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Spectral concentration for Hamiltonian systems

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Abstract. We consider a sequence of operators T_n arising from Hamiltonian systems and require that $T_n \to T$, where T corresponds to a fixed Hamiltonian system. Letting $\tau_n(\lambda)$ and $\tau(\lambda)$ denote the respective spectral functions, we give sufficient conditions that $\tau_n(\lambda_2)-\tau_n(\lambda_1) \to \tau(\lambda_2)-\tau(\lambda_1)$ at continuity points λ_1 and λ_2 of $\tau(\lambda)$.

1. Introduction

'Spectral concentration' is a term arising from perturbation theory when the spectrum $\sigma(T_0)$ of an operator T_0 acting in a Hilbert space H is compared to the spectrum $\sigma(T_\varepsilon)$ of a perturbed operator T_ε, where $\varepsilon > 0$ is a small parameter and $T_\varepsilon \to T_0$ (in a suitable sense) as $\varepsilon \to 0$. In the classical cases of physical interest $\sigma(T_\varepsilon)$ is qualitatively different from, but quantitatively near to, $\sigma(T_0)$.

The expression itself originated with work of Titchmarsh ([22], [23], [24]); also see [20, p. 60]) on the mathematical formulation of the Stark effect in hydrogen. Here one considers the Sturm–Liouville eigenvalue problem

$$-y''+\left(\frac{l(l+1)}{x^2}-\frac{c}{x}-\varepsilon x\right)y=\lambda y, \qquad 0<x<\infty, \tag{1.1}$$

where l is a positive integer, c is a constant, $\varepsilon > 0$ and λ is the spectral parameter. In [23] the more general equation with εx replaced by $\varepsilon\sigma(x)$ is actually treated, where $\sigma(x) \to \infty$ as $x \to \infty$, and in its forerunner [22] the simpler equation $-y''+\varepsilon xy=\lambda y$, $0 \leq x < \infty$, is studied. For $\varepsilon = 0$ (1.1) is the Hamiltonian in one dimension for a hydrogen atom ([7], [8]) and the perturbed case $\varepsilon > 0$ is the corresponding Hamiltonian in a constant, weak electric field of strength ε. The unperturbed equation has continuous spectrum in $[0, \infty)$ and a sequence of isolated negative eigenvalues clustering at $\lambda = 0$ ([7], [15]). But for any positive ε the spectrum of (1.1) continuously fills the entire λ-axis, with no imbedded eigenvalues ([23], [15]). This apparent discrepancy throughout the negative energy spectrum is most easily explained in terms of the spectral function $\rho_\varepsilon(\lambda)$ for (1.1) (we give precise definitions of all relevant terms form spectral theory in §2 below). For $\lambda < 0$ the unperturbed spectral function $\rho_0(\lambda)(\varepsilon = 0)$ is a nondecreasing step function with jumps at the eigenvalues. But while $\rho_\varepsilon(\lambda)$ is continuously differentiable ([15]) on $-\infty < \lambda < \infty$, the "spectral density" function $\rho_\varepsilon'(\lambda)$ has sharp peaks near the unperturbed eigenvalues and is small elsewhere for $\lambda < 0$.

Thus the main strength of the negative spectrum is concentrated near the unperturbed eigenvalues; see [22], [23], [24], [20, §12.5 and *Notes*] and also [19] for another example and further discussion.

Titchmarsh develops in [22] and [23] a theory which attributes the sharp peaks in $\rho_\varepsilon'(\lambda)$ to poles of the perturbed $m_\varepsilon(\lambda)$ function (defined in §3 below) lying near the eigenvalues but just below the negative real axis in the complex λ-plane. These 'perturbed poles", also called "resonances" ([7]) and "pseudo-eigenvalues" ([19]), transmit their strength to the spectral function through the Titchmarsh–Kodaira formula ([3])

$$\rho_\varepsilon(\lambda_2)-\rho_\varepsilon(\lambda_1)=\lim_{\delta\to 0^+}\pi^{-1}\int_{\lambda_1}^{\lambda_2}\operatorname{Im} m_\varepsilon(\mu+i\delta)\,d\mu, \tag{1.2}$$

which is valid at points of continuity λ_1 and λ_2 of $\rho_\varepsilon(\lambda)$. The resonances have exponentially small (as $\varepsilon\to 0$) imaginary parts in the case of (1.1) ([7], [8]) and are given approximately by Titchmarsh's classical real eigenvalue perturbation formula in [22]. Less well-behaved perturbing potentials may produce resonances which do not satisfy such relations ([7], [6]).

