Dilations of a quantum measurement

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Abstract. When a measurement is made on a quantum mechanical system the state suffers a reduction. It is shown that one can always find an environment interacting with the system and measuring instrument in such a way that the reduction due to a finite measurement is approached arbitrarily closely by the ordinary quantum-mechanical evolution. The precision with which a given interaction approximates the reduction depends upon the properties of the measuring instrument.

1. Introduction

The source of most of the unease with the quantum theory of measurement can be traced back to the reduction of the density operator from ρ to $\sum P_\lambda \rho P_\lambda$ during the measurement of an observable whose spectral projections are $\{P_\lambda\}$. Pure states collapse from superpositions into mixtures, and it is this loss of coherence even when no useful information appears to have been gained which most offends the classical intuition. (The further reduction of the mixed state to a single term just parallels the conditioning of a classical probability measure following a precise measurement of a random variable.)

It is well known that a pure state of a closed system cannot evolve into a mixture in the usual formalism. However, a system which undergoes a measurement must certainly interact with the apparatus and its environment. This forms the basis for some of the recent attempts to derive the reduction of the state from a continuous evolution, [1], [2], [3], and it is the approach which we shall be taking in this paper. The coherence of the original system is transferred to radiation emanating from the interaction between the system and the measuring apparatus, and there it is no longer apparent. This can be compared to the apparent loss of information when a classical gas is allowed to expand out of a confined volume. Initial correlations in the positions of the gas molecules are translated into correlations of a different sort, of which we are not immediately sensible.

If one wishes to entertain such a picture one has to show that it is in principle capable of producing the kind of reduction of state which one observes, and then to look for experiments to test whether nature achieves the effect in that way. In [1] Hepp showed that certain exactly soluble models could be interpreted as a system interacting with its environment in such a way that the ordinary quantum mechanical evolution would lead asymptotically towards a reduction of the state of the system. Further refinements were introduced in the case of a spin-lattice system by Frigerio, [2], and Emch and Whitten-Wolfe, [3], [4]. (Bell in [5] has stressed the important distinction between actual reduction and an asymptotic

approach to a reduced state. In the Coleman–Hepp model one has only the latter. The model considered by Whitten-Wolfe and Emch incorporates an interaction so singular that the collapse is immediate.)

In this paper we shall show how for any system it is possible to reconstruct an environment and interaction which will bring one arbitrarily close to a given reduction of the state. The main tool which we shall use is the dilation theory expounded in [6], [7], and [8], which offers a powerful technique for reconstructing the environment from its effects on a subsystem. This will be done in three stages. We start by recalling how the reduction of the state occurs as a consequence of bringing the system and measuring apparatus into correlation. Next, by considering the corresponding classical system one can produce a plausible evolution which drives it irreversibly into a correlated state. The dilation theory then provides a classical environment and interaction which achieve the same effect. Finally one can show that in the quantised version of this model the time evolution of the system, apparatus and environment together lead asymptotically to a state of the system close to the reduced state. It can be made to come arbitrarily close by making the energy needed to disturb the measurement very much larger than the energy of interaction and thermal agitation. This particular approach has the advantage that the physical meaning of each step is quite transparent, although ultimately all that matters is the behaviour of the final model.

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2. The measuring apparatus

Let $\mathcal{H}$ be the Hilbert space associated with the system, and $\{P_\lambda : \lambda \in \Lambda\}$ be the spectral projections for an observable which is being measured (we assume for the moment that Λ is finite). The consequences of the measurement become easier to understand if one includes the pointer of the measuring instrument in the description. To this end we introduce $\mathcal{K} = l^2(\Lambda) \otimes \mathcal{H} = l^2(\Lambda, \mathcal{H})$, the space of $\mathcal{H}$-valued functions on Λ. (We may consider $l^2(\Lambda)$ to be the Hilbert space associated to the pointer.)

