hat{A}_e(\alpha) = \tfrac{1}{2}a \cdot \Omega_e\alpha \tag{2.59}$$

with Ω_e given by (2.53) with $\Sigma = \mathbf{1}$. If $\mathcal{M}^c$ is not transverse to $\mathcal{V}_e$, (2.51) has no solution and $\hat{A}_e$ does not exist in this description. (See, for example, cases (a) and (b), Table 1.)

3. Four well-known and one recent descriptions of canonical maps

Four well-known descriptions of canonical maps [2] by means of generating functions are nearly always used. They are listed in Table 1. The choices (a) and (b) are not adequate to describe the map identity Φ_e. The fifth description, case (e) of Table 1, makes use of the graph $\mathcal{V}_e$ of Φ_e as reference manifold. This was

Table 1

	ψ	$\mathscr{V}$	$d\hat{A}$	$\hat{A}=A+\frac{1}{2}\beta\cdot\alpha$	$\hat{A}_e$
(a)	$\alpha=(q,q')$ $\beta=(p,-p')$	$p=\frac{\partial W}{\partial q}(q,\bar{q})$ $\bar{p}=\frac{\partial W}{\partial \bar{q}}(q,\bar{q})$	$dW=p\cdot dq-\bar{p}\cdot d\bar{q}$	$W=A+\frac{1}{2}(q\cdot p-\bar{q}\cdot\bar{p})$	No
(b)	$\alpha=(p,p')$ $\beta=(-q,q')$	$q=-\frac{\partial G}{\partial p}(p,\bar{p})$ $\bar{q}=\frac{\partial G}{\partial \bar{p}}(p,\bar{p})$	$dG=-q\cdot dp+\bar{q}\cdot d\bar{p}$	$G=A+\frac{1}{2}(-p\cdot q+\bar{p}\cdot\bar{q})$	No
(c)	$\alpha=(q,p')$ $\beta=(p,q')$	$p=\frac{\partial S}{\partial q}(q,\bar{p})$ $\bar{q}=\frac{\partial S}{\partial \bar{p}}(q,\bar{p})$	$dS=p\cdot dq+\bar{q}\cdot d\bar{p}$	$S=A+\frac{1}{2}(q\cdot p+\bar{q}\cdot\bar{p})$	$\bar{p}\cdot q$
(d)	$\alpha=(p,q')$ $\beta=(-q,-p')$	$q=-\frac{\partial F}{\partial p}(p,\bar{q})$ $\bar{p}=-\frac{\partial F}{\partial \bar{q}}(p,\bar{q})$	$dF=-q\cdot dp-\bar{p}\cdot d\bar{q}$	$F=A-\frac{1}{2}(q\cdot p+\bar{q}\cdot\bar{p})$	$-p\cdot\bar{q}$
(e)	$\alpha=\frac{1}{2}(x+x')$ $\beta=L(x'-x)$	$L(\bar{x}-x)$ $=\nabla g\left(\frac{\bar{x}+x}{2}\right)$	$dg=-\frac{1}{2}(\bar{x}-x)\cdot L(d\bar{x}+dx)$	$g=A-\frac{1}{2}(\bar{p}\cdot q-\bar{q}\cdot p)$	0
$\mathscr{M}=\{u(\alpha,\beta)\in\mathscr{E}\mid\beta=0\}$ $\mathscr{M}^c=\{u(\alpha,\beta)\in\mathscr{E}\mid\alpha=0\}$			Conventions: $u(x,x')\in\mathscr{E}$ $u(x,\bar{x})\in\mathscr{V}$		

first published by Marinov [3] who looked for a symplectic invariant description. We happened upon it independently in connection with projective representations of the symplectic group [4, 5]. Choosing for the complementary manifold $\mathscr{M}^c=\mathscr{V}_{-e}$, the graph of the parity $\Phi_{-e}:x\mapsto -x$, the involution (2.23) is

$$I_0=\begin{pmatrix}0 & \mathbf{1}_{2n}\\ \mathbf{1}_{2n} & 0\end{pmatrix} \tag{3.1}$$

and the linear change of coordinates (2.33)

$$U_0=\begin{pmatrix}\mathbf{1}_{2n} & -\frac{1}{2}\Lambda\\ \mathbf{1}_{2n} & \frac{1}{2}\Lambda\end{pmatrix},\qquad \Lambda=L^{-1}. \tag{3.2}$$

The parametric equations (2.40) defining $\mathscr{V}$ become

$$\begin{matrix}x=\alpha-\frac{1}{2}\Lambda\beta(\alpha)\\ \bar{x}=\alpha+\frac{1}{2}\Lambda\beta(\alpha)\end{matrix}\quad \beta(\alpha)=\nabla g(\alpha). \tag{3.3}$$

The potential $\hat{A}$ associated with $\mathscr{M}=\mathscr{V}_e$ has been here conventionally called g as in Table 1. This generating function defines $\bar{x}=\Phi(x)$ implicitly by

$$\bar{x}-x=\Lambda\nabla g\left(\frac{\bar{x}+x}{2}\right). \tag{3.4}$$

This picture will be referred to as the *standard* one in what follows.

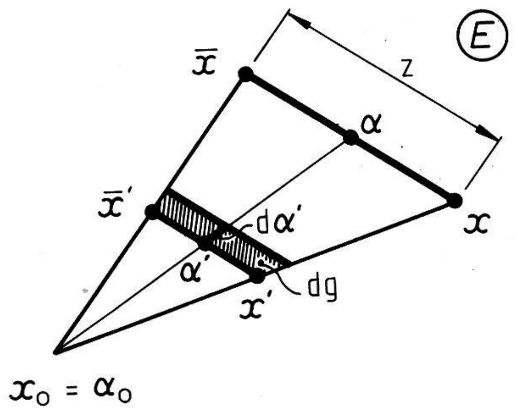

Figure 2
The canonical map Φ maps x onto $\bar{x}$ and leaves x_0 fixed. In the canonical chart, the point α is the middle point of the side $z = \bar{x} - x$.