Another feature of the concentration phenomenon, also introduced by Titchmarsh ([24]), seeks to measure the extent of concentration of the spectrum on small intervals about unperturbed eigenvalues. Suppose that T_0 and T_ε are the operators, having spectra $\sigma(T_0)$ and $\sigma(T_\varepsilon)$, respectively, where $\sigma(T_0)$ is discrete and $\sigma(T_\varepsilon)$ is continuous. Let $f\in H=L^2[0,\infty)$ and let $g_\varepsilon(\lambda)$ be the transform of f; see [3, chapter 7], [24] and Remark 5 in §6 below. Then one has the Parseval formula

$$\int_0^\infty |f(x)|^2\,dx=\int_{-\infty}^\infty |g_\varepsilon(\lambda)|^2\,d\rho_\varepsilon(\lambda),$$

and the theory asserts that there are open intervals $\Delta_n=\Delta_n(\varepsilon)$, each Δ_n containing the eigenvalue λ_n and whose diameters $|\Delta_n(\varepsilon)|\to 0$, as $\varepsilon\to 0$, such that if $\Delta=(\bigcup_{n=1}^\infty \Delta_n)$ then

$$\int_0^\infty |f(x)|^2\,dx=\int_\Delta |g_\varepsilon(\lambda)|^2\,d\rho_\varepsilon(\lambda)+0(1),\qquad \varepsilon\to 0. \tag{1.3}$$

It is then a question of how large the intervals Δ_n must be in order that (1.3) should hold. One says, for example, that the spectrum is concentrated to order p if the widths can be taken to be $2\varepsilon^p$; see [4], [19], [24] and the *Notes*, pp. 64–68, of [20] where there are numerous references.

The present paper takes a somewhat different approach to the subject. Although a great deal is known about convergence of the spectral family $E_\varepsilon(\lambda)$ of T_ε to the unperturbed family $E_0(\lambda)$ of T_0, when $T_\varepsilon\to T_0$ in some appropriate way (see [25, p. 286]), there does not appear to be a proof that the spectral jumps $[\rho_\varepsilon(\lambda)]_{\lambda_1}^{\lambda_2}=\rho_\varepsilon(\lambda_2)-\rho_\varepsilon(\lambda_1)$ converge to the corresponding unperturbed expression $[\rho_0(\lambda)]_{\lambda_1}^{\lambda_2}$, as $\varepsilon\to 0$, even for the scalar equation (1.1). The proof of such a convergence theorem, in a systems context which includes (1.1), is the main purpose of this paper.

The convergence of $[\rho_\varepsilon(\lambda)]_{\lambda_1}^{\lambda_2}$ to $[\rho_0(\lambda)]_{\lambda_1}^{\lambda_2}$ is a qualitative measure of concentration. It implies, for example, that in the typical case where $\rho_\varepsilon(\lambda)$ is differentiable in a neighborhood of an isolated unperturbed eigenvalue λ_0, $\rho'_\varepsilon(\lambda)$ must necessarily be small except for a sharp peak very near λ_0.

It will be convenient to take as our basic equation the Hamiltonian system of [5] (and also [1], [10], [11]), which includes scalar equations as one case, on an interval $[a, b)$, $-\infty<a<b\leq\infty$. Our principal tool is a generalization (§4) of a deep result of Atkinson in [2] which links $\rho(\lambda)$ to $m(\lambda)$ in an alternative way to (1.2). Our main convergence result (Theorem 1) is stated in §2, following the introduction of some notation and terminology, and is proved in §5.

A concluding section (§6) contains examples, a discussion of two singular endpoint problems and a connection between the convergence theorem and a type of convergence of spectral families.

2. Hamiltonian systems

This section contains background material on Hamiltonian systems and their spectral theory, included here for completeness.

We consider the $2k\times 1$ first order system ([5], [1])

$$Jy'=[\lambda A(x)+B(x)]y, \qquad a\leq x<b\leq\infty, \qquad \lambda \text{ complex}, \tag{2.1}$$

where J, A and B are $2k\times 2k$ complex matrices, $A^*=A$ and $B^*=B$ (conjugate transpose), $A\geq 0$ (nonnegative definite) and $J=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}$, I being the $k\times k$ identity. We regard (2.1) as regular at $x=a$, singular at $x=b$ and take the coefficients to be locally integrable so that the usual existence and uniqueness properties hold. Let A be decomposable as

$$A(x)=\begin{pmatrix}A_1(x) & 0\\ 0 & 0\end{pmatrix}, \tag{2.2}$$

where $A_1(x)$ is $r\times r$ and invertible, $1\leq r\leq 2k$, $A=A_1$ in case $r=2k$, and finally let us assume the definiteness hypothesis ([1, p. 253])

$$\int_\alpha^\beta y^*(x)A(x)y(x)\,dx>0, \qquad a<\alpha<\beta<b, \tag{2.3}$$

for all solutions y of (2.1) which do not vanish identically.

