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The Ginzburg Landau free energy for a Josephson array

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Abstract. We derive microscopically the Ginzburg Landau free energy for a Josephson array in the case where the magnetic field is zero. This free energy depends on two order parameters and displays two successive phase transitions.

1. Introduction

Much interest has been focused recently on inhomogeneous superconductors [1]. The detailed nature of the superconducting state in such systems, the occurrence of interesting cooperative effects, (Kosterlitz-Thouless transition), the effects of the dimensionality and of the disorder, the influence of vortex pinning and the charging of the grain are among the questions which are still open in this field. We concentrate our work on three dimensional Josephson arrays, which can be viewed as a class of inhomogeneous superconductors. Under Josephson array we understand a regular array of superconducting grains connected by Josephson weak links.

In the limit where the grains are large enough to be independently superconducting and where the coupling between them *is* weak enough, it is generally accepted that this system undergoes (in 3 dimensions) two phase transitions. At T_{co}, the grains become superconducting and at $T_{cJ} < T_{co}$ the phases of the grains become ordered. To describe such a system by a Ginzburg-Landau free energy, one needs therefore two order parameters. One, ψ_i, describing the transition in the ith grain, the other Ψ the phase ordering transition of the whole system.

The purpose of this paper is to derive microscopically a form for the Ginzburg-Landau function for the case of zero magnetic field. In a first section, we define the order parameters which we shall use in the calculations. An analogy with the problem of the granular ferromagnet shows that the usual *BCS* order parameter (definite phase, indefinite number of particles) is not convenient for the microscopic analysis of this problem. We are led instead to work in the constant particles number representation discussed by Anderson [2]. The second part of the paper is devoted to establishing the equations for these order parameters, using Feynman's diagrams. Finally we shall integrate these equations to get a form for the Ginzburg-Landau function, depending on the order parameters defined in the

constant particles number representation. The integration constants are determined by comparison with the *BCS* theory and by high temperature expansions. It will then be possible to come back to more convenient phase dependent order parameters to give an expression for the free energy of the Josephson array.

Actually it has been the custom, in the field of inhomogeneous superconductors, to write the Ginzburg Landau free energy as [3].

$$F=\sum_i \left\{a_i\,|\psi_i|^2+b_i\,|\psi_i|^4+\sum_j c_{ij}\,|\psi_i-\psi_j|^2\right\} \tag{1}$$

It must be emphasized at this point that the expression (1), if indeed a Ginzburg Landau form for the upper transition, plays the role of an Hamiltonian for the lower (phase coherent) transition. The recent work of C. Ebner and D. Stroud, calculating the lower transition by numerical methods [4] takes explicitly into account this remark.

2. The order parameters

An academical system which shows many analogies with the Josephson array is the granular ferromagnet [5]. It is constituted of N identical ferromagnetic grains, each of these grains containing n spin $\vec{S}$. One assumes that the spins within a grain are coupled by a nearest neighbour exchange interaction j and that the total spin of each grain is coupled to its neighbours by an exchange interaction J. One expects then two critical temperatures, one at which each grain becomes ferromagnetic

$$T_{c0}\sim jS(S+1)$$

and another when all grains become ferromagnetically oriented

$$T_{cJ}\sim J(nS)(nS+1)$$

Note that necessarily $T_{cJ}<T_{c0}$.

This system shows clearly the conceptual difficulty we encounter in an inhomogeneous superconductor. If one chooses to break the symmetry, say in the z direction, at T_{c0} the moments of all the grains line up in the z direction, and nothing special can happen at T_{cJ}. If however one considers the spins of each grain as forming a giant spin in an arbitrary direction, then the collection of these weakly coupled giant spins should undergo a ferromagnetic transition at T_{cJ}. One possible solution to this problem would be to introduce a symmetry breaking field, the direction of which varies from grain to grain. However, as discussed elsewhere [6], this two steps analysis can be avoided by defining a rotationally invariant order parameter as the norm of the effective moments, $|\vec{m}|$ for the grain, $|\vec{M}|$ for the whole system.

