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Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **56 (1983)**

Heft 5

PDF erstellt am: **24.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-115432

Helvetica Physica Acta, Vol. 56 (1983) 1046–1052 0018–0238/83/051046–07$1.50 + 0.20/0

A Loewner-type interpolation problem applied to partial-wave dispersion relations

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(15. XII. 1982)

Abstract. The one channel, η-unitarity, general l partial-wave dispersion relation with the left hand cut approximated by an arbitrary finite number of poles is viewed as a particular case of a higher order Loewner-type interpolation problem. The solvability conditions and the general solution are given in terms of an iterative procedure resembling the Nevanlinna procedure for the Pick–Schur–Nevanlinna interpolation problem.

1. Introduction

It has long been realized that the results and techniques of Pick–Schur–Nevanlinna-type interpolation theory provide a useful method for studying partial-wave dispersion relations [1–4]. These methods are extremely powerful in giving the conditions for a solution to exist and in providing the description of all solutions. In the one channel case using R-unitarity, this has been done for the pole approximation and general l in [2]. In this note we point out how to deal with the threshold condition for η-unitarity in the pole approximation and for general l in the one channel case. More specifically, it is shown how this problem is related to a particular case of a higher order Loewner-type interpolation problem. By an iterative procedure resembling the Nevanlinna procedure for the Pick–Schur–Nevanlinna interpolation problem, we give the solvability conditions and the general solution.

After fixing the notation in Section 2 we will present in Section 3 the analogue of the Blaschke-transform for our problem. Section 4 describes the iterative procedure and Section 5 contains some simple explicit examples.

2. Description of the problem

Let

$$S_l(s) = 1 + 2iq(s)f_l(s)$$

be the partial wave scattering matrix for elastic scattering in the state of angular momentum l. Since l is kept fixed, it will be dropped as an index from the functions. The center-of-mass momentum $q(s)$ is defined in [2]. The function $S(s)$

has the following properties:

(α) $S(s)$ is analytic in the s-plane with a cut from 1 to ∞ and a finite number of simple poles at $s_i < 1$ with residues Γ_i $(i = 1, 2, \ldots N)$;

(β) $S(s) = \bar{S}(\bar{s})$;

(γ) $S(x+i0)$ is continuous for $x > 1$ and at ∞, $|S(x+i0)| = 1$ for $x \geqq 1$ and $S(1) = 1$;

(δ) $S(s)$ has the behaviour $S(s) = 1 + 0((s-1)^{l+1/2})$ near $s = 1$.

(γ) is the elastic unitarity condition. The general inelastic case can be reduced to the elastic one by a Froissart transformation (see e.g. [1]).

It is convenient to map the cut s-plane onto the unit disc by

$$z = \frac{1-(1-s)^{1/2}}{1+(1-s)^{1/2}}$$

We define

$$A(z) = \prod_{i=1}^{N} \frac{z - z_i}{1 - z z_i} S\left(\frac{4z}{(1+z)^2}\right),$$

where

$$z_i = \frac{1-(1-s_i)^{1/2}}{1+(1-s_i)^{1/2}},$$

and state the conditions on $A(z)$, analytic for $|z| < 1$:

(α') $A(z_i) \equiv \gamma_i = \Gamma_i \prod_{j \neq i} \frac{z_i - z_j}{1 - z_i z_j} \frac{1}{1 - z_i^2} \frac{dz}{ds}\Big|_{s=s_i}$;

(β') $A(z) = \bar{A}(\bar{z})$;

(γ') $|A(e^{i\theta})| = 1$, $A(1) = 1$ and $A(z)$ is continuous for $|z| \leqq 1$.

Let B^n be the class of Blaschke products of the form

$$\prod_{k=1}^{n} \frac{z - a_k}{1 - a_k z},$$

where the a_i are either real or appear in complex conjugate pairs. B^0 contains by definition only the function $f = 1$. Let $B = \bigcup_{n=0}^{\infty} B^n$. Then ($\alpha'$), ($\beta'$) and ($\gamma'$) imply that $A(z) \in B$. Since all these $A(z)$ are analytic in a neighborhood of $z = 1$, we have

$$(\delta') \quad \left(\frac{d^j A}{dz^j}\right)(1) \equiv A^{(j)}, \quad j = 1, 2, \ldots, 2l,$$

with the $A^{(j)}$ given explicitly in terms of the z_i. We remark that because of (γ') and (β') only l of the $A^{(j)}$ are independent of each other (for example, $A^{(2)} = A^{(1)}(A^{(1)} - 1)$).

Our task is to derive the conditions on the z_i's and Γ_i's (or γ_i's) for a function satisfying (α')–(δ') to exist and to give the general formula for all the functions satisfying these conditions. This is a higher order loewner-type interpolation problem. One possibility for solving this problem was kindly communicated to us in a letter by Prof. M. G. Krein. It is due to A. A. Nudelman and consists in reducing it to the Stieltjes moment problem. Another solution might be hidden in

the powerful recent results in [5] and the references quoted therein. In the following sections we develop a fairly elementary method of solution. To be more explicit, for $l=0$ the condition (δ') does not exist and one can obtain the complete solution by removing the condition (α') via the Nevanlinna method using the Blaschke-transformation [1]. For $l\neq 0$ the same method can be used to remove condition (α') and Nevanlinna's method can be generalized to remove also the condition (δ'). This iterative procedure is very well suited to practical applications.