In the canonical chart (tool number 1) chosen for the description, the quantity

$$\alpha = \tfrac{1}{2}(\bar{x} + x) \tag{3.5}$$

can be visualized as the middle point between $\bar{x}$ and x, and

$$z = \bar{x} - x = \Lambda \nabla \beta(\alpha) \tag{3.6}$$

as their 'difference'. The differential

$$dg = \beta \cdot d\alpha = -l(z, d\alpha) \tag{3.7}$$

represents an element of symplectic area, or equivalently the action integral computed along the boundary (Fig. 2). Assuming that the map Φ generated by g has a *fixed point* $x_0 : \Phi(x_0) = x_0$, or $\bar{x}_0 = x_0 = \alpha_0$ and $z_0 = 0$, the function $g(\alpha)$ can be normalized in a natural way by integrating dg starting from x_0. Then $g(\alpha)$ is equal to the action relative to the 'triangle' $(x_0, x, \bar{x})$. The form of one of the 'sides' $x_0 x$ or $x_0 \bar{x}$ is irrelevant.

This normalization convention makes sense in more general cases where the fixed points of Φ form a *connected* sub-manifold:

$$\mathscr{F} \ni y, \qquad \Phi(y) = y.$$

The reason is that g is *constant* on $\mathscr{F}$ since $\nabla g(y) = 0$. The argument fails if two fixed points are isolated or contained in two disconnected parts of $\mathscr{F}$.

4. Time dependent generating functions

Let Φ_t be a time dependent canonical map

$$\Phi_t : x \mapsto x_t(x), \tag{4.1}$$

solution of the Hamilton equation

$$\dot{x}_t = \Lambda \nabla h \tag{4.2}$$

with initial conditions

$$x_0(x) = \Phi_0(x). \tag{4.3}$$

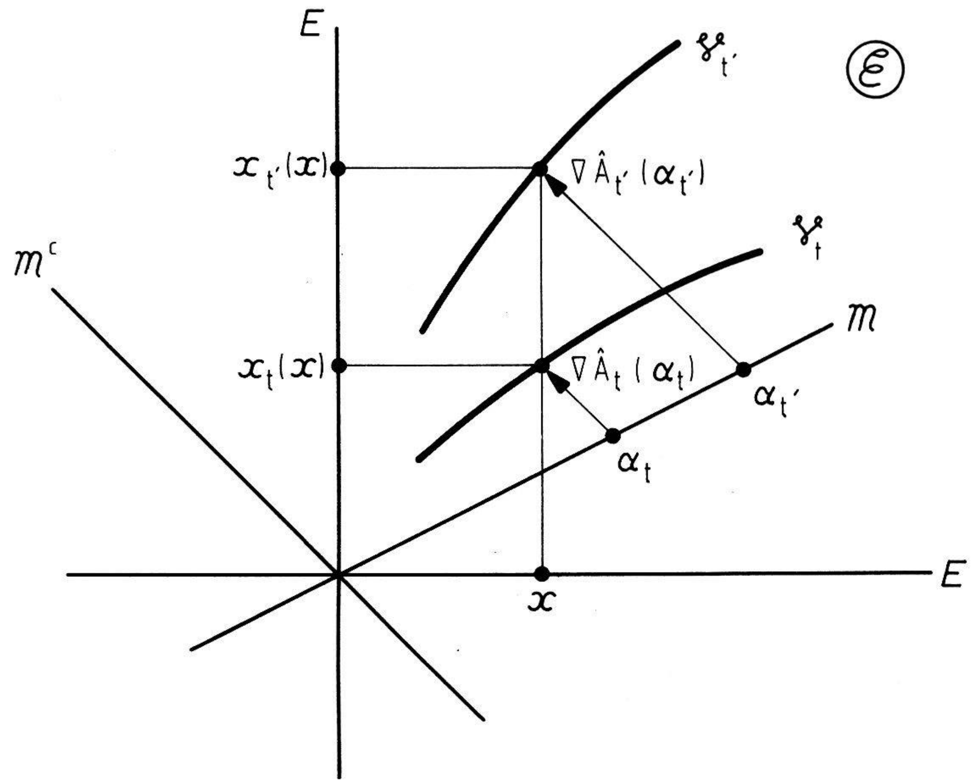

Figure 3
Schematic view in $\mathscr{E}$ of the graphs of canonical maps at two different times t and t'.

The Hamilton function h may be time dependent, and the initial map Φ_0 is not necessarily the identity Φ_e.

The graph $\mathscr{V}_t$ of Φ_t is a Lagrangian manifold, moving in $\mathscr{E}$ as a function of t (see Fig. 3). Applying the results of Section 2 with a time independent reference manifold $\mathscr{M}$, one has at any time

$$\mathscr{V}_t \ni u_t = \begin{pmatrix} x \\ x_t(x) \end{pmatrix} = U\begin{pmatrix} \alpha_t \\ \nabla\hat{A}_t(\alpha_t) \end{pmatrix}, \tag{4.4}$$

where $\hat{A}_t$ is the time dependent generating function of Φ_t. The coordinates α_t also depend on t since the initial points x are fixed (Fig. 3). In order to get an equation of evolution for $\hat{A}_t$, we differentiate (4.4) with respect to t:

$$\begin{pmatrix} 0 \\ \dot{x}_t \end{pmatrix} = U\begin{pmatrix} \dot{\alpha}_t \\ \nabla_\alpha(\partial_t\hat{A}_t(\alpha_t) + \dot{\alpha}_t \cdot \nabla_\alpha\hat{A}_t(\alpha_t)) \end{pmatrix} = \dot{u}_t. \tag{4.5}$$

On the other hand, tangent vectors of $\mathscr{V}_t$ read explicitly

$$du_t = \begin{pmatrix} \dfrac{\partial x}{\partial\alpha}(\alpha_t) \cdot d\alpha \\ \dfrac{\partial\bar{x}}{\partial\alpha}(\alpha_t) \cdot d\alpha \end{pmatrix} = U\begin{pmatrix} d\alpha \\ (d\alpha \cdot \nabla_\alpha)\nabla_\alpha\hat{A}_t(\alpha_t) \end{pmatrix}. \tag{4.6}$$

Now, the value of $\mathscr{L}(du_t, \dot{u}_t)$ computed with coordinates $u = (x, x')$ or $\psi = (\alpha, \beta)$

must be the same. This leads successively, using (4.5–6), to

$$\mathcal{L}(du_t, \dot{u}_t) = -l\left(\frac{\partial x_t}{\partial \alpha} \cdot d\alpha,\ \Lambda \nabla h(x_t)\right) = -d\alpha \cdot \frac{\partial}{\partial \alpha_t} h(x_t(\alpha_t), t)$$
$$= (d\alpha, (d\alpha \cdot \nabla_\alpha)\nabla_\alpha \hat{A}_t(\alpha_t))\Gamma\begin{pmatrix} \dot{\alpha}_t \\ \nabla_\alpha(\partial_t \hat{A}_t + \dot{\alpha}_t \cdot \nabla_\alpha \hat{A}_t)\end{pmatrix}$$
$$= -d\alpha \cdot \frac{\partial}{\partial \alpha_t}(\partial_t \hat{A}_t(\alpha_t)).$$