This choice is well known in the theory of superconductivity, where it has been shown by Anderson [2] that one can work either in the representation where the phase is fixed and the number of particles undetermined (*BCS* representation) or in the representation where the number of particles is fixed and the phase is random (Anderson representation). In this latter representation, the order parameter is $|\Delta|$, the modulus of the BCS order parameter.

The analogy with the granular ferromagnet is particularly clear in the pseudo-spin model of superconductivity, which has been discussed in detail in the literature [7]. In this model the superconducting transition appears as a ferromagnetic transition in the $x-y$ plane. The operator $B_i^+ = \sum_k a_{ik\uparrow}^+ a_{i-k\downarrow}^+$ (where $a_{ik\uparrow}^+$ creates an electron of momentum k and spin up in the ith grain) corresponds to the total spin operator in the $x-y$ plane, while the modulus of the order parameter corresponds to the norm of the magnetization.

To formalize the above arguments, we choose to work in the Anderson representation, where $\langle\langle B_i^+\rangle\rangle = 0$, and $\langle\langle A\rangle\rangle = \mathrm{Tr}\, e^{\beta(\Omega - H + \mu N)} A$. We are therefore led to define the following quantity for the single grain:

$$\delta_i^2 = \langle\langle B_i^+ B_i\rangle\rangle \tag{2}$$

and for the description of the phase correlations in the array (3D system)

$$\chi^2 = \lim_{N\to\infty} \frac{1}{N^2} \sum_{ij} \langle\langle B_i^+ B_j\rangle\rangle \tag{3}$$

The choice of δ_i^2 and χ^2 as order parameters will permit us to have a coherent description of the whole system. In the *BCS* theory, where $V\langle\langle B_i^+\rangle\rangle = \Delta^*$, ($V$ is the *BCS* coupling constant), (2) reads $V^2\delta^2 = |\Delta_{BCS}|^2 + V^2(\langle\langle B^+B\rangle\rangle - \langle\langle B^+\rangle\rangle\langle\langle B\rangle\rangle)$ so that the definition of δ_i^2 contains both the order parameter and the fluctuations. They will be separated by an appropriate choice of the diagrams. In the same way, (3) becomes

$$V^2\chi^2 = \lim_{N\to\infty} \frac{1}{N^2} |\Delta_{BCS}|^2 \sum_{ij} e^{i\varphi_i - i\varphi_j}$$

This expression shows clearly the conceptual difficulty discussed above: in this *BCS* representation, in order to discuss the ferrocoherent transition, it is necessary to introduce a posteriori a rather artificial phase average.

$$e^{i\varphi_i - i\varphi_j} \to \langle e^{i\varphi_i - i\varphi_j}\rangle_{\text{phase average}}$$

3. The equations for the array

To evaluate δ^2 and χ^2 we start from a microscopic Hamiltonian for a Josephson array which can be written as the sum of the individual BCS Hamiltonians for the grains and the tunneling interaction, which is taken as a pair tunneling interaction

$$\begin{aligned} H = &\sum_i \sum_{ks} \varepsilon_{ik} a_{iks}^+ a_{iks} \\ &+ \frac{1}{2} \sum_i \sum_{\substack{kk'q \\ ss'}} V(k-k') a_{ik+qs}^+ a_{i-ks'}^+ a_{i-k's'} a_{ik'+qs} \\ &+ \frac{1}{2} \sum_{ij} \sum_{\substack{kk'q \\ ss'}} v_{ij}(k-k') a_{ik+qs}^+ a_{i-ks'}^+ a_{j-k's'} a_{jk'+qs} \end{aligned} \tag{4}$$

To simplify the calculations we choose for V the *BCS* reduced interaction and

give the same functional form to $v_{ij}(k-k')$. This preserves the essential features of the superconductivity within the grains and describes the Josephson effect between the grains.