3. The analogue of the Blaschke-transformation

Let

$$B_a^n=\left\{f\in B^n \,\middle|\, \frac{df}{dz}(1)=a(>0)\right\} \quad \text{and} \quad B_a=\bigcup_{n=1}^{\infty} B_a^n.$$

Proposition. *The formula*

$$f=\frac{2(1+zh)-(1-z)(1+h)a}{2(1+zh)+(1-z)(1+h)a} \tag{1a}$$

and its inverse

$$h=\frac{2(1-f)-(1+f)(1-z)a}{-2z(1-f)+(1+f)(1-z)a} \tag{1b}$$

give a one-to-one correspondence between all $h\in B$ and all $f\in B_a$.

Proof. During the proof we freely use the notation and the results of [6]. Let

$$f(z)=\frac{i-G(\zeta)}{i+G(\zeta)}, \qquad \zeta=i\frac{1-z}{1+z}. \tag{2}$$

Let R^{finite} (S^{finite}) be the class of functions $R(S)$ defined in [6], with the restriction that the measure in the integral representation of the functions has only a finite number of points of growth not including the point 0. Clearly $f\in B$ is equivalent to $G\in R^{\text{finite}}$ and $G(\zeta)=-\bar{G}(\bar{\zeta})$. From [6] it follows that G can be represented as

$$G(\zeta)=\zeta F(\zeta^2), \tag{3}$$

with $F\in S$ and in our case in fact $F\in S^{\text{finite}}$. Thus we have by (2) and (3) a one-to-one correspondence between B and S^{finite}.

Let (for $a>0$) $S_a=\{F\in S \mid F(0)=a\}$ and S_a^{finite} be defined analogously. It is clear that (2) and (3) give a one-to-one correspondence between B_a and S_a^{finite}. The transformation (first step in the continued fraction expansion)

$$F(\xi)=\frac{a}{1-\xi H(\xi)}, \qquad H(\xi)=\frac{F(\xi)-a}{\xi F(\xi)}, \tag{4}$$

gives a one-to-one correspondence between S_a^{finite} and S^{finite}. To see this, let $H\in S$. It follows from [6] that $\xi H(\xi)\in R$, and so

$$F(\xi)=\frac{a}{1-\xi H(\xi)}\in R.$$

Moreover $F(\xi)>0$ for $\xi<0$, so that $F\in S_a$. Now let $F\in S_a$. Then $\xi H(\xi)=1-\frac{a}{F(\xi)}\in R$ and is negative for $\xi<0$, that is $\xi H(\xi)\in S^{-1}$. It follows from [6] that $H\in S$. We thus have a one-to-one correspondence between S_a and S given by (4). In our case this correspondence is obviously between S_a^{finite} and S^{finite}.

If $h\in B$ is the function corresponding to $H\in S^{\text{finite}}$ via (2) and (3), a simple calculation gives (1a). This completes the proof of the proposition.

Remark. A closer look at the proof reveals a more refined version of the proposition, namely that (1a) gives a one-to-one correspondence between B^n and B_a^{n+1}.

4. The iterative procedure

After having removed (see for example [1]) the conditions (α') via the Nevanlinna method (assuming that z_i and γ_i fulfil the solvability conditions for that procedure), one ends up with a function $\varphi(z)\in B$, whose derivatives

$$\left(\frac{d^j\varphi}{dz^j}\right)(1)\equiv\alpha_j \qquad j=1,2,\ldots,2l,$$

are given in terms of z_i and γ_i. In order that such a φ exists a necessary condition is that either

$$\alpha_j=0, \qquad j=1,2,\ldots,2l, \tag{5a}$$

or

$$\alpha:1>0 \tag{5b}$$

holds. For the first alternative the only solution is $\varphi(z)=1$. If the second alternative holds, we perform the inversion (1b) to give

$$\varphi_1=\frac{2(1-\varphi)-(1+\varphi)(1-z)\alpha_1}{-2z(1-\varphi)+(1+\varphi)(1-z)\alpha_1}. \tag{6}$$

Then $\varphi_1\in B$ and its first $2(l-1)$ derivatives β_j are prescribed in terms of the α_j. In order to see this, let us consider the Taylor expansion of $\varphi_1(z)$ around the point $z=1$. Indeed one can show by a straightforward calculation that the numerator and the denominator in (6) are of the form

$$\alpha_1(z-1)^2+\sum_{n=3}^{\infty}c_n(z-1)^n,$$

where the c_n for $n=3,4,\ldots,2l$ are given in terms of the α_j. Again for such a φ_1 to exist a necessary condition is either $\beta_j=0$ for $j=1,2,\ldots,2(l-1)$ or $\beta_1>0$. This completes the description of the iterative steps.