These equalities hold for all $d\alpha$ and α_t. Dropping the subscript t of α_t and integrating with respect to α, one obtains

$$\frac{\partial}{\partial t}\hat{A}_t(\alpha) = h(x_t(\alpha), t) + \chi(t), \tag{4.7}$$

where $\chi(t)$ is the integration constant in α for each t. This differential equation for the *time dependent generating function* is a generalized form of the Hamilton–Jacobi equation. This equation is usually expressed with respect to one of the four reference manifolds (a) to (d) mentioned in Table 1. The initial condition at $t=0$ is

$$\hat{A}_t|_{t=0} = \hat{A}_0 = \text{generating function of } \Phi_0. \tag{4.8}$$

A more explicit form is obtained by the insertion of (4.4) into (4.7):

$$\partial_t \hat{A}_t(\alpha) = h(U_{11}\alpha + U_{12}\nabla \hat{A}_t(\alpha), t) + \chi(t). \tag{4.7'}$$

The solution of equations (4.7–8) is unique, but it may describe only part of Φ_t if this map is not diffeomorphic. This is evident from the discussion of Section 2 when Φ_0 itself is not a diffeomorphism. On the other hand, when $\Phi_0 = \Phi_e$, for instance, it may happen that Φ_t induces trajectories $x_t(x_0)$ in E which go to infinity within a *finite* time $t_1(x_0)$. Then, the graphs $\mathcal{V}_t$ have more than one branch for any finite but arbitrarily small t. Other solutions of (4.8), singular in t at $t=0$, are needed to describe the branches of $\mathcal{V}_t$ that 'fade out' at $t=0$ (see Section 6, Example (b)).

In the case of particle submitted to velocity independent potentials, one traditionally takes advantage of the special form of h by choosing the descriptions (a) or (c). Equation (4.7′) is then a partial differential equation in n variables only. Practically, this is a big advantage, but the full geometrical meaning of the theory is obscured. Changing the reference manifold allows one to distinguish a physical catastrophy from a coordinate singularity produced by a perspective effect.

For small times, one has to first order

$$\hat{A}_{\Delta t}(\alpha) = \hat{A}_0(\alpha) + \Delta t(h(U_{11}\alpha + U_{12}\nabla \hat{A}_0(\alpha), 0) + \chi(0)). \tag{4.9}$$

If the initial map is the identity, $\hat{A}_0$ is the quadratic function $\hat{A}_e$ (2.59). The *standard description* is then particularly convenient since $\hat{A}_e = 0$. Calling $\hat{A}_t = g_t$ as in Section 3 and putting $\chi = 0$:

$$\partial_t g_t(\alpha) = h(\alpha + \tfrac{1}{2}\Lambda \nabla g_t(\alpha)), \qquad g_0(\alpha) = 0, \tag{4.10}$$

and

$$\alpha_t = \tfrac{1}{2}(x_t + x), \qquad \beta_t = x_t - x = \nabla g_t(\alpha_t). \tag{4.11}$$

For small t

$$g_{\Delta t}(\alpha) = h(\alpha, 0)\Delta t. \tag{4.12}$$

The linear map induced by Φ_t in tangent space ((2.47–52–54)),

$$\Sigma_t(x) = \left(\frac{\partial x_t}{\partial x}\right) = \frac{\mathbf{1} + \frac{1}{2}\Lambda\Omega_t(x)}{\mathbf{1} - \frac{1}{2}\Lambda\Omega_t(x)} \in \mathrm{Sp}\,(TE_x), \tag{4.13}$$

is now the inverse Cayley transform of the matrix $\frac{1}{2}\Lambda\Omega_t$ which belongs to the Lie-algebra sp $(2n, \mathbf{R})$ of Sp $(2n, \mathbf{R})$.

When h does not depend on time, the family $\{\Phi_t \mid t \in \mathbf{R}\}$ is an Abelian semi-group, $\Phi_{t'} \circ \Phi_t = \Phi_{t'+t}$, and the relation between its generator h and the generating functions g_t is

$$h(\alpha) = \partial_t g_t(\alpha)|_{t=0}. \tag{4.14}$$

Owing to the additional variable t, the transversality of $\mathcal{M}^c$ to $\mathcal{V}_t$ has to be studied on the basis of the new set

$$S'' = \{S_t' \mid t \in \mathbf{R}\}$$

where S_t' is defined by (2.55). The statement of Section 2 remains valid here with S'' replacing S'.

Time depending reference manifolds $\mathcal{M}_t$ and $\mathcal{M}_t^c$ can be included in the present formalism. This subject lying somewhat outside the scope of this paper, we only mention here the simple case of the *interaction picture.* For a Hamiltonian of the form

$$h(x, t) = h^{(0)}(x) + h^{(1)}(x, t)$$

where $h^{(0)}$ is a second degree polynomial associated with 'free' motion and $h^{(1)}$ a 'perturbation', it is practical to use a time dependent $\mathcal{M}_t = \mathcal{V}_t^{(0)}$, the graph of $\Phi_t^{(0)}$ generated by $h^{(0)}$. This amounts to setting

$$x_t = \Sigma(t)y_t + v_t$$

(see Section 6(a)) and proceeding as above with the interaction Hamiltonian

$$h_I(y, t) = h^{(0)}(\Sigma(t)y + v_t, t)$$

in place of h.

5. Composition law of generating functions

Let Φ_i, $i = 1, 2$, be two canonical maps, and $\hat{A}_i$ their generating functions relative to the same reference manifold $\mathcal{M}$. The product

$$\Phi_3 = \Phi_2 \circ \Phi_1 \tag{5.1}$$

has a generating function $\hat{A}_3$ which can be expressed as a composition of the first two,

$$\hat{A}_3 = \hat{A}_2 \top \hat{A}_1. \tag{5.2}$$

The law $\top$ naturally depends on the choice of I (2.23).