Scalar equations $-(pu')'+qu=\lambda wu$ are special cases of (2.1), as can be seen from the identity

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}u\\ pu'\end{pmatrix}'=\left[\lambda\begin{pmatrix}w & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}-q & 0\\ 0 & p^{-1}\end{pmatrix}\right]\begin{pmatrix}u\\ pu'\end{pmatrix}. \tag{2.4}$$

The Dirac systems studied in [18] comprise another special case.

Associated with (2.1) is the linear space ([1]) $L^2_A[a, b)$ of equivalence classes of measurable functions $f(x)$ defined on $[a, b)$ such that $\int_a^b f^*(x)A(x)f(x)\,dx<\infty$, and it is natural to introduce the inner product $(f, g)_A=\int_a^b f^*Ag$ for $f, g\in L^2_A[a, b)$. There is induced the seminorm $\|f\|_A=(f, f)_A^{1/2}$, which is a norm in the event that A is invertible. Let P_r be the $2k\times 2k$ matrix $\begin{pmatrix}I_r & 0\\ 0 & 0\end{pmatrix}$, where I_r is the identity matrix which has the size of $A_1(x)$ in (2.2), and for $g \in L^2_A[a, b)$ define $g_1 = P_r g$. Letting $P_r L^2_A[a, b) = \{P_r g \mid g \in L^2_A[a, b)\}$, we have $\|f\|^2_A = \int_a^b f_1^* A_1 f_1$ for all $f \in P_r L^2_A[a, b)$. Since A_1 is invertible then $\|\cdot\|_A$ restricted to $P_r L^2_A[a, b)$ is a norm; hence $P_r L^2_A[a, b)$ is a Hilbert space under the inner product $(\cdot, \cdot)_A$.

The limit point-limit circle theory of (2.1) is well-known; see [21], [10] and [13]. We know that the number of linearly independent solutions of (2.1) lying in $L^2_A[a, b)$ is independent of λ in each halfplane $\mathrm{Im}(\lambda) > 0$ and $\mathrm{Im}(\lambda) < 0$, and can range from k to $2k$. If there are exactly k linearly independent solutions for $\mathrm{Im}(\lambda) \neq 0$ we say that (2.1) is of limit-point type at $x = b$. We take as a basic hypothesis in this paper that

$$\text{system (2.1) is of limit point type at } x = b. \tag{2.5}$$

This hypothesis implies, for example, that certain boundary value problems associated with (2.1) have unique solutions. To discuss one particular type of boundary problem it will be convenient to introduce the $2k \times 2k$ matrices E_α given by

$$E_\alpha = \begin{pmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{pmatrix}$$

where the α_j are $k \times k$ and satisfy $\mathrm{rank}[\alpha_1, \alpha_2] = k$, $\alpha_1 \alpha_2^* = \alpha_2 \alpha_1^*$ and $\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* = I$. A typical boundary condition placed on a vector y at $x = a$ is

$$[\alpha_1, \alpha_2] y(a) = v,$$

where v is a $k \times 1$ constant vector. In [10] and [12] the present authors proved under assumption (2.5) that the problem $Jy' = (\lambda A + B)y + Af$, where $f \in L^2_A[a, b)$, $\mathrm{Im}(\lambda) \neq 0$ and $[\alpha_1, \alpha_2] y(a) = v$, has a unique solution $y \in L^2_A[a, b)$. This result has operator-theoretic consequences which we now proceed to describe (see [11], [12]). Define an operator T with domain $D(T) \subset L^2_A[a, b)$ by saying $y \in D(T)$ provided (i) $y \in L^2_A[a, b)$ (ii), y is locally absolutely continuous, (iii) $[\alpha_1, \alpha_2] y(a) = 0$, (iv) $\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} (Jy' - By) \in P_r L^2_A[a, b)$, (v) $\begin{pmatrix} 0 & 0 \\ 0 & I_{2k-r} \end{pmatrix} (Jy' - By) = 0$, and then defining $T: D(T) \to P_r L^2_A[a, b)$ by the equation

$$(Ty)(x) = \begin{pmatrix} A_1^{-1}(x) & 0 \\ 0 & 0 \end{pmatrix} (Jy'(x) - B(x) y(x)). \tag{2.6}$$