If before performing l steps in the iterative procedure, it turns out that at an intermediate step neither (5a) nor (5b) is fulfilled, a solution does not exist. If (5a) is fulfilled at one of the intermediate steps, the solution is unique.

Suppose now that after the first $l-1$ steps (5b) is fulfilled. Then one can perform the lth (and final) step and clearly the solution is not unique and all

solutions are in one-to-one correspondence with the functions $h \in B$. The solution corresponding to $h = 1$ is called the isolated solution.

5. Explicit examples

5.1. *The case* $N = 1, l = 1$

We have $S(1) = 1, S'(1) = 0$. In our case

$$A(z) = \frac{z - z_1}{1 - zz_1} S\left(\frac{4z}{(1+z)^2}\right)$$

and

$$A(1) = 1, \qquad A'(1) = \frac{1 + z_1}{1 - z_1},$$

$$A(z_1) = \frac{1}{4} \Gamma_1 \frac{(1+z_1)^2}{(1-z_1)^2}.$$

The solvability condition for the Nevanlinna procedure is

$$|A(z_1)| \leq 1. \tag{7}$$

If this is fulfilled, we define

$$\varphi(z) = \frac{1 - z_1 z}{z - z_1} \frac{A(z) - A(z_1)}{1 - A(z_1)A(z)}.$$

Thus (α') has been removed by the Nevanlinna procedure. It follows that

$$\varphi(1) = 1, \qquad \varphi'(1) = \frac{2A(z_1)}{1 - A(z_1)} \frac{1 + z_1}{1 - z_1} \equiv \alpha.$$

In order that a nonunique solution exists, it is necessary that

$$\alpha > 0. \tag{8}$$

We can then remove (δ') by our transformation (6) or its inverse (1a), namely

$$\varphi(z) = \frac{2(1 + zh(z)) - (1 - z)(1 + h(z))\alpha}{2(1 + zh(z)) + (1 - z)(1 + h(z))\alpha}.$$

Conditions (7) and (8) are necessary and sufficient for a non-unique solution to exist. The isolated solution is given by

$$\varphi_{is}(z) = \frac{(1 + z) - (1 - z)\alpha}{1 + z + (1 - z)\alpha}.$$

5.2. *The case* $N = 2, l = 1$

We have $S(s = 1) = 1, S'(s = 1) = 0$. Defining

$$A(z) = \frac{z - z_1}{1 - z_1 z} \frac{z - z_2}{1 - z_2 z} S\left(\frac{4z}{(1+z)^2}\right)$$

we have

$$A(1)=1, \qquad A'(1)=\frac{1+z_1}{1-z_1}+\frac{1+z_2}{1-z_2},$$

$$A(z_1)=\frac{1}{4}\Gamma_1\frac{z_1-z_2}{1-z_1z_2}\frac{(1+z_1)^2}{(1-z_1)^2}\equiv\gamma_1,$$

$$A(z_2)=\frac{1}{4}\Gamma_2\frac{z_2-z_1}{1-z_1z_2}\frac{(1+z_2)^2}{(1-z_2)^2}\equiv\gamma_2.$$

The solvability condition for the Nevanlinna procedure is

$$|\gamma_1|\leq 1 \quad \text{and} \quad |\gamma_2|\leq 1. \tag{9}$$

We then can define

$$\varphi_1(z)=\frac{1-z_1z}{z-z_1}\frac{A(z)-\gamma_1}{1-\gamma_1A(z)}$$

and calculate

$$\varphi_1'(1)=2\frac{\gamma_1}{1-\gamma_1}\frac{1+z_1}{1-z_1}+\frac{1+\gamma_1}{1-\gamma_1}\frac{1+z_2}{1-z_2},$$

$$\varphi_1(z_2)=\frac{1}{4}\frac{\Gamma_2\left(\frac{1+z_2}{1-z_2}\right)^2-\Gamma_1\left(\frac{1+z_1}{1-z_1}\right)^2}{1-\gamma_1\gamma_2}\equiv\gamma.$$

For a nonunique solution to exist, it is necessary that

$$\varphi_1'(1)>0 \quad \text{and} \quad |\gamma|\leq 1. \tag{10}$$

We then can define

$$\varphi(z)=\frac{1-z_2z}{z-z_2}\frac{\varphi_1(z)-\gamma}{1-\gamma\varphi_1(z)}.$$

Thus (α') has been removed by the Nevanlinna procedure. It follows that

$$\varphi'(1)=-\frac{1+z_2}{1-z_2}+\frac{1+\gamma}{1-\gamma}\varphi_1'(1)$$

For a nonunique solution to exist, it is necessary that

$$\varphi'(1)>0. \tag{11}$$

We can remove (δ') as in the last step of 5.1 by our transformation (6) or its inverse (1a). Conditions (9)–(11) are necessary and sufficient for a nonunique solution to exist.

5.3. *The case* $N=1, l=2$

Taking for convenience $s_1=0$ one can show by a straightforward but lengthy calculation that in this case no solution exists; it turns out that in the final step the

derivative at $z = 1$ is negative. This result should be compared to [2], where it is shown generally that $N \geqslant l$ for a solution to exist in the problem treated there.

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