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Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **54 (1981)**

Heft 3

PDF erstellt am: **23.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-115219

Helvetica Physica Acta, Vol. 54 (1981), Birkhäuser Verlag, Basel

The vacuum structure of the Schwinger model and its external field problem[1])

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(25. VI. 1981)

Abstract. An exact treatment of the effects of external charges localized in a finite region of two dimensional space-time is used to get a better understanding of massless QED_2. Among these effects we find: creation and absorption of massive bosons, complete shielding of static external charges and cancellation of constant external electic fields. Furthermore, if the external charges divide space-time into domains, one has a distinct θ-vacuum in each of them. The difference between the values of θ in two neighbouring domains is related to the strength of the external charge localized in their common boundary.

1. Introduction

In this note we continue the study of massless quantum electrodynamics in two space-time dimensions (QED_2) in the covariant Landau gauge that was begun in [1]. The analysis carried out in [1] emphasized the role of the gauge symmetry in determining the remarkable vacuum structure of the model. The well known disappearance of the fermions from the observable spectrum was seen to be a consequence of the vacuum structure and consequently of the gauge symmetry. We saw that this vacuum structure is described by a family of sectors labelled by an angle θ. The gauge symmetry of the model implies that these sectors are superselection sectors.

In the present work we study the dynamical origin of this special vacuum or superselection structure, our method being to introduce an external classical current J_μ and to study the various responses it induces. This has been the subject of the well known work of Casher et al. [2], but, since they were mainly interested in studying the quark-parton model of electroproduction, there is not much overlap. Our main tool is a straightforward generalisation of the Lowenstein–Swieca solution of the 'free' Schwinger model [3] (we shall frequently use the work 'free' to mean $J_\mu = 0$) to this case. An important feature of this extension is that it does not affect the θ-structure of the model. This will become clear in Section 2 where our solution is described in detail.

In Section 3 we consider the transitions between asymptotic states induced by an external current of zero total charge and with finite space-time support. We

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find that such a current produces and absorbs Σ-particles, the massive, pseudoscalar 'photons' of the theory. It is amusing to note that while this resembles classical radiation, in fact radiation is not classically possible in one space dimension.

In Sections 4 and 5 we study the response to the external current J_μ in the space-time region where J_μ is nonvanishing. We take J_μ to be the current due to two charges $+\alpha e$ and $-\alpha e$ which are separated initially and then brought together again after a finite time. In Section 4 we show that the external charges are completely shielded by the polarization-induced charges even if α is not an integer. This implies that the constant external electric field between the external charges is always exactly cancelled [4, 5, 6]. It follows that the only electric field one can observe macroscopically is the field carried by the Σ-particles. Since a constant external field does not produce such particles, it has no observable effects which can be detected by an electric field measurement.

This does not mean that a constant external field has no observable effects at all. If we consider the properties of our system in suitable domains Δ, localized either outside or inside the space-time loop Γ defined by the external charges, we find a correspondence between its pure states Ψ and observables $\mathcal{O}$ and the states and observables Ψ_{free}, $\mathcal{O}_{\text{free}}$ of the 'free' Schwinger model. For each Ψ there exists a $\Psi^\Delta_{\text{free}}$ such that:

$$(\Psi, \mathcal{O}^\Delta\Psi) = (\Psi^\Delta_{\text{free}}, \mathcal{O}^\Delta_{\text{free}}\Psi^\Delta_{\text{free}})$$

where the operators $\mathcal{O}^\Delta$ and $\mathcal{O}^\Delta_{\text{free}}$ represent the same observable localized in Δ, resp. with nonvanishing and vanishing external current. At fixed Ψ, the Δ-dependence of $\Psi^\Delta_{\text{free}}$ is such that if $\Psi^\Delta_{\text{free}}$ belongs to the sector θ when Δ is outside the loop Γ, $\Psi^\Delta_{\text{free}}$ will be in the sector $(\theta+\alpha\pi)$ when Δ is inside Γ. Consequently the θ-sensitive chirality carrying observables have different values outside and inside the loop and reveal the presence of the constant external electric field of strength $-\alpha e$ inside the loop. There is a transmutation of this field into a chiral angle $\alpha\pi$ which establishes a connexion between the fractional part of the external charges and the angle labelling the superselection sectors. Pairs of external charges induce a domain structure of space time defined by the coexistance of distinct sectors.

2. The solution of the equations of motion

In the covariant Landau gauge ($\partial_\mu A^\mu = 0$), the model is defined by the following coupled Dirac and Maxwell equations[4]):

$$\gamma^\mu\,\partial_\mu\psi(x) = \frac{ie}{2}\lim_{\varepsilon\to 0}\,[\gamma^\mu A_\mu(x+\varepsilon)\psi(x) + \gamma^\mu\psi(x)A_\mu(x-\varepsilon)], \tag{2.1}$$

$$\Box A_\mu(x) = -e(j_\mu(x) + J_\mu(x)). \tag{2.2}$$

In (2.2), $j_\mu(x)$ is the quantized current determined by means of a gauge invariant point splitting of $\bar\psi\gamma_\mu\psi$ [3] and $J_\mu(x)$ is a given c-number external current.