Then the existence and uniqueness result cited above may be rephrased by saying that $(T - \lambda P_r): D(T) \to P_r L^2_A[a, b)$ is a one-to-one and onto map whenever $\mathrm{Im}(\lambda) \neq 0$. It also happens ([10], [11], [12]) that $T - \lambda P_r$ has a bounded inverse for $\mathrm{Im}(\lambda) \neq 0$, and in fact

$$\|(T - \lambda P_r)^{-1} f\|_A \leq |\mathrm{Im}(\lambda)|^{-1} \|f\|_A. \tag{2.7}$$

The *resolvent set* (see [11], [12]) $\rho(T)$ of T is defined as the set of all complex λ such that $(T - \lambda P_r)^{-1}: P_r L^2_A[a, b) \to L^2_A[a, b)$ exists and is bounded; thus $\rho(T)$ contains all nonreal numbers by (2.7). The *spectrum* $\sigma(T)$ of T is the complement of $\rho(T)$ in the set of complex numbers. The set of isolated points of $\sigma(T)$ is called the *point spectrum*, and is denoted by $P(T)$. The set $\sigma(T) - P(T)$ is called the *essential spectrum* and the subset $PC(T) \subset E(T)$ consisting of imbedded eigenvalues is termed the *point-continuous spectrum*; see [11]. We call $C(T) =$

$E(T)-PC(T)$ the *continuous spectrum* of T. We cannot appeal to the standard definitions of the above terms because T is not in the strict sense a Hilbert space operator.

The spectral function $\tau(\lambda)$ of the operator T is a $2k\times 2k$ matrix function defined on $-\infty<\lambda<\infty$ which is right-continuous, nondecreasing (in the nonnegative definite sense) and normalized by $\tau(0)=0$; see [11] and [12]. It is constant on that part of the resolvent intersecting the real axis, so that its points of increase comprise $\sigma(T)$, there are jump discontinuities at both isolated and imbedded eigenvalues, and $\tau(\lambda)$ is continuous on $C(T)$. There is an analogue of the Tichmarsh–Kodaira formula (1.2); we give this in §3. In the scalar case where (2.1) reduces to (2.4) $\tau_{22}(\lambda)$ is the usual scalar spectral function.

The convergence theorem we prove will be for spectral functions $\tau_n(\lambda)$, $n=1,2,3,\ldots$, corresponding to operators T_n given by (2.6). By means of the imbedding (2.4) our result will include spectral functions for scalar equations. Thus let there be given sequences $\{A_n(x)\}_{n=1}^{\infty}$, $\{B_n(x)\}_{n=1}^{\infty}$ and $(E_{n,\alpha}\}_{n=1}^{\infty}$ such that the systems

$$Jy'=[\lambda A_n(x)+B_n(x)]y,\qquad a\leqslant x<b\leqslant\infty, \tag{2.1$_n$}$$

satisfy the same hypotheses (2.2)–(2.6) as system (2.1). Let T_n and τ_n denote the corresponding operators and spectral functions. As regards the convergence of T_n to T, we assume specifically that $A_n\to A$ and $B_n\to B$ in the L^1-norm on compact subsets of $[a,b)$; i.e.,

$$\lim_{n\to\infty}\int_c^d\|A_n(x)-A(x)\|\,dx=\lim_{n\to\infty}\int_c^d\|B_n(x)-B(x)\|\,dx=0, \tag{2.8}$$

where $a\leqslant c<d<b$ and where $\|\cdot\|$ denotes the matrix operator norm $\|A(x)\|=\sup\{\|A(x)v\|\mid\|v\|=1\}$. Let $E_{n,\alpha}\to E_\alpha$. Then our principal result may be stated as follows.

Theorem 1. Let λ_1 and λ_2 be points of continuity of the spectral function $\tau(\lambda)$. Then

$$\lim_{n\to\infty}(\tau_n(\lambda_2)-\tau_n(\lambda_1))=\tau(\lambda_2)-\tau(\lambda_1). \tag{2.9}$$

This is proved in §5. The method of proof will cover the case of a continuous perturbation parameter ε; i.e., $A_\varepsilon\to A$ and $B_\varepsilon\to B$, as $\varepsilon\to 0$, in the sense of (2.8). Hence we include operators of the type (1.1) restricted to $0<a\leqslant x<\infty$. Two singular endpoint problems will be discussed in §6.