The original space can be identified with the subspace of constant functions, that is we define $I : \mathcal{H} \to \mathcal{K}$ by $(Iv)(\lambda) = v$ for each v in $\mathcal{H}$ and λ in Λ. The adjoint I^* maps ψ in $\mathcal{K}$ to $I^*\psi = \sum \psi(\lambda)$. There is also a distinguished projection P on $\mathcal{K}$ defined by

$$(P\psi)(\lambda) = P_\lambda \psi(\lambda)$$

for ψ in $\mathcal{K}$ and λ in Λ. It projects down into the subspace where the pointer position is correlated with the state of the measured system.

For each bounded operator A on $\mathcal{H}$ one can define an operator $(1 \otimes A)$ on $\mathcal{K}$ by

$$((1 \otimes A)\psi)(\lambda) = A\psi(\lambda).$$

We then have

$$I^*P(1 \otimes A)PI = \sum P_\lambda A P_\lambda.$$

The right-hand side is just the conditioning of A brought about by the measurement. On the left-hand side the operators I and I^* just identify $\mathcal{H}$ with its image in $\mathcal{K}$, so that this conditioning is brought about by a single projection P. In this way reduction of the state of a system is equivalent to bringing the system and measuring instrument into correlation. (Strictly speaking there is still some ambiguity in the interpretation of this correlation. Zurek has recently proposed that this too can be eliminated by considering the interactions between the apparatus and its environment [10].) In order to understand the reduction it will be sufficient to show how a continuous evolution can lead to the correlating projection P. Moreover, since the range of P is isomorphic to $\mathcal{H}$, P must be equivalent to an operator of the form $Q \otimes 1$ on $l^2(\Lambda) \otimes \mathcal{H}$, where Q is a one-dimensional projection. The essence of the problem is therefore to explain how one can arrive at Q, and it is to this that we now turn.

3. The environment

In this section we shall describe a classical model of the system and apparatus, from which it will be easy to reconstruct that part of the environment with which they interact. The classical system, apparatus and environment can then all be quantised together to provide a quantum model of the interactions which drive the system and apparatus into correlation as the measurement takes place.

We may as well assume that Λ has 2^N elements for some integer N, since we can always adjoin extra zero projections to the $\{P_\lambda\}$ to make up the number. Then $l^2(\Lambda)$ carries an irreducible representation the Clifford algebra, $Cl(M)$, of a $2N$-dimensional real inner product space M. We shall write $\gamma(m)$ for both the algebra element and the operator on $l^2(\Lambda)$ corresponding to m in M. Thus, if round brackets denote the inner product on M,

$$[\gamma(m), \gamma(n)]_+ = 2(m, n)1$$

for all m and n in M. We can regard M as the phase space of a classical model of the measurement, and $l^2(\Lambda)$ as its Fermi quantised version.

If the action of $Cl(M)$ is appropriately chosen then the one-dimensional projection Q will project onto a normalised vacuum vector Ω. Such a vector can be characterised in terms of a complex structure J on M by

$$i\gamma(Jm)\Omega = \gamma(m)\Omega.$$

Writing φ for the state defined by Ω we have

$$\begin{aligned}
\varphi([\gamma(m), \gamma(n)]) &= \langle \Omega, [\gamma(m), \gamma(n)]\Omega \rangle \\
&= \langle \Omega, \gamma(m)\gamma(n)\Omega \rangle - \langle \gamma(n)\Omega, \gamma(m)\Omega \rangle \\
&= i(\langle \Omega, \gamma(m)\gamma(Jn)\Omega \rangle + \langle \gamma(Jn)\Omega, \gamma(m)\Omega \rangle) \\
&= i(\langle \Omega, [\gamma(m), \gamma(Jn)]_+\Omega \rangle) \\
&= 2i(m, Jn).
\end{aligned}$$

(For consistency J is skew-adjoint with respect to the inner product on M.) These two-point correlations completely determine the quasi-free state φ.