[4]) Our notations are conventional:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1, \quad \varepsilon_{01} = \varepsilon^{10} = 1.$$

The solution of the above equations differs only minimally from the covariant Lowenstein–Swieca solution [3] of the Schwinger model ($J_\mu = 0$) and can be expressed in terms of the latter, which we denote by ($\psi^{\text{L.S.}}$, $A_\mu^{\text{L.S.}}$):

$$\begin{aligned} \psi(x) &= \exp(i\gamma^5\phi(x))\psi^{\text{L.S.}}(x), \\ A_\mu(x) &= -(1/e)\varepsilon_{\mu\nu}\,\partial^\nu\phi(x) + A_\mu^{\text{L.S.}}(x), \end{aligned} \tag{2.3}$$

$\phi(x)$ is a c-number solution of an inhomogeneous Klein–Gordon equation:

$$(\Box + m^2)\phi(x) = eE_{\text{ext}}(x), \tag{2.4}$$

where $m^2 = (e^2/\pi)$ and E_{ext} is the electric field generated by the external current according to the Maxwell equations:

$$\varepsilon_{\mu\nu}\,\partial^\nu E_{\text{ext}}(x) = eJ_\mu(x). \tag{2.5}$$

A general expression of E_{ext} is:

$$E_{\text{ext}}(x) = -e\int_{-\infty}^{x^1} dy^1 J_0(x^0, y^1) + E_0. \tag{2.6}$$

Coleman [5] has pointed out that the constant E_0 appearing in (2.6) needs not be zero in one space dimension and can be associated with ‘charges at infinity’. We will nevertheless put $E_0 = 0$ for the present and later show in Section 4 that a nonvanishing E_0 does not change the physical content of our results.

We recall that [3]:

$$\begin{aligned} \psi^{\text{L.S.}}(x) &= :\exp[i\sqrt{\pi}\gamma^5(\tilde{\Sigma}(x) + \tilde{\eta}(x))]:\psi_0(x) \\ A_\mu^{\text{L.S.}}(x) &= -(\sqrt{\pi}/e)\varepsilon_{\mu\nu}\,\partial^\nu(\tilde{\Sigma}(x) + \tilde{\eta}(x)) \end{aligned} \tag{2.7}$$

where $\psi_0(x)$ and $\tilde{\Sigma}(x)$ are, respectively, a canonical massless Dirac field and a canonical massive boson field; $\tilde{\eta}(x)$ is a massless scalar field with an indefinite metric:

$$\gamma^\mu\,\partial_\mu\psi_0(x) = 0, \qquad (\Box + m^2)\tilde{\Sigma}(x) = 0, \qquad \Box\tilde{\eta}(x) = 0. \tag{2.8}$$

While (2.1) is satisfied identically by the solution (2.3), the Maxwell equation (2.2) only holds on the subspace of states satisfying the subsidiary condition:

$$j_{l,\mu}^-(x)\Psi \equiv (j_{0,\mu}(x) - (1/\sqrt{\pi})\varepsilon_{\mu\nu}\,\partial^\nu\tilde{\eta}(x))^-\Psi, \tag{2.9}$$

where $j_{l,\mu}^-$ denotes the negative frequency part of the longitudinal current $j_{l,\mu}$ and $j_{0,\mu}$ is the current associated with ψ_0. An important feature of condition (2.9) is that it does not depend on the external current J_μ. Accordingly the analysis of [1] can be taken over to this case.

In the subspace of states fulfilling the condition (2.9), the quantized current reduces to

$$j_\mu(x) = -(1/\sqrt{\pi})\varepsilon_{\mu\nu}\,\partial^\nu(\tilde{\Sigma}(x) + (1/\sqrt{\pi})\phi(x)) \tag{2.10}$$

With $E_0 = 0$, the total electric field is:

$$E(x) \equiv (\partial_0 A_1 - \partial_1 A_0)(x) = E_{\text{ext}}(x) - (e/\sqrt{\pi})(\tilde{\Sigma}(x) + (1/\sqrt{\pi})\phi(x)) \tag{2.11}$$

In the following we will always assume that the external current vanishes outside a bounded space-time region $[0, T]\times[-L, L]$, i.e.:

$$J_\mu(x)=0 \quad \text{if either} \quad x^0<0, \qquad x^0>T \quad \text{or} \quad |x^1|>L. \tag{2.12}$$

Then by conservation of the total external charge, this latter is at all times zero:

$$\int_{-\infty}^{+\infty} dx^1 J_0(x^0, x^1)=0 \tag{2.13}$$

As a result of (2.6) we see that E_{ext} also vanishes outside the domain $[0, T]\times[-L, +L]$.

In order to define $\phi(x)$ uniquely we choose the retarded solution of (2.4):

$$\phi(x)=\phi_{\text{ret}}(x)\equiv e\int(dy)^2 D_{\text{ret}}(x-y)\,E_{\text{ext}}(y). \tag{2.14}$$

Here $D_{\text{ret}}(x)$ is the retarded Green's function of the Klein–Gordon operator. Thus $\phi_{\text{ret}}(x)=0$ for $x^0<0$ and (ψ, A_μ) reduces to $(\psi^{\text{L.S.}}, A_\mu^{\text{L.S.}})$ for negative times. With this choice the solution (2.3) of the field equations (2.1), (2.2) reduces to the Lowenstein–Swieca fields for negative times, the latter thereby playing the role of incoming fields. We indicate this explicitly by replacing the lable 'L.S.' by 'in' in equations (2.3) (also $\tilde{\Sigma}$ by $\tilde{\Sigma}_{\text{in}}$ (2.7)) which become:

$$\begin{aligned} \psi(x)&=\exp\,(i\gamma^5\phi_{\text{ret}}(x))\psi^{\text{in}}(x),\\ A_\mu(x)&=-(1/e)\varepsilon_{\mu\nu}\,\partial^\nu\phi_{\text{ret}}(x)+A_\mu^{\text{in}}(x). \end{aligned} \tag{2.15}$$