3. Preliminaries

Titchmarsh–Weyl functions for Hamiltonian systems may be defined in terms of the initial value matrices E_α of §2. Following [10] and [12], let $Y(x,\lambda)$ be the unique fundamental matrix solution of (2.1) which satisfies for all λ the initial condition $Y(a,\lambda)=E_\alpha$, and let us partition $Y(x,\lambda)$ into $k\times k$ blocks by writing

$$Y(x,\lambda)=\begin{pmatrix}\Theta(x,\lambda) & \Phi(x,\lambda)\\ \hat{\Theta}(x,\lambda) & \hat{\Phi}(x,\lambda)\end{pmatrix}. \tag{3.1}$$

Then under hypothesis (2.5) the limit

$$M(\lambda) = -\lim_{x\to b} \Phi^{-1}(x, \lambda)\Theta(x, \lambda), \qquad \mathrm{Im}\,(\lambda) \neq 0, \tag{3.2}$$

exists and defines a $k \times k$ matrix analytic function of λ in the two half λ-planes. We suppress in our notation the dependence of $M(\lambda)$ and $Y(x, \lambda)$ on the initial value matrix E_α. For the scalar case arising via (2.4), $M(\lambda)$ reduces to the usual scalar $m(\lambda)$ function as given, for example, in [3].

Thus there is associated with the operator (2.6) the uniquely defined Titchmarsh–Weyl coefficient (3.2). The $M(\lambda)$ function carries complete information on the spectrum $\sigma(T)$ of T. To quote only part of the main result of [11] (see also [12]) we can say that $\lambda \in \rho(T)$ if and only if M is analytic at λ, $\lambda \in P(T)$ if and only if M has a simple pole at λ and $\lambda \in C(T)$ if and only if M is not analytic at λ and we have $\lim_{\nu\to 0} \nu M(\lambda + i\nu) = 0$; for the characterization of $PC(T)$, see [12].

There is a Titchmarsh–Kodaira formula analogous to (1.2), but for this we need to mention the 'characteristic function' $F(\lambda)$, introduced by F. V. Atkinson in [1], for the operator T. This is a $2k \times 2k$ matrix function which may be shown ([10], [11]) to equal

$$F(\lambda) = \begin{pmatrix} 0 & (\tfrac{1}{2})I \\ (\tfrac{1}{2})I & M(\lambda) \end{pmatrix}. \tag{3.3}$$

In (3.3) we have reversed the sign of $F(\lambda)$ from the way it appears in [10] and [11]. In [11] we prove the following relations between $F(\lambda)$ and $\tau(\lambda)$:

$$\lim_{\delta\to 0+} \pi^{-1} \int_{\lambda_1}^{\lambda_2} \mathrm{Im}\, F(\mu + i\delta)\, d\mu = \tau(\lambda_2) - \tau(\lambda_1), \tag{3.4}$$

($\mathrm{Im}\, F = (F - F)^*/2i$) at points λ_1 and λ_2 of continuity of τ;

$$F(\lambda) = \int_{-\infty}^{\infty} \left(\frac{1}{\mu - \lambda} - \frac{\mu}{1 + \mu^2}\right) d\tau(\mu) + K_1\lambda + K_2, \qquad \mathrm{Im}\,(\lambda) \neq 0, \tag{3.5}$$

for constant Hermitian matrices $K_1 \geq 0$ and K_2. Recall in (3.5) that $d\tau(\mu) \geq 0$.

Let $Y_n(x, \lambda)$ be the fundamental matrix solution of $(2.1)_n$ which satisfies initial values $Y_n(a, \lambda) = E_{n,\alpha}$, for all λ, where

$$E_{n,\alpha} = \begin{pmatrix} \alpha^*_{1,n} & -\alpha^*_{2,n} \\ \alpha^*_{2,n} & \alpha^*_{1,n} \end{pmatrix} \to E_\alpha, \qquad n \to \infty,$$

and where the $\alpha_{i,n}$ satisfy the same hypotheses as the α_i in E_α. Let us denote by $M_n(\lambda)$ and $F_n(\lambda)$ the corresponding Titchmarsh–Weyl coefficient and characteristic function for the operator T_n arising via (2.6) from equation $(2.1)_n$ and initial values $E_{n,\alpha}$. We are going to establish that $M_n(\lambda) \to M(\lambda)$, $n \to \infty$, uniformly on compact subsets of $\mathrm{Im}\,(\lambda) \neq 0$ (so that the same is true for $F_n \to F$ by (3.3)) and the first step in doing so is the following.