The Green's function related to δ_i^2 and χ^2 can be written as

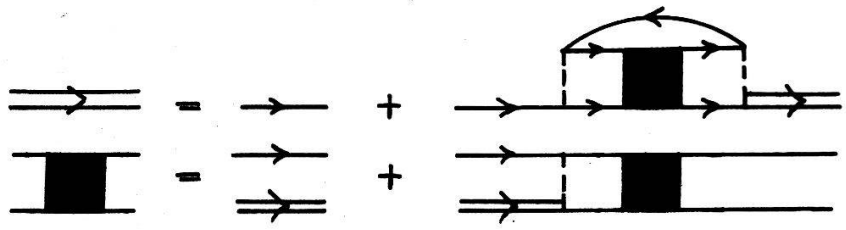
$$G_{ij}^{(2)}(k, k', i\omega_\nu) = \int_0^{1/T} \ll T_\tau\{a_{ik\uparrow}^+(\tau)a_{i-k\downarrow}^+(\tau)a_{j-k'\downarrow}(0)\cdot a_{jk'\uparrow}(0)\} \gg e^{i\omega_\nu\tau}\, dt \tag{5}$$

As discussed above, the *BCS* theory must first be reformulated in the Anderson representation. In this representation, the anomalous propagator

$$F_i(k, \tau) = -\langle\langle T_\tau(a_{ik\uparrow}(\tau)a_{i-k\downarrow}(0))\rangle\rangle$$

is identically zero, and the usual diagrammatic formulation of the *BCS* theory is inappropriate for our problem.

We formulate the *BCS* theory in the Anderson representation by introducing for the quantities of interest a set of self consistent equations which we represent diagrammatically by



(6)

Figure 1

In this $\rightarrow$ is $Go(k, \omega_\nu)$, the non-renormalized one-particle Green's function $=$ is $G(k, i\omega_\nu)$ the renormalized one-particle Green's function, $\vdots$ is V the superconducting interaction and $\blacksquare$ is $G^{(2)}(k, k', i\omega_\nu)$.

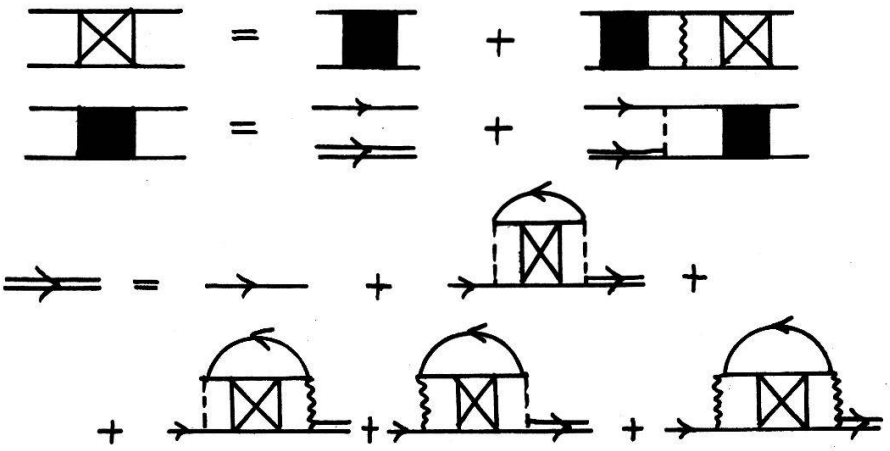
The solution of the equation (6) is straight forward and yields

$$V\Delta_0(T) = \Delta_{BCS}(T)$$

for the quantity

$$\Delta_0^2 = \sum_{kk'} G^2(k, k', t=0)$$

The set of equations (6) shall be a guideline for the array problem. We introduce the intergrain interaction $v = \wr$ and write the following set of equations



(7)

(8)

(9)

Figure 2

In this $\mathbb{X}$ gives the renormalized $G_{ij}^{(2)}(k, k', i\omega_\nu)$ and $\mathbb{I}$ is a quantity closely related but not completely identical to $G_{ii}^{(2)}(k, k', iw_\nu)$ calculated for an isolated grain, the difference being that the one-particle Green's function entering (8) is the fully renormalized $G(k, i\omega_\nu)$ defined by the equation (9).