The significance of $\phi_{\text{ret}}(x)$ is seen by computing the expectation value of the total electric field $E(x)$ in the incoming vacuum, the state in which there are no incoming Σ-particles; (2.11) gives:

$$\langle E(x)\rangle_{\text{in}}=E_{\text{ext}}(x)-(e/\pi)\phi_{\text{ret}}(x). \tag{2.16}$$

This shows that $\phi_{\text{ret}}(x)$ measures the polarization produced by the external current. Our later discussion makes this observation more precise.

3. Transitions between incoming and outgoing configurations

In this short Section we consider the transition from an ingoing configuration, observed at negative times, to an outgoing configuration detected after the external current has been switched off ($x^0>T$). We show that the external current absorbs incoming Σ-particles and emits outgoing ones. These processes are described by a rotation in the Fock space of the Σ-particles.

For $x^0>T$, $\phi_{\text{ret}}(x)$ satisfies the free Klein–Gordon equation. Let $\phi_{\text{out}}(x)$ be the solution of this homogeneous equation which coincides with $\phi_{\text{ret}}(x)$ for $x^0>T$:

$$(\Box+m^2)\phi_{\text{out}}(x)=0 \quad \forall x, \qquad \phi_{\text{ret}}(x)=\phi_{\text{out}}(x) \quad \text{for} \quad x^0>T. \tag{3.1}$$

Since ϕ_{out} satisfies the same equation as the Σ-field, we can define an outgoing field:

$$\tilde{\Sigma}_{\text{out}}(x) = \tilde{\Sigma}_{\text{in}}(x) + (1/\sqrt{\pi})\phi_{\text{out}}(x) \tag{3.2}$$

and write

$$(\psi(x), A_\mu(x)) = (\psi^{\substack{\text{in}\\ \text{out}}}(x), A_\mu^{\substack{\text{in}\\ \text{out}}}(x)) \quad \text{for} \quad \begin{cases} x^0 < 0 \\ x^0 > T \end{cases}, \tag{3.3}$$

where:

$$\begin{aligned} \psi^{\substack{\text{in}\\ \text{out}}}(x) &= :\exp\left[i\sqrt{\pi}\gamma^5(\tilde{\Sigma}_{\text{in}}\ (x) + \tilde{\eta}(x))\right]: \psi_0(x), \\ A_\mu^{\substack{\text{in}\\ \text{out}}}(x) &= -(\sqrt{\pi}/e)\varepsilon_{\mu\nu}\,\partial^\nu(\tilde{\Sigma}_{\substack{\text{in}\\ \text{out}}}\ (x) + \tilde{\eta}(x)). \end{aligned} \tag{3.3}$$

The incoming and outgoing Σ-fields differ by a square integrable function and consequently are related by a unitary transformation U:

$$\tilde{\Sigma}_{\text{out}}(x) = U^\dagger \tilde{\Sigma}_{\text{in}}(x) U. \tag{3.5}$$

The operator U is nothing but the S-matrix of the transitions induced by the external current. It is easily seen that it has the form:

$$U = \exp\left[(i/\sqrt{\pi}) \int dx^1 \phi_{\text{out}}(x)\, \overleftrightarrow{\partial}_0 \tilde{\Sigma}_{\text{out}}(x)\right]. \tag{3.6}$$

Equations (3.5) and (3.6) show that the incoming vacuum Ω_{in}, the state with no incoming Σ-particles, is a coherent superposition of outgoing Σ-particles, since $\Omega_{\text{in}} = U\Omega_{\text{out}}$.

The interaction lagrangian describing the coupling of an external current J_μ with a quantized electromagnetic vector potential A_μ being equal to $-eJ^\mu A_\mu$, one expects the in-out transitions produced by the external current to be described by [7]:

$$U = \exp\left[-ie \int (dx)^2 J^\mu(x) A_\mu(x)\right]. \tag{3.7}$$

It is quite easy to verify that (3.6) and (3.7) are indeed equivalent, up to a c-number phase factor. One has to use the relations (2.5) and (2.7) between the Σ-field and A_μ as well as the connection of ϕ_{out} with J_μ.

It is amusing to note that we have here an example of radiation in one space dimension, though with massive 'photons', by an external current. This is a purely quantum phenomenon with no classical counterpart. Classically, the electric field produced by the external current would be trapped inside the bounded region $[0, T]\times[-L, +L]$. In fact, we have a nonvanishing electric field in the causal shadow of the external current as well as a nonvanishing current. For $x^0 > T$; it follows from (2.15) and (2.9) that:

$$\begin{aligned} \langle E(x)\rangle_{\text{in}} &= -(e/\pi)\phi_{\text{out}}(x), \\ \langle j_\mu(x)\rangle &= -(1/\pi)\varepsilon_{\mu\nu}\,\partial^\nu\phi_{\text{out}}(x). \end{aligned} \tag{3.8}$$

Looking at the results of this Section from the point of view of the state space of the Schwinger model [1], we observe that the effects we have considered exhibit no θ-dependence. We have merely a rotation in the Σ-Fock space. We consequently turn our attention to the region inside the support of the external current.