Lemma 3.1. *Let $[c, d]$ be a compact subinterval of $[a, b)$ and let K be a compact subset of the λ-plane. Then $Y_n(x, \lambda) \to Y(x, \lambda)$, $n \to \infty$, uniformly on $[c, d] \times K$.*

Proof. We have for $a \leqslant x \leqslant d$ and $\lambda \in K$ that

$$\begin{aligned} Y(x,\lambda)-Y_n(x,\lambda) &= (E_\alpha - E_{n,\alpha}) + J^{-1}\int_a^x [(\lambda A+B)Y-(\lambda A_n+B_n)Y_n]\,dt \\ &= (E_\alpha - E_{n,\alpha}) + J^{-1}\int_a^x (\lambda A_n + B_n)(Y-Y_n)\,dt + J^{-1} \\ &\quad \times \int_a^x [\lambda(A-A_n)+(B-B_n)]Y\,dt. \end{aligned}$$

It follows that

$$\begin{aligned} \|Y(x,\lambda)-Y_n(x,\lambda)\| &\leqslant \|E_\alpha - E_{n,\alpha}\| + \int_a^d (|\lambda|\,\|A-A_n\|+\|B-B_n\|)\,\|Y\|\,dt \\ &\quad + \int_a^d (|\lambda|\,\|A_n\|+\|B_n\|)\,\|Y-Y_n\|\,dt \\ &= \varepsilon_n + \int_a^d r_n(t)\,\|Y-Y_n\|\,dt \end{aligned}$$

where ε_n stands for the first two terms on the right of the inequality and $r_n(t)$ is the first part of the integrand in the third. From the Gronwall inequality ([9, p. 24])

$$\|Y-Y_n\| \leqslant \varepsilon_n \exp \int_a^d r_n(t)\,dt,$$

and this expression clearly tends to 0 uniformly over K because of (2.8) and the fact that $\|Y(x,\lambda)\|$ is uniformly bounded on $[c,d]\times K$. This completes the proof.

Lemma 3.2. *Let K^+ be a compact subset of* $\operatorname{Im}(\lambda)>0$. *Then $M_n(\lambda)\to M(\lambda)$ uniformly on K^+ (and therefore $F_n(\lambda)\to F(\lambda)$ uniformly on K^+).*

Proof. We will represent $M_n(\lambda)$ and $M(\lambda)$ by the matrix circle method of [13]. For $a<d<b$ and $\operatorname{Im}(\lambda)>0$ let

$$\begin{pmatrix} \mathscr{A} & \mathscr{B}^* \\ \mathscr{B} & \mathscr{D} \end{pmatrix}(d,\lambda) = -iY^*(d,\lambda)JY(d,\lambda) \tag{3.6}$$

and then let

$$\begin{aligned} C_d(\lambda) &= -\mathscr{D}^{-1}(d,\lambda)\mathscr{B}(d,\lambda), \\ R_d^1(\lambda) &= \mathscr{D}^{-1/2}(d,\lambda), \\ R_d^2(\lambda) &= [\mathscr{B}^*\mathscr{D}^{-1}\mathscr{B}-\mathscr{A}]^{1/2}(d,\lambda); \end{aligned} \tag{3.7}$$

see [13] for proofs that these expressions exist. Let $\mathscr{A}_n$, $\mathscr{B}_n$, etc, be the corresponding expressions generated by the fundamental matrix $Y_n(d,\lambda)$. Then there are matrices $V_d(\lambda)$ and $V_{n,d}(\lambda)$ such that $V_d^*V_d \leqslant I$, $V_{n,d}^*V_{n,d} \leqslant I$ (see [13]) and

$$\begin{aligned} M &= C_d + R_d^1 V_d R_d^2, \\ M_n &= C_{n,d} + R_{n,d}^1 V_{n,d} R_{n,d}^2. \end{aligned}$$

Therefore

$$\|M(\lambda)-M_n(\lambda)\| \leq \|C_d(\lambda)-C_{n,d}(\lambda)\| + \|R_d^1(\lambda)\|\,\|R_d^2(\lambda)\| + \|R_{n,d}^1(\lambda)\|\,\|R_{n,d}^2(\lambda)\|. \tag{3.8}$$

We know from [13] that $R_d^1(\lambda)$ and $R_d^2(\lambda)$ are continuous in d and λ and decreases to 0 as d increases to b; hence $R_d^1(\lambda)\to 0$ and $R_d^2(\lambda)\to 0$ uniformly on K^+ as $d\to b$. Thus given $\varepsilon>0$ we may choose d sufficiently near b that $\|R_d^i(\lambda)\|<\varepsilon$. Having fixed d we then note the formulas (3.6) and (3.7) and invoke Lemma 3.1 to conclude $\|C_d(\lambda)-C_{n,d}(\lambda)\|<\varepsilon$ and $\|R_d^i(\lambda)-R_{n,d}^i(\lambda)\|<\varepsilon$ for n sufficiently large. Then from (3.8) we get $\|M(\lambda)-M_n(\lambda)\|<\varepsilon+\varepsilon^2+4\varepsilon^2$, when n is large and $\lambda\in K^+$, and from there the conclusion of the lemma.