By the definition of Φ_3, any initial point x of E and its images $\bar{x}$, $\bar{\bar{x}}$ must form a 'triangle' with edges

$$x, \qquad \bar{x}=\Phi_1(x), \qquad \bar{\bar{x}}=\Phi_2(\bar{x})=\Phi_3(x). \tag{5.3}$$

Points of the graphs $\mathscr{V}_i$, $i=1,2,3$, are respectively

$$u_1=(x,\bar{x}), \qquad u_2=(\bar{x},\bar{\bar{x}}), \qquad u_3=(x,\bar{\bar{x}}) \tag{5.4}$$

and tangent vectors to $\mathscr{V}_i$ are

$$du_1=(dx,d\bar{x})=(dx,\Phi_1'(x)\,dx), \qquad du_2=(d\bar{x},dx), \qquad du_3=(dx,d\bar{\bar{x}}). \tag{5.5}$$

It can be verified that the 'potential' $\mathscr{A}$ (2.22) taken at these points of $T\mathscr{E}$ is additive,

$$\mathscr{A}(u_3\mid du_3)=\mathscr{A}(u_1\mid du_1)+\mathscr{A}(u_2\mid du_2). \tag{5.6}$$

By splitting $\mathscr{A}$ with respect to $\mathscr{M}$ as in (2.31), the equality (5.6) becomes

$$\hat{\mathscr{A}}_3(u_3\mid du_3)-\tfrac{1}{4}d(u_3\cdot\mathscr{L}Iu_3)=\sum_{i=1,2}[\hat{\mathscr{A}}(u_i\mid du_i)-\tfrac{1}{4}d(u_i\cdot\mathscr{L}Iu_i)] \tag{5.7}$$

or, in coordinates $\psi^{(i)}=(\alpha^{(i)},\beta^{(i)})=U^{-1}u_i$ adapted to $\mathscr{M}$,

$$\beta^{(3)}\cdot d\alpha^{(3)}-\tfrac{1}{2}d(\beta^{(3)}\cdot\alpha^{(3)})=\sum_{i=1,2}[\beta^{(i)}\cdot d\alpha^{(i)}-\tfrac{1}{2}d(\beta^{(i)}\cdot\alpha^{(i)})]. \tag{5.8}$$

Taking the canonicity of Φ_1 and Φ_2 into account, $\beta^{(i)}=\nabla\hat{A}_i$, it follows from (5.8) that $\beta^{(3)}$ is also the gradient of an $\hat{A}_3$. The integration of (5.8) is straightforward and yields, putting the integration constant equal to zero,

$$\hat{A}_3(\alpha^{(3)})=\tfrac{1}{2}\alpha^{(3)}\cdot\beta^{(3)}+\sum_{i=1,2}[\hat{A}_i(\alpha^{(i)})-\tfrac{1}{2}\alpha^{(i)}\cdot\beta^{(i)}]. \tag{5.9}$$

This equation defines $\hat{A}_3$ implicitly and consequently the product (5.2). (We call the operation $\top$ a product rather than a sum because it is non-commutative.) A useful geometrical picture of the situation is obtained in the product space $(E\times)^6=\{(u_1,u_2,u_3)\}$ parametrized by the variables $\boldsymbol{\alpha}=(\alpha^{(1)},\alpha^{(2)},\alpha^{(3)})$ and $\boldsymbol{\beta}=(\beta^{(1)},\beta^{(2)},\beta^{(3)})$. The definitions (5.4) regarded as constraints in this space can be rewritten

$$B\boldsymbol{\beta}=C\boldsymbol{\alpha} \tag{5.10}$$

with

$$B=\begin{pmatrix} U_{22} & -U_{12} & 0 \\ 0 & U_{22} & -U_{22} \\ -U_{12} & 0 & U_{12}\end{pmatrix} \qquad C=\begin{pmatrix} -U_{21} & U_{11} & 0 \\ 0 & -U_{21} & U_{21} \\ U_{11} & 0 & -U_{11}\end{pmatrix}.$$

The canonicity of Φ_1 and Φ_2 is expressed by the two additional constraints

$$\beta^{(i)}(\alpha^{(i)})=\nabla\hat{A}_i(\alpha^{(i)}), \qquad i=1,2. \tag{5.11}$$

The relations (5.10) and (5.11) define implicitly a $2n$-dimensional sub-manifold of $(E\times)^6$ on which equation (5.9) is identically satisfied. By choosing the set $\alpha^{(3)}$ as independent variables and letting $\alpha_0^{(i)}(\alpha^{(3)})$, $i=1,2$, $\beta_0^{(j)}(\alpha^{(3)})$, $j=1,2,3$ be the solutions of (5.10) and (5.11), one obtains immediately from (5.9):

$$\hat{A}_3(\alpha^{(3)}) = \tfrac{1}{2}\alpha^{(3)} \cdot \beta_0^{(3)}(\alpha^{(3)}) + \sum_{i=1,2} [\hat{A}_i(\alpha_0^{(i)}(\alpha^{(3)})) + \tfrac{1}{2}\alpha_0^{(i)}(\alpha^{(3)}) \cdot \beta_0^{(i)}(\alpha^{(3)})] \quad (5.12)$$

Noting that the matrices B and C as well as the sub-matrices U_{ij} may be irregular, we first set down a procedure for the calculation of the unknowns $\boldsymbol{\alpha}_0(\alpha^{(3)})$, $\boldsymbol{\beta}_0(\alpha^{(3)})$ valid in all cases. Using the property (2.35) of U, one can rewrite (5.10) as

$$\beta^{(3)} = \tilde{U}_{21}LU_{21}(\alpha^{(2)} - \alpha^{(1)}) + \tilde{U}_{21}LU_{22}\beta^{(2)} - \tilde{U}_{11}LU_{12}\beta^{(1)} \quad (5.13)$$

$$\alpha^{(3)} = \tilde{U}_{12}LU_{11}\alpha^{(1)} - \tilde{U}_{22}LU_{21}\alpha^{(2)} + \tilde{U}_{12}LU_{12}(\beta^{(1)} - \beta^{(2)}) \quad (5.14)$$

$$0 = U_{21}\alpha^{(1)} - U_{11}\alpha^{(2)} + U_{22}\beta^{(1)} - U_{12}\beta^{(2)}, \quad (5.15)$$

or, taking (5.11) into account,

$$\beta^{(3)} = \tilde{U}_{21}LU_{21}(\alpha^{(2)} - \alpha^{(1)}) + \tilde{U}_{21}LU_{22}\nabla\hat{A}_2(\alpha^{(2)}) - \tilde{U}_{11}LU_{12}\nabla\hat{A}_1(\alpha^{(1)}) \quad (5.16)$$

$$\left.\begin{aligned} \tilde{U}_{12}LU_{11}\alpha^{(1)} - \tilde{U}_{22}LU_{21}\alpha^{(2)} + \tilde{U}_{12}LU_{12}(\nabla\hat{A}_1(\alpha^{(1)}) - \nabla\hat{A}_2(\alpha^{(2)})) = \alpha^{(3)} \\ U_{21}\alpha^{(1)} + U_{22}\nabla\hat{A}_1(\alpha^{(1)}) - U_{11}\alpha^{(2)} - U_{12}\nabla\hat{A}_2(\alpha^{(2)}) = 0 \end{aligned}\right\}. \quad (5.17)$$

For fixed $\alpha^{(3)}$ the solution of the system (5.17) yields the desired $\alpha_0^{(1)}(\alpha^{(3)})$ and $\alpha_0^{(2)}(\alpha^{(3)})$. The remaining ones, $\beta_0^{(j)}(\alpha^{(3)})$, are obtained by inserting the arguments $\alpha_0^{(i)}(\alpha^{(3)})$ into (5.11) and (5.16).