4. Charge screening and electric field cancellation

We saw in Section 3 that the transitions from free ingoing configurations to free outgoing ones is described by the unitary transformation (3.6), 'free' meaning $J_\mu = 0$. In this Section and the next one we study the physical properties of a state in the space-time region where the current J_μ and the electric field E_{ext} are not zero. The external current we consider is obtained, as shown in Fig. 1, by taking two equal but opposite charges at the origin $x = (0, 0)$, separating them by a macroscopic distance for a macroscopic time and bringing them together again at $x = (T, 0)$ (macroscopic = large compared to $(1/m)$). We thus have a large space-time loop Γ defined by the external charges.

In this Section the charges are kept fixed at $x^1 = \pm L$ during the large time interval $[T_1, T_2]$. This enables us to study the charge shielding mechanism in terms of the static response to the external charges. Over the interval $[T_1, T_2]$ the

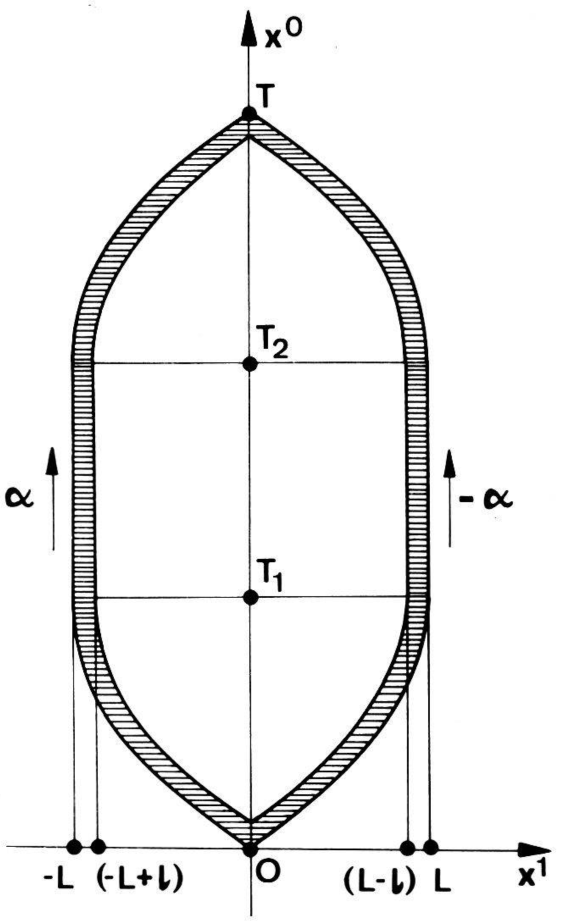

Figure 1
Two extended external charges of strength α and $-\alpha$ (in units of e) and width l are separated at $x = (0, 0)$, kept at rest at $\mp L$ during the time interval (T_1, T_2) and brought together again at $x = (0, T)$. This set-up leads to a simple static response to the external charges.

current J_μ describes a static charge distribution:

$$J_\mu(x) = \delta_{\mu,0}\rho(x^1), \qquad \rho(x^1) = \rho^+(x^1) + \rho^-(x^1), \tag{4.1}$$

where $\rho^\pm(x^1)$ are the smooth densities of the positive and negative charges localized at $x^1 = -L$ and $x^1 = +L$. Specifically, $\rho^+(x^1)$ and $\rho^-(x^1)$ are nonvanishing only inside the intervals $[-L, -L+l]$ and $[L-l, L]$ respectively ($L \gg l \gg (1/m)$). The total charge of each distribution is $\int dx^1 \rho^\pm(x^1) = \pm\alpha$ in units of e.

The general solution written down in (2.15) is, of course, valid for $x^0 \in [0, T]$. However, $\phi_{\text{ret}}(x)$ does not satisfy the homogeneous Klein–Gordon equation as in Section 3 but the inhomogeneous equation (2.4). We show in the Appendix that $\phi_{\text{ret}}(x)$ can be decomposed uniquely into a static part $\phi_{\text{stat}}(x^1)$ and a solution $\phi_I(x)$ of the homogeneous Klein–Gordon equation:

$$\phi_{\text{ret}}(x) = \phi_{\text{stat}}(x) + \phi_I(x) \quad \text{for} \quad x^0 \in [T_1, T_2], \tag{4.2}$$

where

$$\phi_{\text{stat}}(x^1) = (\sqrt{\pi}/2)\int_{-\infty}^{+\infty} dy^1 \exp(-m\,|x^1 - y^1|)E_{\text{ext}}(y^1) \tag{4.3}$$

$$\phi_I(x) = -(e^2/2)\int_{-\infty}^{+\infty} \frac{dk}{\omega_k^2}[\tilde{J}^1(\omega_k, k)\exp[-i(\omega_k x^0 - kx^1)] + \text{c.c.}] \tag{4.4}$$

Here $\tilde{J}^1(k^0, k^1)$ is the Fourier transform of $\theta(T_1 - x^0)J^1(x)(\omega_k = (k^2+m^2)^{1/2})$. We have written $E_{\text{ext}}(y)$ as a function of y^1 only in (4.3) since it is static for $y^0 \in [T_1, T_2]$.