In this ground state the system and instrument are perfectly correlated, and if the instrument is to be of any use this should be a stable configuration. Near a point of classical stable equilibrium one expects small harmonic oscillations. A very simple and convenient model of such behaviour is provided by taking the time evolution in phase space to consist of the rotations which take m in M to

$$m_t = (\cos \omega t + \sin \omega t J)m.$$

In other words we take a linear evolution with generator ωJ. (This particular choice of generator ensures that the evolution preserves the natural symplectic structure defined on M by J.)

We must expect the environment to dampen the motion by adding a frictional term to the classical generator. In our model we shall take this extra term to have the form $-fp$, where $f > 0$ is the strength of the frictional force and p is an N-dimensional projection on M. This can be thought of as the projection onto the momentum subspace. Then the projection onto the coordinate subspace should be $(1-p)$, and to be consistent with our interpretation of ωJ as the generator of simple harmonic motion we need

$$J(1-p) = pJ.$$

The damped classical motion is now described by a contraction semigroup T_t, whose generator is $Z = \omega J - fp$. The nature of the damping is clear from the identity

$$Z^2 + fZ + \omega^2 1 = 0,$$

which follows from that linking p and J.

The classical environment can now be reconstructed from T_t using the dilation theory of [6], [7], and [8].

Proposition 1. *Let* $V = L^2(\mathbb{R}, pM)$, *on which the unitary group* U_t *acts by sending* v *in* V *to*

$$(U_t v)(x) = v(x - t).$$

Let j *be the map which embeds* M *into* V *by taking* m *to the function whose value at* x *is*

$$(jm)(x) = (2f)^{1/2}\theta(-x)(pT_{-x}m),$$

where θ *is the Heaviside step function. Then for all* $t \geq 0$

$$j^* U_t j = T_t.$$

Proof. The semigroup T_t contracts strongly to zero so we may apply [7] Section 9.4 or [8] Theorem 3.13. On substituting the details of our model the result follows immediately

Clearly the enlarged phase space V looks like a space of waves interacting with M, and the time evolution U_t keeps these moving along with constant velocity. Setting $t = 0$ in the equation relating U and T we see that j is an

isometry and thence that jj^* is a projection. This allows us to decompose V into $jM \oplus N$, where $N = (1 - jj^*)V$ can be interpreted as the phase space of the classical environment.

In calculations later it will be more convenient to work in the Fourier transformed space where the time evolution acts as a multiplication operator. There

$$(Fjm)(s) = (2\pi)^{-1/2} \int \exp(-isx)(jm)(x)\, dx$$

$$= (f/\pi)^{1/2} \int_0^\infty \exp(isx)(pT_x m)\, dx.$$

When s is different from ω this gives

$$(Fjm)(s) = (f/\pi)^{1/2} p(Z - is)^{-1} m.$$

By using the quadratic identity for Z this can be further simplified to

$$(Fjm)(s) = (f/\pi)^{1/2}(s^2 - \omega^2 - ifs)^{-1} p(Z + f + is)m$$
$$= (f/\pi)^{1/2}(s^2 - \omega^2 - ifs)^{-1} p(\omega J + is)m.$$

4. The collapse

The model which we now have for the system and apparatus coupled to their environment resembles that of a statistical mechanical system coupled to a heat bath. In the latter case one expects the system to be driven into thermal equilibrium, and Maassen has verified this for a boson system, [9]. We shall now show how in our fermion system the environment similarly drives the system and measuring apparatus into correlation.

The Clifford algebra, $Cl(V)$, of V is generated by elements $\Gamma(v)$ depending linearly on v in V and subject to the relations

$$[\Gamma(u), \Gamma(v)]_+ = 2(u, v),$$

where, since j is an isometry, we have taken the liberty of using the same round brackets for the inner products on V and M. The time evolution U_t defines an automorphism α_t of this algebra.

We now take a β-K.M.S. state Φ on $Cl(V)$ corresponding to a finite temperature of the environment, [11].