The generating function of the inverse of a canonical map can be calculated following a similar procedure. One has in this case $\Phi_2 = \Phi_1^{-1}$, $\Phi_3 = \Phi_e$, i.e. $x = \bar{\bar{x}}$. Consequently, equation (5.6) becomes

$$0 = \mathscr{A}_2(u_2 \mid du_2) + \mathscr{A}_1(u_1 \mid du_1) \quad (5.18)$$

and equation (5.9) reads

$$\hat{A}_2(\alpha^{(2)}) - \tfrac{1}{2}\beta^{(2)} \cdot \alpha^{(2)} = -\hat{A}_1(\alpha^{(1)}) + \tfrac{1}{2}\beta^{(1)} \cdot \alpha^{(1)}. \quad (5.19)$$

The constraints (5.10) and (5.11) are now

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} \alpha^{(1)} \\ \beta^{(1)} \end{pmatrix} = \begin{pmatrix} U_{21} & U_{22} \\ U_{11} & U_{12} \end{pmatrix}\begin{pmatrix} \alpha^{(2)} \\ \beta^{(2)} \end{pmatrix} \quad (5.20)$$

$$\beta^{(1)} = \nabla\hat{A}_1(\alpha^{(1)}). \quad (5.21)$$

The remaining calculation is similar to the previous one.

If the manifold $\mathcal{M}^c$ is not transverse to $\mathscr{V}_3$, the system (5.17) has no solution. If $\mathcal{M}^c$ is transverse to $\mathscr{V}_3$, this same system may have more than one solution. This is the case for instance when Φ_3 is not a diffeomorphism of E; the various solutions are necessary to give Φ_3 within corresponding domains of E, in which it is well defined (see Section 6, example (b)). These difficulties reflect the local character of the theory.

The above general treatment contains many expressions which are much simpler when $\mathcal{M}$ is one of the reference manifolds of Table 1. As an illustration, we take the case $\mathcal{M} = \mathscr{V}_e$, the graph of the identity. With $U = U_0$ (3.2) the matrices B and C are regular and (5.10) admits the simple solution

$$\beta^{(i)} = 2L(\alpha^{(j)} - \alpha^{(k)}) \quad (5.22)$$

where ijk are even permutations of 123. The system (5.17) becomes

$$\nabla \hat{A}_2(\alpha^{(2)}) + 2L\alpha^{(1)} = -\nabla \hat{A}_1(\alpha^{(1)}) + 2L\alpha^{(2)} = 2L\alpha^{(3)}. \tag{5.23}$$

Replacing the $\beta^{(i)}$'s in equation (5.9) by their values (5.22), one has

$$\begin{aligned}\hat{A}_3 &= \Psi(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)})\\ &= \hat{A}_1(\alpha^{(1)}) + \hat{A}_2(\alpha^{(2)}) - 2(\alpha^{(1)} - \alpha^{(3)}) \cdot L(\alpha^{(2)} - \alpha^{(3)}).\end{aligned} \tag{5.24}$$

The function $\sum_{i=1}^{3} \beta^{(i)} \cdot \alpha^{(i)}$ can be expressed using the invariant l only. The equations (5.23) are just the stationary conditions for Ψ with respect to arbitrary variations of $\alpha^{(1)}$ and $\alpha^{(2)}$. In summary, the generating function $\hat{A}_3(\alpha)$ of $\Phi_2 \circ \Phi_1$ is given by (5.24) for values $\alpha_0^{(1)}(\alpha)$ and $\alpha_0^{(2)}(\alpha)$ such that

$$\frac{\partial \Psi}{\partial \alpha^{(1)}}(\alpha^{(1)}, \alpha^{(2)}, \alpha) = \frac{\partial \Psi}{\partial \alpha^{(2)}}(\alpha^{(1)}, \alpha^{(2)}, \alpha) = 0. \tag{5.25}$$

The expression of the inverse $\Phi_2 = \Phi_1^{-1}$ of Φ_1 is obtained from (3.3) by permuting x and $\bar{x}$. One has obviously $\alpha^{(1)} = \alpha^{(2)}$, $\beta^{(1)} = -\beta^{(2)}$; in other words

$$\hat{A}_2(\alpha) = -\hat{A}_1(\alpha). \tag{5.26}$$

The т-product law (5.24–25) has a simple geometrical interpretation when Φ_i, $i = 1, 2, 3$ have a single fixed point each. These points are located in $x_{(i)} = \alpha^{(i)}$ (see Section 3). Therefore, (5.24–25) prescribes that the value of $\hat{A}_3$ at the fixed point $x_{(3)}$ is equal to the sum of the values of $\hat{A}_i$, $i = 1, 2$, at the fixed points $x_{(i)}$ plus half the action associated with the triangle $(x, \bar{x}, \bar{\bar{x}})$. By iteration for a triple product of maps, one shows easily that the т-product is *associative* if each factor $\hat{A}_i$ has a single critical point.