We use the decomposition (4.2) to define a new Σ-field in the interval $[T_1, T_2]$:

$$\tilde{\Sigma}_I(x) = \tilde{\Sigma}_{\text{in}}(x) + (1/\sqrt{\pi})\phi_I(x). \tag{4.5}$$

This field is unitarily related to the in-field as in Section 3. Thus the non-static part of ϕ_{ret} is associated with absorption and production of Σ-particles. We are mainly interested in the static part in order to isolate the charge shielding produced by the polarization of the vacuum.

Using (2.10), (4.1) and (4.5) we see that the total current is given by:

$$j_\mu^{\text{tot}}(x) = J_\mu(x) + j_\mu(x) = \delta_{\mu,0}[\rho(x^1) + (1/\pi)\,\partial_1\phi_{\text{stat}}(x^1)] \\ -(1/\sqrt{\pi})\varepsilon_{\mu\nu}\,\partial^\nu\tilde{\Sigma}_I(x) \tag{4.6}$$

Our interest lies in the c-number static part of (4.6) Inserting (4.1), (4.3) and recalling (2.6), this term becomes:

$$j_\mu^{\text{stat}}(x^1) = \delta_{\mu,0}[\rho^+(x^1) + \rho^+_{\text{pol}}(x^1) + \rho^-(x^1) + \rho^-_{\text{pol}}(x^1)], \tag{4.7}$$

where

$$\rho^\pm_{\text{pol}}(x^1) = -\tfrac{1}{2}m\int_{-\infty}^{+\infty} dy^1 \exp[-m\,|x^1 - y^1|]\rho^\pm(y^1) \tag{4.8}$$

are the static polarization charges induced by the external charges. It follows immediately from (4.8) that these polarization charges shield the external charges exactly. The charge density $\rho^+_{\text{pol}}(x^1)$ has an exponentially decreasing tail outside the interval $I_- = [-L, -L+l]$ and so the polarization charge induced in any larger interval $[-L-a, -L+l+a]$ cancels the external charge in I_- up to a term of order $\exp(-ma)$, i.e.:

$$\int_{-L-a}^{-L+l+a} dx^1 \rho_{\text{pol}}^+(x^1) = -\alpha + 0(e^{-ma}). \tag{4.9}$$

The shielding of the external charges implies that the total electric field (2.11) reduces to the sum of a residual static field and the field due to the Σ-particles;

$$E(x) = E_{\text{stat}}(x^1) + (e/\sqrt{\pi})\tilde{\Sigma}_I(x), \tag{4.10}$$

where:

$$E_{\text{stat}}(x^1) = \tfrac{1}{2}e \int_{-\infty}^{+\infty} dy^1 \varepsilon(x^1 - y^1) \exp[-m\,|x^1 - y^1|]\rho(y^1). \tag{4.11}$$

This field is negligibly small at macroscopic distances from the external charges.

Equations (4.9) and (4.11) display the charge shielding phenomenon of massless QED_2. They show that this shielding is complete for all values of the external charge strength α, as is well known [4, 5, 6]. We remark that the picture of charge shielding as arising from the creation of a definite number of electron-positron pairs would lead one to suppose that only the integer part of α was shielded and that a finite residual constant electric field survived.

We can now replace in (2.6) the constant background field E_0 that we had dropped earlier. Examining (2.14) we see that we should add a term $(\pi/e)E_0$ to the expression for $\phi_{\text{ret}}(x)$ we used until now. We see, however, that the total electric field as given in (2.11) remains unchanged, since it involves the difference $E_{\text{ext}}(x) - (e/\pi)\phi_{\text{ret}}(x)$. Consequently, equation (4.10) above is unchanged and thus polarization effects cancel a constant external background field as well as the field produced by the external charges $\pm\alpha$. We conclude that a nonvanishing E_0 cannot be detected by an electric field measurement.

5. External charges and vacuum structure

We saw in Section 3 that a time-varying external field of compact support resulted in the absorption and production of Σ-particles. In comparison, a constant external electric field E_0 introduces merely a constant phase-factor, $\exp(i\gamma^5(\pi/e)E_0)$ in the fermion field of the 'free' (i.e. $J_\mu = 0$) Schwinger model. Let us consider now a point inside the space-time loop of Section 4. In a neighbourhood Δ of this point lying wholly inside the loop Γ, we observe that we have both situations arising. In Δ we have a constant background external field E_Γ and, consequently, we should expect to be essentially considering the 'free' Schwinger model. We should, however, also see the Σ-particles produced and miss the particles absorbed on that part of the boundary of the space-time loop Γ which lies in the backward causal shadow of Δ. While the external electric field has the same value E_Γ at all points inside the loop, the Σ-particles detected in Δ will depend on the location and shape of this domain. We will show that ϕ_{ret} has a decomposition in Δ of the form $\phi_{\text{ret}}(x) = (\pi/e)E_\Gamma + \phi_\Delta(x)$ where $\phi_\Delta(x)$ satisfies the massive Klein–Gordon equation. We no longer require the special form of the