Proposition 2. *Let Φ be a β-K.M.S. state on $Cl(V)$. Then Φ is the quasi-free state determined by the two-point correlation functions*

$$\Phi([\Gamma(u), \Gamma(v)]) = \int th(\tfrac{1}{2}\beta\hbar s)((Fv(-s), Fu(s)) - (Fu(-s), Fv(s)))\, ds.$$

Proof. If H is the infinitesimal generator of U_t then we may proceed as in [11] Example 5.3.2 to show that Φ is that quasi-free state which satisfies

$$\Phi(\Gamma(u)\Gamma(v)) = 2(u, (1 + \exp(\beta H))^{-1} v)$$

$$= 2\int (u(x), ((1 + \exp(\beta H))^{-1} v)(x))\, dx$$

$$= 2\int ((Fu)(-s), (F(1 + \exp(\beta H))^{-1} v)(s))\, ds,$$

(the last line following from Plancherel's Theorem). Using the explicit form of U_t and thence of H this gives

$$\Phi(\Gamma(u)\Gamma(v)) = 2 \int (1 + \exp(\beta\hbar s))^{-1}((Fu)(-s), (Fv)(s)) \, ds$$

$$= \int (1 + \exp(\beta\hbar s))^{-1}(Fu(-s), Fv(s)) \, ds$$

$$- \int (1 + \exp(-\beta\hbar s))^{-1}(Fv(-s), Fu(s)) \, ds.$$

The stated result now follows by elementary manipulation.

The first important step towards establishing the asymptotic behaviour of the time evolution is the mixing property.

Proposition 3. *For any normal state Φ' on the G.N.S. representation space defined by Φ and for any A in $Cl(V)$ one has*

$$\lim_{t \to \infty} \Phi'(\alpha_t(A)) = \Phi(A).$$

Proof. This result which depends mainly on the fact that Φ is separating together with an application of the Riemann–Lebesgue Lemma can be established by minor modifications of the boson proof given by Maassen in [9], I.9.3 and II.5.2.

To link the limiting value given here with the correlated state of the system and pointer described in the previous section we must first give some thought to the relative values of the various parameters which appear. A reliable measuring instrument should not be seriously disturbed by thermal agitation. In other words the energy $\hbar\omega$ required to shift the system and apparatus out of correlation should be significantly larger than the thermal energy $1/\beta$. Similarly it makes sense to consider the case in which the classical evolution is underdamped, that is when $\omega \gg f$. If we introduce the dimensionless parameters $\kappa = f/\omega$ and $\sigma = \beta\hbar\omega$ then we are interested in the behaviour when κ is very small and σ is very large.

The algebra of linear operators on $l^2(\Lambda)$ can be identified with $Cl(M)$. Furthermore the embedding j of M into V gives rise to a homomorphism from $Cl(M)$ to $Cl(V)$ which sends $\gamma(m)$ to $\Gamma(jm)$. We shall write A' for the image in $Cl(V)$ of an element A in $Cl(M)$. In this way observables associated with the smaller phase space can be considered as observables associated to the larger space as well. Since Φ is quasi-free the asymptotic values assigned to these observables by the mixing property can all be expressed as polynomials in quantities of the form $\Phi([\Gamma(jm), \Gamma(jn)])$. The following result shows that in the range which concerns us this differs but little from the correlated state $\varphi([\gamma(m), \gamma(n)])$.

Theorem 1. *For m and n in M and κ, σ as above*

$$\Phi([\Gamma(jm), \Gamma(jn)]) = \varphi([\gamma(m), \gamma(n)])(1 + O(\kappa) + O(\sigma^{-2})).$$

Proof. Recalling the earlier expressions for Φ and Fjm we see that

$\Phi([\Gamma(jm), \Gamma(jn)])$ is equal to

$$(f/\pi) \int th(\tfrac{1}{2}\beta\hbar s)((s^2 - \omega^2)^2 + f^2 s^2)^{-1}$$

$$\times \{(p(\omega J - is)n, p(\omega J + is)m) - (p(\omega J - is)m, p(\omega J + is)n)\} \, ds.$$