6. Examples

(a) *Generating functions of one-parameter sub-groups of symplectic maps*

For simplicity, we assume that E is an affine symplectic manifold, homeomorphic to $\mathbf{R}^{2n}$. The coordinates $x = (x', \ldots, x^{2n})$ are assumed to be linear and to belong to a canonical chart in which the coefficients of L are given by the usual matrix

$$L = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{6.1}$$

Let h be a second degree function that may be written in two convenient forms

$$h(x) = \tfrac{1}{2}x \cdot \Omega x + A \cdot x + h_0 = \tfrac{1}{2}l(x, Mx) + l(x, a) + h_0 \tag{6.2}$$

where

$$\Omega = LM = \tilde{\Omega}, \qquad A = La. \tag{6.3}$$

The solutions of the equations of motion

$$\dot{x}_t = \Lambda \nabla h = Mx_t + a \tag{6.4}$$

define a 1-parameter sub-group of canonical maps

$$\Phi_t : x_0 \mapsto x_t(x_0) \tag{6.5}$$

which belongs to the inhomogeneous symplectic group $I\,\mathrm{Sp}\,(E)$. Explicitly:

$$x_t(x_0) = \Sigma(t)x_0 + v(t) \tag{6.6}$$

where

$$\Sigma(t) = e^{tM} \tag{6.7}$$

and

$$v(t) = \Sigma(t)\int_0^t dt'\Sigma^{-1}(t')a = \int_0^t dt'\Sigma(t')a. \tag{6.8}$$

$\Sigma(t)$ belongs to the symplectic group $\mathrm{Sp}\,(E)$ and $M = \Lambda\Omega$ to its Lie-algebra $\mathrm{sp}\,(E)$.

We choose the reference manifold $\mathcal{M} = \mathcal{V}_e$ and define accordingly the new variables

$$\begin{aligned} \alpha_t &= \tfrac{1}{2}(x_t + x_0) = \tfrac{1}{2}[(\Sigma(t) + \mathbf{1})x_0 + v(t)] \\ \beta_t &= L(x_t - x_0) = (\Sigma(t) - \mathbf{1})x_0 + v(t). \end{aligned} \tag{6.9}$$

The elimination of x_0 between the two right hand sides yields

$$\beta_t(\alpha_t) = 2Lc(t)\alpha_t + L(\mathbf{1} - c(t))v(t), \tag{6.10}$$

where

$$c(t) = \frac{\Sigma(t) - \mathbf{1}}{\Sigma(t) + \mathbf{1}} = th\frac{t}{2}M \tag{6.11}$$

is the Cayley transform of $\Sigma(t)$. Because $c(t) \in \mathrm{sp}\,(E)$, Lc is symmetric and $\beta_t(\alpha_t)$ is the gradient

$$\beta_t(\alpha_t) = \nabla g_t(\alpha_t) \tag{6.12}$$

of the generating function

$$g_t(\alpha) = l(\alpha, c(t)\alpha) + l(\alpha, (\mathbf{1} - c(t))v(t)) + \gamma(t). \tag{6.13}$$

This function must be a solution of the standard Hamilton–Jacobi equation (4.10). In order to fix the arbitrary function $\gamma(t)$ we require that g_t be solution of (4.10) with introducing g_t into this equation and using the key relation

$$2L\dot{c} = (\mathbf{1} + \tilde{c})\Omega(\mathbf{1} + \tilde{c})$$

one obtains

$$g_t(\alpha) = l(\alpha - \tfrac{1}{2}v(t), c(t)(\alpha - \tfrac{1}{2}v(t))) + l(\alpha, v(t)) + \tfrac{1}{2}l(a, w(t)) + h_0 t \tag{6.14}$$

where $v(t)$ is the vector defined in (6.8) and

$$w(t) = \int_0^t dt'\Sigma^{-1}(t')v(t') = \int_0^t dt'\int_0^{t'} dt''\Sigma^{-1}(t'')\alpha. \tag{6.15}$$

Critical points b of h,

$$\nabla h(b) = 0 = L(Mb + a) \tag{6.16}$$

are fixed points of Φ_t for any time and, consequently, critical points of g_t ($\beta_t = \nabla g_t = 0$ if $\Phi(b) = b$). There exist three possible cases:
(i) h has no critical point.
This happens when $\det M = 0$ and a has non-vanishing components in the sub-space cancelled by M.
(ii) h has a unique critical point.
This is the case whenever $\det M \neq 0$; $b = -M^{-1}a$. Using b to parametrize the Hamiltonian,

$$h(x) = \tfrac{1}{2}l(x-b, M(x-b)) + h_b, \qquad h_b = h_0 - \tfrac{1}{2}l(b, Mb), \tag{6.17}$$

one obtains the simple form for g_t (6.14)

$$g_t(\alpha) = l(\alpha - b, c(t)(\alpha - b)) + h_b t. \tag{6.18}$$

(iii) h has a continuous manifold of critical points.
In this case, the vector a must be of the form $a = Mz$ and M must be non-regular; $\det M = 0$. The critical manifold is the set of points

$$b = -z + y, \text{ with } y \text{ such that } My = 0. \tag{6.19}$$

The forms (6.17) for h and (6.18) for g_t are applicable and do not depend on the choice of y.
The graphs $\mathcal{V}_t$ of Φ_t, $t \in \mathbf{R}$, are linear manifolds of $\mathscr{E}$. The sets S'_t (2.55) reduce to the single points

$$\frac{\partial x_t}{\partial x} = \Sigma(t) \tag{6.20}$$

and

$$S'' = \{\Sigma(t) \mid t \in \mathbf{R}\}. \tag{6.21}$$

If M (6.2) has a purely imaginary eigenvalue m_μ, S'' is not homotopic to zero and a universal $\mathcal{M}^c$ does not exist for all t's. With the present choice (Σ_0 (2.57) equals to $-\mathbf{1}$), this fact is apparent in (6.11). For $t_1 = ((2k+1)/m_\mu)\pi$, $k \in Z_1$, $\Sigma(t_1)$ has an eigenvalue equal to -1 and $c(t_1)$ is no longer defined.

(b) *Canonical maps leaving the configuration space invariant*

In a given canonical chart of E in which the matrix L is given by (6.1), we distinguish two sub-sets of coordinates $x = (q, p)$ denoting q a position and p a momentum. The manifolds

$$\begin{aligned} \mathcal{M}_{q_0} &= \{(q, p) \mid q = q_0 \text{ fixed}\} \\ \mathcal{M}_{p_0} &= \{(q, p) \mid p = p_0 \text{ fixed}\} \end{aligned} \tag{6.22}$$

are complementary Lagrangian sub-manifolds of E. The equivalence class

$$E_q = E/\mathcal{M}_{q_0=0}$$

is the usual configuration space and

$$E_p = E/\mathcal{M}_{p_0=0}$$

the momentum space.