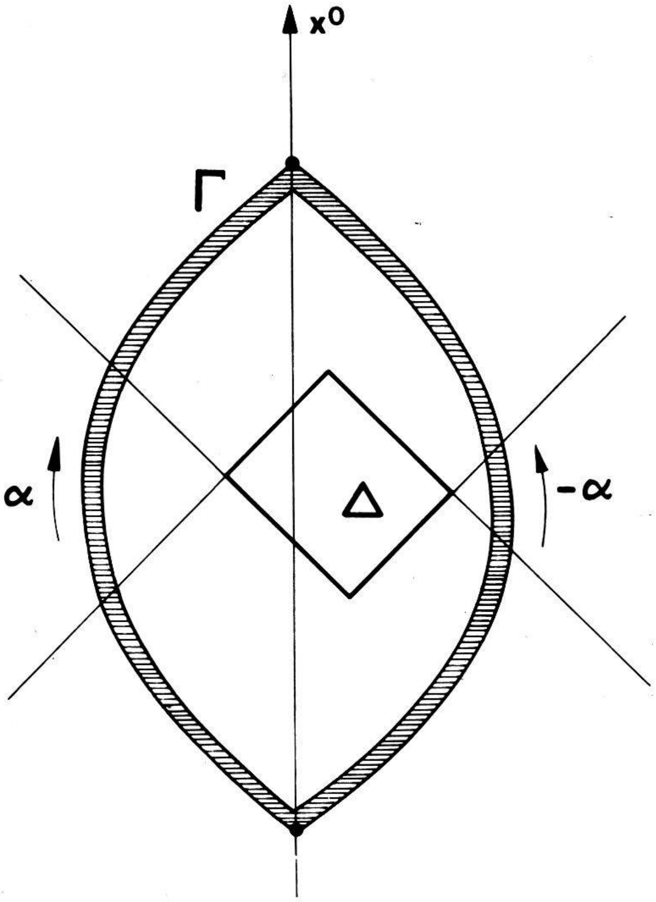

Figure 2
The external charges α and $-\alpha$ define a loop Γ in space and time. A state exhibits, inside and outside Γ, the properties of distinct states of the 'free' Schwinger model, belonging to distinct θ-sectors. This is obtained from an analysis of measurements performed in domains like Δ.

space-time loop of Section 4, which was selected to allow the extraction of a static response. In the calculations below we maintain the convention that the constant term E_0 in (2.6) is zero; consequently $E_\Gamma = -\alpha e$.

Figure 2 shows the space-time loop Γ and the domain Δ. We choose Δ to be a diamond-shaped region, the intersection of a forward and a backward cone, the diamond lying wholly inside the loop. We can rewrite (2.14) as:

$$\phi_{\text{ret}}(x) = (\pi/e)\left[E_{\text{ext}}(x) - \int (dy)^2 D_{\text{ret}}(x-y)(\Box E_{\text{ext}})(y)\right]. \tag{5.1}$$

The support of $\Box E_{\text{ext}} = \varepsilon_{\mu\nu}\,\partial^\mu J^\nu$ is the rim of the loop, shadowed in Fig. 2. If we restrict x to lie in Δ, the effective support of $\Box E_{\text{ext}}$ in (5.1) is the intersection of its support with the backward causal shadow of Δ. Therefore, we don't change $\phi_{\text{ret}}(x)$ if we replace $\Box E_{\text{ext}}$ by a different function outside this region. Define $\eta(x)$ such that:

(i) $\eta(x) = -(\pi/e)\ \Box E_{\text{ext}}(x)$ if x is in the backward causal shadow of Δ
(ii) $\eta(x) = 0$ if x is in the forward causal shadow

and $\eta(x)$ is smooth everywhere. This is possible since the regions (i) and (ii) are disjoint. Define for x in Δ:

$$\phi_\Delta(x)=(\pi/e)\int(dy)^2D(x-y)\eta(y) \tag{5.2}$$

where $D(x)=D_{\text{ret}}(x)-D_{\text{adv}}(x)$. From Fig. 2 it is clear that $\phi_\Delta(x)$ reduces to the second term in (5.1) for x in Δ. Moreover, for x in Δ we have $E_{\text{ext}}(x)=-e\alpha$ so that we can write (5.1) as:

$$\phi_{\text{ret}}(x)=-\alpha\pi+\phi_\Delta(x),\qquad x\in\Delta \tag{5.3}$$

We note that $\phi_\Delta(x)$ satisfies the same equation as $\tilde{\Sigma}_{\text{in}}(x)$, i.e. the free massive Klein–Gordon equation. This comes about since D is the difference of two Green's functions.

The representation (5.3) tells us that inside the domain Δ the solution (2.15) has the following form:

$$\psi(x)=\exp(-i\gamma^5\alpha\pi)\psi^\Delta(x),\qquad A_\mu(x)=A_\mu^\Delta(x), \tag{5.4}$$

where $(\psi^\Delta, A_\mu^\Delta)$ is the Lowenstein–Swieca type solution obtained from $(\psi^{\text{in}}, A_\mu^{\text{in}})$ through the substitution:

$$\tilde{\Sigma}_{\text{in}}\to\tilde{\Sigma}_\Delta=\tilde{\Sigma}_{\text{in}}+(1/\sqrt{\pi})\phi_\Delta. \tag{5.5}$$

The transformation (5.5) corresponds to a unitary transformation of the Fock space of the Σ-particles and is effected by a unitary operator U_Δ given by an expression analogous to (3.6).