By using the skew-adjointness of J and the self-adjointness of p the expression in curly brackets can be simplified to

$$(m, (\omega J + is)p(\omega J + is)n) - (m, (\omega J - is)p(\omega J - is)n) = 2i\omega s(m, (pJ + Jp)n)$$
$$= 2i\omega s(m, Jn).$$

Substituting this back into the integral we obtain

$$\Phi([\Gamma(jm), \Gamma(jn)]) = (2i\omega f/\pi)(m, Jn) \int sth(\tfrac{1}{2}\beta\hbar s)((s^2 - \omega^2)^2 + f^2 s^2)^{-1} \, ds$$

$$= (\kappa/\pi)\varphi([\gamma(m), \gamma(n)]) \int uth(\tfrac{1}{2}\sigma u)((u^2 - 1)^2 + \kappa^2 u^2)^{-1} \, du$$

where we have set $s = \omega u$.

The integral can be rewritten as

$$\int |u| \, ((u^2 - 1)^2 + \kappa^2 u^2)^{-1} \, du - 2 \int |u| \, (1 + \exp(\sigma |u|))^{-1}((u^2 - 1)^2 + \kappa^2 u^2)^{-1} \, du.$$

The first of these two integrals can be evaluated by substituting for u^2 and gives

$$\kappa^{-1}(1 - \tfrac{1}{4}\kappa^2)^{-1/2}(\pi - \tan^{-1}(\kappa(1 - \tfrac{1}{4}\kappa^2)^{1/2}/(1 - \tfrac{1}{2}\kappa^2))) = \kappa^{-1}(\pi - \kappa + O(\kappa^2)).$$

The leading terms in the asymptotic expansion of the second integral for large σ are easily obtained by Laplace's method and give $2\sigma^{-2}$. Substitution of these values back into the formula for $\Phi([\Gamma(jm), \Gamma(jn)])$ now gives the result.

Corollary (i). *For any A in $Cl(M)$ the expectation values $\Phi(A')$ and $\varphi(A)$ differ only by terms of order κ and σ^{-2}.*

Proof. Since both Φ and φ are quasi-free states the expectation values of general observables are given by identical polynomials in the respective two-point correlation functions. The result therefore follows immediately from the theorem.

Corollary (ii). *For any A in $Cl(M)$ and any normal state Φ' of $Cl(V)$ on the G.N.S. space defined by Φ*

$$\lim_{t \to \infty} \Phi'(\alpha_t(A')) = \varphi(A)(1 + O(\kappa) + O(\sigma^{-2})).$$

Proof. This follows immediately from Proposition 3 and Corollary (i).

This result shows that the asymptotic values of observables associated with the system and pointer are close to their values in the correlated state.

5. The reduction of the system and pointer

For a more precise description of the effects of the collapse we must reinstate the space $\mathcal{H}$ and make explicit some of the identifications which we have used. First let us choose unitary operators W_λ on $l^2(\Lambda)$ such that

$$(W_\lambda \Omega)(\mu) = \delta_{\lambda\mu}$$

for all λ and μ in Λ. If we define $W = \sum W_\lambda \otimes P_\lambda$ on $l^2(\Lambda) \otimes \mathcal{H}$ then bearing in mind the fact that Q is the projection onto Ω we see that

$$W(Q \otimes 1)W^* = \sum W_\lambda Q W_\lambda^* \otimes P_\lambda$$
$$= P.$$

So W provides an explicit equivalence between $Q \otimes 1$ and P. Since each W_λ can be identified with an element of $Cl(M)$ it can also be mapped to an element W_λ' in $Cl(V)$. We shall write W' for the element $\sum W_\lambda' \otimes P_\lambda$ in the (algebraic) tensor product $Cl(V) \otimes \mathcal{B}(\mathcal{H})$, (where, as usual, $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$). Just as W connects $l^2(\Lambda) \otimes \mathcal{H}$ with the formalism of the last two sections, W' can be used to interpret the results of those sections by allowing the inclusion of $\mathcal{H}$ as well.