Writing $\alpha-(\xi,\eta)$, the relation (3.3) reads more explicitly

$$q=\xi-\frac{1}{2}\frac{\partial g}{\partial\eta}\qquad p=\eta+\frac{1}{2}\frac{\partial g}{\partial\xi}$$
$$\bar q=\xi+\frac{1}{2}\frac{\partial g}{\partial\eta}\qquad \bar p=\eta-\frac{1}{2}\frac{\partial g}{\partial\xi}\tag{6.23}$$

or

$$\tfrac{1}{2}(\bar q+q)=\xi,\qquad \tfrac{1}{2}(\bar p+p)=\eta\tag{6.24}$$

$$\bar q-q=\frac{\partial g}{\partial q}(\xi,\eta),\qquad \bar p-p=-\frac{\partial g}{\partial\xi}(\xi,\eta).\tag{6.25}$$

There are two (local) subgroups of canonical maps, ϕ^q and ϕ^p, which map E_q and E_p into themselves respectively. The invariance condition is clearly

$$\bar q=\text{function of } q \text{ only}\tag{6.26}$$

in the first case, and

$$\bar p=\text{function of } p \text{ only}\tag{6.27}$$

in the second one. The canonical map $q\mapsto p$, $p\mapsto -q$ connects the two cases; we need only discuss the first one, i.e. canonical maps of E induced by arbitrary maps of E_q. It is clear from (6.25) that (6.26) holds if and only if $(\partial g/\partial\eta)(\xi,\eta)$ no longer depends on η. Locally, the more general generating function of a ϕ^q is thus

$$g(\xi,\eta)=f(\xi)+\eta\cdot b(\xi),\tag{6.28}$$

where f is an arbitrary C^2 function and b any C^2 vector field on E_q. The relations (6.25) then read

$$\bar q-q=b\left(\frac{\bar q+q}{2}\right),\tag{6.29}$$

defining $\bar q(q)$ implicity, and

$$\bar p(q,p)=-(\mathbf{1}+\tilde\rho)^{-1}\nabla f+\frac{\mathbf{1}-\tilde\rho}{\mathbf{1}+\tilde\rho}p=-\tfrac{1}{2}(\mathbf{1}+\tilde R^{-1})\nabla f+\tilde R^{-1}p.\tag{6.30}$$

Here, the function f, the matrix ρ

$$(\rho_{lk})=\frac{1}{2}\left(\frac{\partial b^l}{\partial q^u}\right)=\frac{R-\mathbf{1}}{R+\mathbf{1}}\tag{6.31}$$

and the Jacobi matrix

$$R=\left(\frac{\partial\bar q}{\partial q}\right)=\frac{\mathbf{1}+\rho}{\mathbf{1}-\rho}\tag{6.32}$$

only depend on q through the variable $\xi=\tfrac{1}{2}(\bar q(q)+q)$.

The composition (5.24–25) of two functions g_i, $i=1,2$, of type (6.28) yields a function of the same type since the product also fulfills (6.26). Explicitly

$$\begin{aligned} g_3(\xi_3, \eta_3) &= g_1(\xi_1, \eta_1) + g_2(\xi_2, \eta_2) \\ &\quad + 2[(\xi_1-\xi_3)\cdot(\eta_2-\eta_3) - (\eta_1-\eta_3)\cdot(\xi_2-\xi_3)] \\ &= f_1(\xi_1) + f_2(\xi_2) + [b_1 - 2(\xi_2-\xi_3)]\cdot \eta_1 \\ &\quad + [b_2 - 2(\xi_1-\xi_3)]\cdot \eta_2 + 2(\xi_2-\xi_1)\cdot \eta_3, \end{aligned}$$

where $\eta_i(\xi_3, \eta_3)$, $\xi_i(\xi_3)$, $i=1,2$, are solution of (5.25). This last equation reads for ξ_i

$$\tfrac{1}{2}b_1(\xi_1) - \xi_2 + \xi_3 = \tfrac{1}{2}b_2(\xi_2) + \xi_1 - \xi_3 = 0. \tag{6.33}$$

The factors of η_i in g_3 are thus zero and one finally has

$$\begin{aligned} g_3(\xi_3, \eta_3) &= f_3(\xi_3) + \eta_3 \cdot b_3(\xi_3) \\ f_3(\xi_3) &= f_1(\xi_1(\xi_3)) + f_2(\xi_2(\xi_3)) \\ b_3(\xi_3) &= b_1(\xi_1(\xi_3)) + b_2(\xi_2(\xi_3)), \end{aligned} \tag{6.34}$$

where $\xi_i(\xi_3)$ satisfy

$$\tfrac{1}{2}b_2(\xi_2) + \xi_1 = -\tfrac{1}{2}b_1(\xi_1) + \xi_2 = \xi_3. \tag{6.35}$$

The generator h of a one-parameter sub-group $\{\phi_t^q \mid t \in \mathbf{R}\}$ has the same form (6.28) as g_t by virtue of (4.12),

$$h(q,p) = \partial_t f_t(q) + p\cdot \partial_t b_t(q)|_{t=0} = h_0(q) + p\cdot v(q). \tag{6.36}$$

To illustrate the possibly local character of the theory, we consider a specific canonical map Φ_q which is not automorphic. For one degree of freedom ($n=1$), the Hamiltonian

$$h(q,p) = q^2 p \tag{6.37}$$

generates the map

$$\Phi_t^q: \begin{aligned} q &\mapsto \bar{q}_t(q) = \frac{q}{1-qt}, \\ p &\mapsto \bar{p}_t(q,p) = (1-qt)^2 p. \end{aligned} \tag{6.38}$$

For fixed $t>0$ this map is well defined in the domains

$$\begin{aligned} &\mathcal{D}_t: p\in\mathbf{R}, \qquad q<\frac{1}{t} \\ &\mathcal{D}_t^c: p\in\mathbf{R}, \qquad q>\frac{1}{t}. \end{aligned} \tag{6.39}$$

The graph $\mathcal{V}_t$ of Φ_t^q goes to $\pm\infty$ as $q\to 1/t\mp 0$. With respect to the axis $\xi = \frac{1}{2}(\bar{q}+q)$, $\mathcal{V}_t$ has one branch above and one below (Fig. 4). This implies that two generating functions, one for each branch, are needed. Indeed, writing (6.24–25)