4. Atkinson inequality

Here we give a matrix version of Atkinson's extension in [2] of the Titchmarsh–Kodaira formula (1.2). In our case we shall be extending (3.4).

Let $\delta>0$, $\lambda_1<\lambda_2$ and define

$$G(\lambda_1,\lambda_2) = \operatorname{Im}\int_{\lambda_1+i\delta}^{\lambda_2+i\delta} F(\lambda)\,d\lambda - [\pi\tau(\mu)]_{\lambda_1}^{\lambda_2} - [\delta \operatorname{Re} F(\mu+i\delta)]_{\lambda_1}^{\lambda_2}, \tag{4.1}$$

where $\operatorname{Re} F=(F+F^*)/2$, and define

$$H(\lambda_1,\lambda_2) = (\pi\delta/2)[\operatorname{Im} F(\lambda_1+i\delta)+\operatorname{Im} F(\lambda_2+i\delta)]. \tag{4.2}$$

We choose for path of integration in (4.1) the horizontal line joining $\lambda_1+i\delta$ to $\lambda_2+i\delta$, so that $d\lambda$ is real and the symbol Im can be moved under the integral sign and attached to $F(\lambda)$.

Lemma 4.1. *The functions G and H are Hermitian, $H>0$ and*

$$-H\leq G\leq H \text{ (nonnegative definite sense).} \tag{4.3}$$

Proof. The Hermitian properties follow from that of τ ([11]) and the definitions of $\operatorname{Re} F$ and $\operatorname{Im} F$. Positive definitemness of H is a consequence of the Pick–Nevanlinna property $\operatorname{Im} F(\lambda)>0$ for $\operatorname{Im}(\lambda)>0$; see [10] and [11].

Emulating Atkinson's proof, we take (3.5) as the starting point for the proof of (4.3). For the purpose of the proof, in fact, it is best to regard $\tau(\mu)$ as given and (3.5) as the definition of $F(\lambda)$, for this opens the way for some preliminary reductions. Note first that the constant K_2 does not appear in either (4.1) or (4.2) due to cancellation. Hence there is no loss in generality in setting $K_2=0$. Furthermore, K_1 does not appear in (4.1), because its effect is cancelled between the first and third terms of (4.1), but it contributes a nonegative definite term to (4.2). Therefore (4.3) is true for general $K_1\geq 0$ if it is true for $K_1=0$. This being the case, we take $K_1=K_2=0$ and prove (4.3) for the special form

$$F(\lambda)=\int_{-\infty}^{\infty}\left(\frac{1}{\mu-\lambda}-\frac{\mu}{1+\mu^2}\right)d\tau(\mu). \tag{4.4}$$

A final preliminary reduction is accomplished by replacing $\tau(\mu)$ by a simplified function $\tau(\mu; R)$ which is nondecreasing and agrees with $\tau(\mu)$ for $-R \leq \mu \leq R$ but is constant outside $[-R, R]$. Let $F(\lambda; R)$ denote the corresponding characteristic function given by (4.4). Then we have $\tau(\mu; R) \to \tau(\mu)$ and $F(\lambda; R) \to F(\lambda)$ as $R \to \infty$. We may assume that R is large enough that the term involving $\tau(\mu)$ in (4.1) is unaffected by the change. If we can prove (4.3) for compactly supported $d\tau(\mu)$, then we obtain the general result by letting $R \to \infty$. Hence we proceed with the proof of (4.3) for the special case (4.4), and in which $\tau(\mu)$ is constant outside some interval $[-R, R]$.