From the equation (8) one can calculate the quantities δ^2 and χ^2 without difficulty and one gets

$$V^2\delta^2 = V^2\Delta_0^2\left(1 + vz\frac{\chi^2}{k_BT}\right) \tag{10}$$

$$V^2\chi^2\left(1 - vz\frac{\Delta_0^2}{k_BT}\right) = \frac{V^2\Delta_0^2}{N} \tag{11}$$

and from (7) we have

$$\frac{V^2\Delta_0^2}{k_BT}(1 - Vg) = V^2 f \tag{12}$$

with

$$f = \overrightarrow{\rightleftarrows}$$

$$= \sum_k \sum_\nu \frac{-1}{\omega_\nu^2 + \varepsilon_k^2 + V^2\delta^2 + 2Vv\frac{z\chi^2}{N^2} + v^2\frac{z^2\chi^2}{N^2}} \tag{13}$$

Neglecting the terms of the order $1/n$ (n number of superconducting electrons in a grain) in (12) and $1/N$ (N number of grains) in (11), (12) and (13) the system of equations becomes

$$\chi^2\left\{1 - \frac{vz}{k_BT}\frac{\delta^2}{\left(1 + \frac{vz}{k_BT}\chi^2\right)}\right\} = 0 \tag{14}$$

$$\Delta^0(1 - Vf(\delta^2, T)) = 0 \tag{15}$$

$$f(\delta^2, T) = \sum_k \sum_\nu \frac{-1}{\omega_\nu^2 + \varepsilon_k^2 + V^2\delta^2} \tag{16}$$

Equation (15) together with the definition (16) is identical with the *BCS* gap equation. Equation (14) has two solutions:

$$\chi^2 = 0 \qquad \text{for} \quad T > T_{cJ}$$

$$\chi^2 = \delta^2 - \frac{k_BT}{vz} \qquad \text{for} \quad T \leqslant T_{cJ} \tag{17}$$

with

$$T_{cJ} = \frac{vz}{k_B}\delta^2(T_{cJ}) \tag{18}$$

So we find three temperature regimes

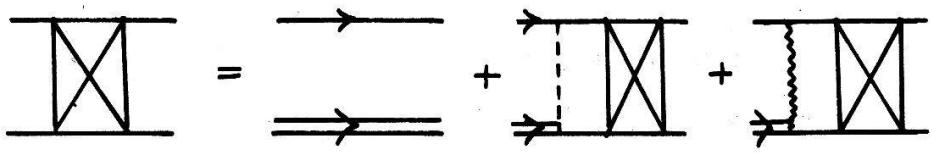
(A) $T > T_{co}$ $\qquad \delta = \chi = 0$

(B) $T_{co} > T > T_{cj}$ $\qquad V\delta = \Delta_{BCS}$ $\qquad \chi = 0$

(C) $T < T_{cJ}$ $\qquad V_\delta = \Delta_{BCS}$ $\qquad \chi^2 = \delta^2 - \frac{k_BT}{vz}$

Similar results have been recently obtained numerically by Monte Carlo calculations [4].

An interesting point comes up when writing the diagrammatic series for the array. Actually (7) and (8) can also be written as



(19)

Figure 3

From (19) we can extract the following equation for the order parameters.

$$\frac{V^2\delta^2}{k_BT} = V^2f + Vf\frac{V^2\delta^2}{k_BT} + vfz\frac{V^2\chi^2}{k_BT} \tag{20}$$

$$\frac{V^2\chi^2}{k_BT} = \frac{V^2f}{N} + Vf\frac{V^2\chi^2}{k_BT} + vfz\frac{V^2\chi^2}{k_BT} \tag{21}$$

Putting the fluctuations terms (the first term in each right hand side) equal to zero in these equations yields obviously the result $\delta^2 = \chi^2$ for all temperatures, which is patently different from (17) and (18). However, if one keeps the fluctuations terms different from zero, one easily derives from (20) and (21) the equation (10) and (11). The root of the difficulty lies therefore in the way one treats the fluctuations terms, i.e. in the way one lets n and N go to infinity. That (17) and (18) are the correct results can be seen from (20) and (21) by keeping n and N finite and looking for the various solutions of the equations.