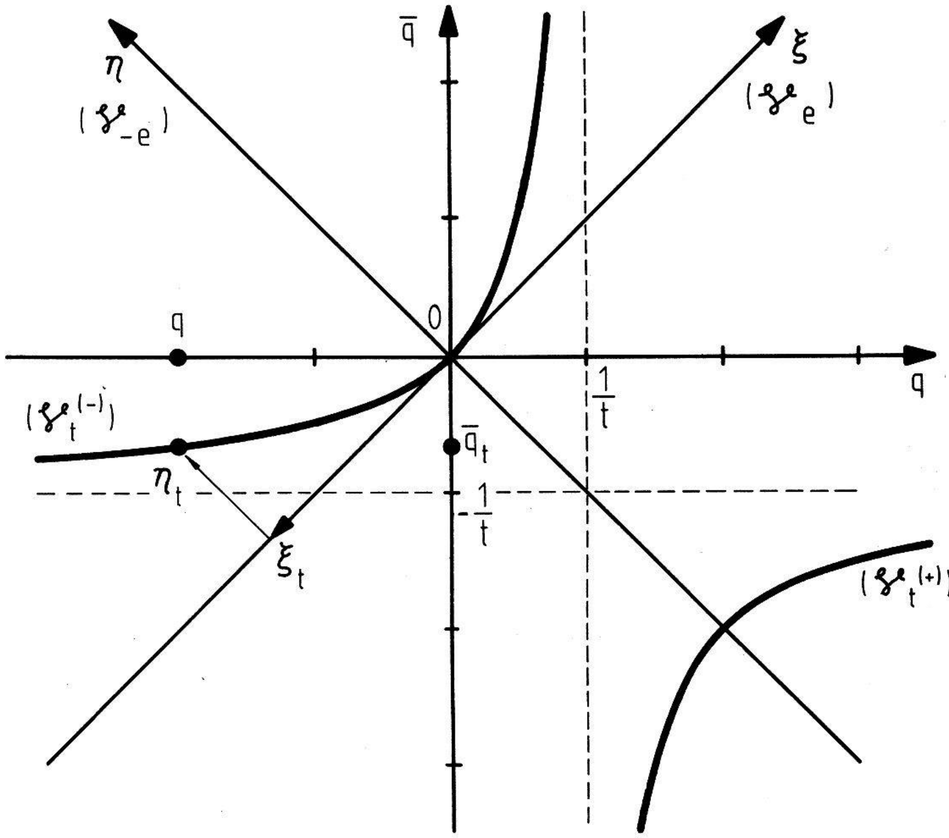

Figure 4
Section at $p = \bar{p} = 0$ of the graph $\mathscr{V}_t$ of Φ_t^q, at a time $t > 0$. $\mathscr{V}_t$ has two branches $\mathscr{V}_t^{(\pm)}$. The ξ-axis is the section of the reference manifold $\mathscr{V}_e$, and the η-axis that of $\mathscr{M}^c = \mathscr{V}_{-e}$.

and solving for g_t, one finds

$$g_t^{(\pm)}(\xi, \eta) = \eta f_t^{(\pm)}(\xi) \tag{6.40}$$

$$f_t^{(\pm)}(\xi) = \frac{2}{t} \pm \frac{2}{t}\sqrt{1 + (t\xi)^2} \underset{t \to 0}{\simeq} \begin{cases} 4/t \\ -\xi^2 \cdot t. \end{cases} \tag{6.41}$$

Both $g_t^{(\pm)}$ are solutions of the standard Hamilton–Jacobi equation (4.10); $g_t^{(-)}$ satisfies the initial condition $g_0^{(-)} = 0$, but $g_t^{(+)}$ becomes singular at $t = 0$. $g_t^{(-)}$ generates Φ_t^q in the domain $\mathscr{D}_t$ which covers E completely at $t = 0$, and $g_t^{(+)}$ generates Φ_t^q in $\mathscr{D}_t^c$ which vanishes at $t = 0$. The standard picture never breaks down in this case because $\mathscr{M}^c = \mathscr{V}_{-e} = \{(x, \bar{x}) \mid \bar{q} = -q, \bar{p} = -p\}$ is, for each t, everywhere transverse to both branches of $\mathscr{V}_t$.

(c) *Examples of products* ⊤

The product ⊤ is in general not unique. The rule (5.24–25) yields as many results as the number of critical values of Ψ, α fixed. But in the following cases, the product is well defined and associative even if the critical points are not unique. (Pig through critical variety, for example [6].)

For λ, λ' real constants and $g(x)$ a real function on E,

$$(\lambda \top g)(x) = \lambda + g(x) \tag{6.42}$$

$$(\lambda p \top g)(x) = \lambda p + g(q - \tfrac{1}{2}\lambda, p) \tag{6.43}$$

$$(g \top \lambda p)(x) = \lambda p + g(q + \tfrac{1}{2}\lambda, p) \tag{6.44}$$

$$(\lambda q \top g)(x) = \lambda q + g(q, p + \tfrac{1}{2}\lambda) \tag{6.45}$$

$$(g \top \lambda q)(x) = \lambda q + g(q, p - \tfrac{1}{2}\lambda) \tag{6.46}$$

$$\lambda p \top \lambda' p = (\lambda + \lambda')p \tag{6.47}$$

$$\lambda q \top \lambda' q = (\lambda + \lambda')q \tag{6.48}$$

$$\lambda p \top \lambda' q = \lambda p + \lambda' q - \tfrac{1}{2}\lambda\lambda' \tag{6.49}$$

$$\lambda' q \top \lambda p = \lambda' q + \lambda p + \tfrac{1}{2}\lambda\lambda' \tag{6.50}$$

$$\lambda p \top \lambda' q \top \lambda'' p = \lambda p + \lambda' q + \lambda'' p + \tfrac{1}{2}\lambda'(\lambda'' - \lambda). \tag{6.51}$$

For differentiable functions, infinitesimal transformations are also well defined.

$$(\varepsilon h \top g)(x) = g(x) + \varepsilon h(x - \tfrac{1}{2}\Lambda\nabla g(x)) + O(\varepsilon^2) \tag{6.52}$$

$$(g \top \varepsilon h)(x) = g(x) + \varepsilon h(x + \tfrac{1}{2}\Lambda\nabla g(x)) + O(\varepsilon^2) \tag{6.53}$$

$$(\varepsilon h \top g \top (-\varepsilon h)(x) = g(x) + \varepsilon[h(x - \tfrac{1}{2}\Lambda\nabla g(x)) - h(x + \tfrac{1}{2}\Lambda\nabla g(x))] + O(\varepsilon^2) \tag{6.54}$$

$$\varepsilon g \top \varepsilon' h \top (-\varepsilon g) \top (-\varepsilon' h) = -\varepsilon\varepsilon'\{g, h\} + O(\varepsilon\varepsilon'^2) + O(\varepsilon'\varepsilon^2). \tag{6.55}$$

The Poisson-bracket is the Lie-product for the composition law of 'infinitesimal' functions.

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