There is a natural time evolution τ_t on $Cl(V) \otimes \mathcal{B}(\mathcal{H})$ defined by

$$\tau_t(C) = W'((\alpha_t \otimes 1)(W'^*CW'))W'^*.$$

The ultimate fate of the system and apparatus in the ensuing evolution is just what one would expect.

Theorem 2. *Let Φ' be a normal state of $Cl(V)$ on the G.N.S. space defined by Φ, let Ψ be a normal state of $B(\mathcal{H})$ on $\mathcal{H}$, and let A_λ be the projection onto $W_\lambda \Omega$ in $l^2(\Lambda)$. Then*

(i) $\displaystyle\lim_{t\to\infty} (\Phi' \otimes \Psi)(\tau_t(1 \otimes B)) = \Psi(\sum P_\lambda B P_\lambda)(1 + O(\kappa) + O(\sigma^{-2}));$

(ii) $\displaystyle\lim_{t\to\infty} (\Phi' \otimes \Psi)(\tau_t(A_\lambda' \otimes 1)) = \Psi(P_\lambda)(1 + O(\kappa) + O(\sigma^{-2})).$

Before giving the proof we just remark that apart from the terms of order κ and σ^{-2} these are precisely the first two conditions suggested by Whitten-Wolfe and Emch, [3], [4]. (Their other three conditions are also satisfied by this model.) The first conclusion tells us that the system is reduced by the measurement in the expected way, the second that there are observables, the A_λ', which enable us to read off the pointer positions.

Proof of the Theorem. For C in $Cl(V) \otimes \mathcal{B}(\mathcal{H})$ we have

$$(\Phi' \otimes \Psi)(\tau_t(C)) = \sum (\Phi' \otimes \Psi)((W_\lambda' \otimes P_\lambda)((\alpha_t \otimes 1)(W'^*CW'))(W_\mu'^* \otimes P_\mu)).$$

Now, by applying Proposition 3 to normal states of the form

$$A \to \Phi'((\sum a_\lambda W_\lambda)A(\sum a_\mu W_\mu)^*)$$

and comparing the coefficients of $a_\lambda \bar{a}_\mu$ for various choices of scalars $\{a_\lambda\}$, we see

that

$$\lim_{t \to \infty} \Phi'(W'_\lambda \alpha_t(A) W''^*_\mu) = \Phi'(W'_\lambda W''^*_\mu) \Phi(A)$$

for any A in $Cl(V)$. We therefore have

$$\lim_{t \to \infty} (\Phi' \otimes \Psi)(\tau_t(C)) = \sum \Phi'(W'_\lambda W''^*_\mu)(\Phi \otimes \Psi)((1 \otimes P_\lambda) W''^* C W'(1 \otimes P_\mu)).$$

Now

$$(1 \otimes P_\lambda) W''^* C W'(1 \otimes P_\mu) = (W''^*_\lambda \otimes P_\lambda) C (W'_\mu \otimes P_\mu).$$

If C has the form $A' \otimes B$ for A in $Cl(M)$ and B in $\mathcal{B}(\mathcal{H})$ then we can apply the first corollary to Theorem 1 to deduce that

$$(\Phi \otimes \Psi)((W''^*_\lambda \otimes P_\lambda) C (W'_\mu \otimes P_\mu))$$

differs from

$$(\varphi \otimes \Psi)((W^*_\lambda \otimes P_\lambda)(A \otimes B)(W_\mu \otimes P_\mu))$$

only by terms of order κ and σ^{-2}. The latter expression simplifies down to

$$(\varphi \otimes \Psi)((W^*_\lambda A W_\mu) \otimes (P_\lambda B P_\mu)) = \varphi(W^*_\lambda A W_\mu) \Psi(P_\lambda B P_\mu).$$

Substituting these back we see that the limit of $(\Phi' \otimes \Psi)(\tau_t(C))$ is close to

$$\sum \Phi'(W'_\lambda W''^*_\mu) \varphi(W^*_\lambda A W_\mu) \Psi(P_\lambda B P_\mu).$$