The constant term in (5.3), however, produces in (5.4) a global chiral transformation of the Lowenstein–Swieca operator ψ^Δ. We will therefore briefly recall the status of the chiral transformations of the Schwinger model as discussed in [1]. We saw that the algebra of observables of the Schwinger model consists of functions of a chirality carrying unitary operator S tensored with the observables associated with $\tilde{\Sigma}(x)$. Consequently, irreducible representations of this algebra are labelled by an angle θ, the chiral angle θ_5 in the notation of [1] which takes the values $0\leqq\theta_5<\pi$. Each representation appears in a sector $\mathscr{H}(\theta)$ which is a copy of the Σ-field Fock space with unique vacuum $\Omega(\theta)$.

The fact that the sectors $\mathscr{H}(\theta)$ are the sectors of a superselection rule follows easily from the analysis of [1]. Consider a superposition $\Psi=c_1\Psi_1+c_2\Psi_2$ ($\Psi_1\in\mathscr{H}(\theta_1)$, $\Psi_2\in\mathscr{H}(\theta_2)$, $\theta_1\neq\theta_2(\text{mod }\pi)$). According to [1] we can find an operator U implementing a strong local gauge transformation such that:

$$U\Psi=c_1e^{i\theta_1}\Psi_1+c_2e^{i\theta_2}\Psi_2. \tag{5.6}$$

This shows that Ψ is not left invariant by all gauge transformations: consequently this vector cannot represent a physical pure state. This argument is, of course, analogous to the well-known argument for the univalence superselection rule [8].

Coming back to the external field problem, we remember that the state space is not affected by the presence of external charges. With the choice $\phi=\phi_{\text{ret}}$ the building blocks of the solution (2.15) are incoming fields defined on incoming sectors $\mathscr{H}_{\text{in}}(\theta)$ with incoming vacua $\Omega_{\text{in}}(\theta)$.

In order to identify the observable implications of the chiral transformation appearing in (5.4) we consider the chirality changing string operators $S_\pm(x, y)$.

Formally:

$$S_{\pm}(x, y) = \tfrac{1}{2}\bar{\psi}(x)(1 \pm \gamma^5) \exp\left[ie \int_y^x dz^\mu A_\mu(z)\right]\psi(y). \tag{5.7}$$

A precise definition is obtained by expressing the quantities appearing in (5.7) in terms of the operator solution (2.15):

$$S_{\pm}(x, y) = F_{\pm}(x, y) S_{\pm}^{\text{in}}(x, y) \tag{5.8}$$

Here $S_{\pm}^{\text{in}}(x, y)$ is the string operator of the incoming fields and $F_{\pm}(x, y)$ is a c-number quantity describing the influence of the external current:

$$F_{\pm}(x, y) = \exp\left\{i\left(\pm[\phi_{\text{ret}}(x) + \phi_{\text{ret}}(y)] - \int_y^x dz^\mu \varepsilon_{\mu\nu}\, \partial^\nu \phi_{\text{ret}}(z)\right)\right\} \tag{5.9}$$

It follows from [1], equation (5.12) that on $\mathcal{H}_{\text{in}}(\theta)$ the operators $S_{\pm}^{\text{in}}(x, y)$ have the form:

$$S_{\pm}^{\text{in}}(x, y) = e^{\pm 2i\theta} T_{\pm}^{\text{in}}(x, y) \tag{5.10}$$

where $T_{\pm}^{\text{in}}$ depends only on $\tilde{\Sigma}_{\text{in}}$ and is given explicitly as:

$$T_{\pm}^{\text{in}}(x, y) = :\exp\left\{i\left(\pm[\tilde{\Sigma}_{\text{in}}(x) + \tilde{\Sigma}_{\text{in}}(y)] - \int_y^x dz^\mu \varepsilon_{\mu\nu} \partial^\nu \tilde{\Sigma}_{\text{in}}(z)\right)\right\}: \tag{5.11}$$

The phase factors $\exp(\pm 2i\theta)$ in (5.10) are eigenvalues of the operators S and $S^\dagger$ mentioned above. The operators $S_{\pm}^{\text{in}}$, and consequently also the observables $S_{\pm}$ leave the θ-sectors invariant and can be used to label them. Let us, therefore, consider $S_{\pm}(x, y)$ when x, y and the path joining them are entirely in the domain Δ of Fig. 2. Then from (5.8)–(5.10) and (5.3):

$$S_{\pm}(x, y) = \exp[\pm 2i(\theta + \alpha\pi)] T_{\pm}^{\Delta}(x, y), \tag{5.12}$$

where $T_{\pm}^{\Delta}$ is obtained from $T_{\pm}^{\text{in}}$ by making the substitution (5.5). Since the transformation (5.5) is effected by the unitary transformation U_Δ, the operator $T_{\pm}^{\Delta}$ is related to $T_{\pm}^{\text{in}}$ by this same transformation. Then, if Ψ is a state of $\mathcal{H}_{\text{in}}(\theta)$:

$$(\Psi, S_{\pm}(x, y)\Psi)_\theta^{\text{in}} = e^{2i(\theta + \alpha\pi)}(\Psi, U_\Delta^\dagger T_{\pm}^{\text{in}}(x, y) U_\Delta \Psi)_\theta^{\text{in}} \tag{5.13}$$

The right-hand side of (5.13) is the mean value of $U_\Delta^\dagger S_{\pm}^{\text{in}} U_\Delta$ in the state of the sector $\mathcal{H}(\theta + \alpha\pi)$ corresponding to Ψ, i.e. the state of $\mathcal{H}(\theta + \alpha\pi)$ with the same Σ-particle component as Ψ. We can write down an equation similar to (5.13) for any gauge invariant observable $\mathcal{O}$ localized in Δ. In an obvious notation:

$$(\Psi, \mathcal{O}\Psi)_\theta^{\text{in}} = (\Psi, U_\Delta^\dagger \mathcal{O}^{\text{in}} U_\Delta \Psi)_{\theta + \alpha\pi}^{\text{in}}. \tag{5.14}$$