Let us first remind the order of magnitude of the different quantities entering in the calculation. V is of the order $1/n$ and f of the order n, we have $Vf = 0(1)$. further if we want that $T_{cJ} < T_{co}$, v has to be of the order $1/n^2$, $vf = 0(1/n)$. The analysis is then similar to the one encountered in the simple *BCS* equations

$$(1 - Vf)\frac{V^2\delta^2}{k_BT} = V^2f$$

which has two solutions

$$(1 - Vf) = 0(1)$$
$$V^2\delta^2 = 0(1/n)$$

and

$$(1 - Vf) = 0(1/n)$$
$$V^2\delta^2 = 0(1)$$

A straightforward analysis of the equation (20) and (21) shows that we have now three regimes.

(A) $T > T_{c0}$

$V^2\delta^2 = 0(1/n)$ $\quad (1 - Vf) = 0(1)$

$V^2\chi^2 = 0(1/nN)$ $\quad (1 - Vf - vf) = 0(1)$

(B) $T_{cJ} < T < T_{co}$

$V^2\delta^2 = 0(1) \qquad (1 - Vf) = 0(1/n)$

$V^2\chi^2 = 0(1/N) \qquad (1 - Vf - vzf) = 0(1/n)$

(C) $T < T_{cJ}$

$V^2\delta^2 = 0(1) \qquad (1 - Vf) = 0(1/n)$

$V^2\chi^2 = 0(1) \qquad (1 - Vf - vzf) = 0(1/nN)$

This shows that it is wrong to let fluctuations term go to zero in equation (20) and (21) when there are two large quantities n and N going to infinity independently.

4. The Ginzburg-Landau function

It is possible to write two expansions for $\delta(T)$ and $\chi(T)$ near the transition temperatures, because $\delta(T_{co}) = 0$ and $\chi(T_{cJ}) = 0$. Expanding $1 - Vf(\delta, T)$ and replacing Δ_0^2 by δ^2, one obtains easily

$$\delta^2\left(1 - \frac{T}{T_{co}} - \frac{V^2\delta^2}{(3.06)^2(k_B T_{co})^2}\right) = 0 \tag{22}$$

which yields, for $T \sim T_{co}$,

$$V\delta(T) = 3.06 k_B T_{co}\left(1 - \frac{T}{T_{co}}\right)^{1/2}$$

These are the well-known *BCS* results.

For $\chi(T)$ the calculation is simpler and gives

$$\chi^2\left(1 + \frac{vz\chi^2}{k_B T} - \frac{vz\delta^2}{k_B T}\right) = 0 \tag{23}$$

where z is the number of nearest neighbours in the array.

In order to find the form of the Ginzburg-Landau function F, we must divide the equation (22) by δ and identify the results with $c_1'(\partial F/\partial\delta)$, where c_1' is an unknown multiplicative constant. A similar procedure is applied to equation (23).

To calculate the multiplicative constants entering in F we first evaluate the order of magnitude of the different parts of the free energy F. A look back at the Hamiltonian shows that the part of the self-energy describing the grain should be proportional to nN (N = number of grains in the array, n = number of electrons in each grain), whereas the part describing the array should be proportional to N. This allows us to write

$$F = F_0 + c_1(1/n)N\frac{T - T_{co}}{T_{co}}\delta^2 + c_1(1/n)N\frac{V^2}{2\Delta_1^2}\delta^4$$
$$+ c_2(N/n^2)\left(1 - \frac{vz}{k_B T}\delta^2\right)\chi^2 + c_2(N/n^2)\frac{vz}{k_B T}\chi^4 \tag{24}$$