To consider observables on the system we set $A = 1$. But

$$\varphi(W^*_\lambda W_\mu) = \langle W_\lambda \Omega, W_\mu \Omega \rangle = \delta_{\lambda\mu}$$

so that the limit of $(\Phi' \otimes \Psi)(\tau_t(1 \otimes B))$ is close to

$$\sum \Phi'(1) \Psi(P_\lambda B P_\lambda) = \Psi\left(\sum P_\lambda B P_\lambda\right).$$

For observables on the pointer we have rather $B = 1$. This time the approximation to the limit is

$$\sum \Phi'(1) \varphi(W^*_\lambda A W_\lambda) \Psi(P_\lambda) = \sum \langle W_\lambda \Omega, A W_\lambda \Omega \rangle \Psi(P_\lambda).$$

On taking A to be the projection onto $W_\lambda \Omega$ the second part of the theorem follows.

6. Refinements and outlook

The model of the measuring process outlined above can be refined various ways. One interesting possibility is to consider the measurement of an observable with continuous spectrum. This is most easily done in the operational formalism which has been given a precise and rigorous form by Davies and Lewis, [12], [13], [14], [15].

For our purposes it is more convenient to transpose the operation and to start with a positive linear map E from $C(\Lambda) \otimes \mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$, where Λ is now

allowed to be any compact topological space. We assume that E is an instrument, that is it maps $1 \otimes 1$ to 1, and that it is completely positive. (In [15] Davies adduces some reasons based on the stability of measurement which might lead one to consider this a plausible requirement. Both conditions hold when Λ is finite and

$$E(f \otimes B) = \sum f(\lambda) P_\lambda B P_\lambda,$$

which is the case which we were considering in Section 2.)

The Stinespring construction, [15], [16], enables us to dilate E and find a space $\mathcal{K}$ on which there is a *-representation R of $C(\Lambda) \otimes B(\mathcal{H})$ and a map I from $\mathcal{H}$ to $\mathcal{K}$ such that

$$I^* R(f \otimes B) I = E(f \otimes B).$$

Taking $f \otimes B = 1 \otimes 1$ we see that I is an isometry, and so $P = I^* I$ is a distinguished projection on $\mathcal{K}$. This provides the same basic data as that with which we started our constructions of the previous three sections.

We may summarise the work presented in this paper and in [1]–[4] as showing that it is possible to find mathematical models in which continuous interaction with the measuring apparatus and environment makes the measured system approach arbitrarily closely to the reduced state. In practice because of the very short time scales of quantum phenomena the asymptotic state could be approached very rapidly indeed, and although coherence is never really lost during continuous evolution it can be swiftly dissipated over vast regions of space.

Unfortunately our model provides no precise estimate of the time-scale involved, and in this its mathematical generality proves to be a practical disadvantage. The most that one can do is to note that the only time scales which appear explicitly are the relaxation time, f^{-1}, the thermal relaxation time, $\hbar \beta$, and ω^{-1}, which is, by hypothesis, much shorter than the other two. Of these $\hbar \beta$ seems most likely to be relevant. Under the most naive assumption that the appropriate environmental temperature is that of the laboratory one then arrives at a time scale of about 10^{-14} seconds. Relaxation times of this order of magnitude have been suggested in a variety of attempts to derive the reduction from some kind of continuous evolution and, in particular, they appear in the hidden variable theory of Bohm and Bub, [17]. The experiment of Papaliolios, [18], found no evidence of gradual decay between measurements of the order of 10^{-13} seconds apart, but this is still too long to eliminate even the most naive decay models. (It would be interesting to repeat the experiment at low temperature with $\hbar \beta$ of the order of 10^{-11} seconds.)

Finally we remark on one curious feature of models of this kind. Since they rely only on the conventional formalism of quantum mechanics their effects should in principle occur under suitable conditions whether or not they are relevant to the measuring process. If experiments vindicate the standard picture of an instantaneous collapse of the wave function then there will be two kinds of reduction in nature which are virtually indistinguishable on the ordinary time scale.

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