The incoming form $\mathcal{O}^{\text{in}}$ of $\mathcal{O}$ is obtained by replacing (ψ, A_μ) by $(\psi^{\text{in}}, A_\mu^{\text{in}})$. Equation (5.14) tells us that, up to effects due to the creation and annihilation of Σ-particles accounted for by U_Δ, the result of a measurement performed in Δ in a state of $\mathcal{H}_{\text{in}}(\theta)$ is the same as the result of the same measurement performed in the absence of external charges in the corresponding state of $\mathcal{H}_{\text{in}}(\theta + \alpha\pi)$. By making the motion of the external charges sufficiently slow, we can reduce the absorption and production of Σ-particles arbitrarily so that U_Δ can be approximated by the identity. In this adiabatic limit:

$$(\Psi, \mathcal{O}\Psi)_\theta^{\text{in}} \simeq (\Psi, \mathcal{O}^{\text{in}}\Psi)_{\theta + \alpha\pi}^{\text{in}}, \tag{5.15}$$

and so states belonging to $\mathcal{H}_{\text{in}}(\theta)$ behave in Δ practically as the corresponding states of $\mathcal{H}_{\text{in}}(\theta+\alpha\pi)$ without external charges.

We have seen that a measurement of the electric field in Δ does not reveal the presence of external charges outside Δ. The appropriate local quantities to measure are the gauge-invariant densities:

$$\left.\begin{aligned}\chi(x) &= \text{“}\bar{\psi}(x)\psi(x)\text{”} = \\ \chi_5(x) &= \text{“}\bar{\psi}(x)\gamma^5\psi(x)\text{”} = \end{aligned}\right\}\lim_{\varepsilon\to 0}\tfrac{1}{2}(S_+(x+\varepsilon,x)\pm S_-(x+\varepsilon,x)). \tag{5.16}$$

In the approximation (5.15) we have:

$$(\Omega,\chi(x)\Omega)_\theta^{\text{in}} \simeq \begin{cases}\text{cst. }\cos(2\theta) & \text{outside }\Gamma,\\ \text{cst. }\cos(2(\theta+\alpha\pi)) & \text{inside }\Gamma.\end{cases} \tag{5.17}$$

with similar expressions for $\chi_5(x)$ with cos replaced by sin.

We have a transmutation of the external electric field $-e\alpha$ inside the loop Γ into a shift $\alpha\pi$ of the chiral angle. This shift depends only on the magnitude of the external charges and not on the geometry of the loop Γ. This is consistent with interpretations of θ as a measure of the charges localized at infinity. Indeed the constant term E_0 that we suppressed from (2.6) would correspond to the presence of such charges. The presence of a nonvanishing E_0 does not affect the above analysis; it merely means that we have to replace θ in the above equations by $\tilde{\theta}=\theta-(\pi/e)E_0$. In other words, it is the angle $\tilde{\theta}$ which is physically relevant, its decomposition into θ and E_0 is arbitrary since E_0 cannot be measured separately.

Our final conclusion is that the superselection sectors of massless QED_2 are distinguished physically by the values of chirality carrying string operators and their local limits χ and χ_5, whether external charges are present or not. The changes of the total charge at the left are related to the shifts of the angle defining the values of χ and χ_5.

Appendix: the decomposition (4.2)

In a first step we rewrite ϕ_{ret} as in (5.1). In the present case the time derivative $f(x)$ of $\Box E_{\text{ext}}(x)$ has a support which breaks into two pieces localized in $(0<x^0<T_1)$ and $(T_2<x^0<T)$. Inserting

$$\Box E_{\text{ext}}(x)=\int_{-\infty}^{x^0} dt f(t,x^1) \tag{A.1}$$

into (5.1) one gets:

$$\begin{aligned}\phi_{\text{ret}}(x)=\pi\Big\{(1/e)E_{\text{ext}}(x)-\frac{1}{2}\int_{y^0<T_1}(dy)^2 f(y)\int_{-\infty}^{+\infty}\frac{dk}{\omega_k^2}\\ \times[1-\cos(\omega_k(x^0-y^0))]e^{-ik(x^1-y^1)}\Big\}\end{aligned} \tag{A.2}$$

if $x^0\in[T_1,T_2]$. The first term in the square bracket of the integral over k and $(1/e)E_{\text{ext}}$ give the static part ϕ_{stat} of ϕ_{ret}. Using:

$$\int_{-\infty}^{T_1} dy^0 f(y^0,y^1)=(\partial/\partial y^1)\rho(y^1) \tag{A.3}$$

and (2.6), integrations by parts lead to (4.3).

The second term in the square bracket of the k-integral in (A.2) produces ϕ_I. It is obviously a solution of the free Klein–Gordon equation. The form (4.4) is obtained if one notices that

$$f(x) = e\, \partial_0 \varepsilon_{\mu\nu}\, \partial^\mu J^\nu = \Box J^1. \tag{A.4}$$

The last equality is a consequence of the external charge conservation $\partial_\mu J^\mu = 0$.

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