where c_1 and c_2 are coefficients of order $1/n$ and N/n^2 respectively and

$$\Delta_1^2 = \frac{8\pi^2}{7\zeta(3)}(k_B T_c)^2.$$

We note first that deriving the equation (24) with respect to δ and multiplying by δ does not give back exactly equation (22) but yields an additional term

$$-2c_2(N/n^2)\frac{vz}{k_BT}\chi^2\delta^2.$$

This term is however smaller by at least an order of $1/n$ compared to the other terms of equation (22), and it presumably would arise from higher order diagrams that we have neglected. In order to determine the values of the constants c_1 and c_2 entering (24), we use the high temperature value of $\langle\chi^2\rangle$ (the fluctuations of χ) given by perturbation theory.

$$\langle\chi^2\rangle=\frac{N\delta^2}{1-\dfrac{T_{cJ}}{T}}\quad T\gg T_{cJ} \tag{25}$$

which determines c_2. One also replaces c_1 by its usual value for *BCS* superconductors. At this point we can come back to complex order parameters ψ and Ψ (corresponding to δ and χ respectively) and choose their definition so that $|\psi|=\Delta_{BCS}$ and $|\Psi|^2|_{T=0}=\Delta^2_{BCS}$.

One then gets finally

$$F=F_0+N\Omega N(0)\left\{\frac{T-T_{c0}}{T_{c0}}|\psi|^2+\frac{0.526}{(\pi k_BT_{co})^2}|\psi|^4\right\}$$
$$+\frac{1}{|\psi|^2}\left\{\left(k_BT-\frac{vz}{V^2}|\psi|^2\right)|\Psi|^2+\frac{vz}{2V^2}|\Psi|^4\right\} \tag{26}$$

In this, N is the number of grains, Ω the volume of each grain, $N(0)$ the density of states, T_{co} the grain transition temperature k_B the Boltzman constant, V the *BCS* interaction constant, v the intergrain interaction, and z the number of nearest neighbours in the array.

To find the phase transitions implied by (26), one must establish the minimum of F with respect to ψ and Ψ. One then recovers the three temperature regimes discussed after equation (18), with $V\delta$ replaced by $|\psi|$ and $V\chi$ by $|\Psi|$.

The equation (26) is the final result of this paper. We note that, as a Ginzburg-Landau free energy for two coupled order parameters, the expression (26) has a form which is quite unexpected. In fact, one writes in general for this kind of free energy (depending on two order parameters x and y) an expression of the type [9].

$$F(x,y)=a_2x^2+a_4x^4+b_2y^2+b_4y^4+\lambda x^2y^2$$

with $\lambda>0$ corresponding to a competition between x and y and $\lambda<0$ to an enhancement.

One can see that the expression (26) is quite different: the coupling between the order parameters occurs in all the terms of the second part of F except in the one defining the transition temperature related to Ψ.

One must stress however that $F(x,y)$ describes a type of situation which is somewhat different from the problem discussed in this paper, namely that of a system subjected to the interplay of several modes of ordering. These modes are

of different nature, for instance, magnetic and superconducting, ferro and antiferromagnetic, etc. In our paper, we deal with the same type of ordering acting successively on different scales.

5. Conclusion

We have derived, using Feynman diagrams, a Ginzburg-Landau free energy function for Josephson arrays. It depends on two order parameters: one describes the intragrain superconducting transition, the other describes the phase coherence transition across the whole system. In principle, by using the same body techniques, this Ginzburg-Landau function can be generalized to include the effects of magnetic fields and the Coulomb charging energy [10]. We are currently working on this problem.

The Ginzburg-Landau function can also be useful to study fluctuations effects in particular the effect of the fluctuations of the grain order parameter (superconducting order parameter) on the phase coherence transition. This effect should be important in an inhomogeneous superconductors constituted by small superconducting grains. This problem will be the subject of a future publication.

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