Introducing

$$f(\lambda, \mu) = \delta\{(\mu - \lambda)^2 + \delta^2\}^{-1},$$

we have $\int_{-\infty}^{\infty} f(\lambda, \mu)\, d\lambda = \pi$ and

$$\int_{\lambda_1+i\delta}^{\lambda_2+i\delta} \operatorname{Im} F(\lambda)\, d\lambda = \int_{-\infty}^{\infty} d\tau(\mu) \int_{\lambda_1}^{\lambda_2} f(\lambda, \mu)\, d\lambda,$$

since the integral on the right is finite in this restricted case of compactly supported $d\tau(\mu)$. Let

$$R(\gamma) = \int_{\gamma}^{\infty} d\tau(\mu) \int_{-\infty}^{\gamma} f(\lambda, \mu)\, d\lambda - \int_{-\infty}^{\gamma} d\tau(\mu) \int_{\gamma}^{\infty} f(\lambda, \mu)\, d\lambda,$$

so that a calculation leads to

$$\int_{\lambda_1+i\delta}^{\lambda_2+i\delta} \operatorname{Im} F(\lambda)\, d\lambda = [\pi\tau(\lambda) + R(\lambda)]_{\lambda_1}^{\lambda_2}.$$

Now by (4.1) $G(\lambda_1, \lambda_2) = [R(\lambda) - \delta \operatorname{Re} F(\lambda + i\delta)]_{\lambda_1}^{\lambda_2}$, so if we let $S(\lambda) = R(\lambda) - \delta \operatorname{Re} F(\lambda + i\delta)$, then what we must prove is

$$-H(\lambda_1, \lambda_2) \leq S(\lambda_2) - S(\lambda_1) \leq H(\lambda_1, \lambda_2). \tag{4.5}$$

The identical argument in Atkinson's proof gives

$$S(\lambda) = \int_{\lambda}^{\infty} d\tau(\mu) h\left(\frac{\mu - \lambda}{\delta}\right) - \int_{-\infty}^{\lambda} d\tau(\mu) h\left(\frac{\lambda - \mu}{\delta}\right), \tag{4.6}$$

where $h(\sigma) = \int_{\sigma}^{\infty} (1 + t^2)^{-1}\, dt - \sigma(1 + \sigma^2)^{-1}$. However, $0 \leq h(\sigma) \leq (\pi/2)(1 + \sigma^2)^{-1}$ and $d\tau(\mu)$ is nonegative. Thus (4.6) implies

$$S(\lambda) \leq (\pi/2) \int_{-\infty}^{\infty} d\tau(\mu)(1 + (\mu - \lambda)^2 \delta^{-2})^{-1}. \tag{4.7}$$

But in (4.4) we can compute $\operatorname{Im} F(\lambda + i\delta)$ by dropping the term $\mu/(1 + \mu^2)$, which is real, and the result is $\operatorname{Im} F(\lambda + i\delta) = \delta \int_{-\infty}^{\infty} d\tau(\mu/((\mu - \lambda)^2 + \delta^2)$. Therefore (4.7) becomes

$$S(\lambda) \leq (\pi\delta/2) \operatorname{Im} F(\lambda + i\delta). \tag{4.8}$$

In (4.6) we could have used the second term to bound $S(\lambda)$ below, and the result would then read

$$S(\lambda) \geq -(\pi\delta/2) \operatorname{Im} F(\lambda + i\delta). \tag{4.9}$$

Combining (4.8) and the reversed negative of (4.9) we get

$$S(\lambda_2)-S(\lambda_1)\leq(\pi\delta/2)[\operatorname{Im} F(\lambda_2+i\delta)+\operatorname{Im} F(\lambda_1+i\delta)].$$

The other half of (4.3) follows from switching the roles of (4.8) and (4.9) as used above. This completes the proof of the lemma.

Since the norm of a Hermitian matrix equals the maximum of the moduli of its eigenvalues, (4.3) gives the useful inequality

$$\|G\|\leq\|H\|. \tag{4.10}$$

5. Proof of main result

Assume the hypotheses of Theorem 1. Then by (4.1) we have for any $\delta>0$

$$\begin{aligned}\pi[\tau(\mu)-\tau_n(\mu)]_{\lambda_1}^{\lambda_2}=\operatorname{Im}\int_{\lambda_1+i\delta}^{\lambda_2+i\delta}&[F(\lambda)-F_n(\lambda)\,d\lambda\\ &-\delta[\operatorname{Re} F(\mu+i\delta)-\operatorname{Re} F_n(\mu+i\delta)]_{\lambda_1}^{\lambda_2}\\ &-G(\lambda_1,\lambda_2)+G_n(\lambda_1,\lambda_2),\end{aligned} \tag{5.1}$$

where G_n is the function of (4.1) formed from τ_n and F_n; we will denote by H_n the corresponding function of (4.2). Each of the first two terms on the right of (5.1) can be made small by making $n\to\infty$, once δ has been fixed, by the uniform convergence of F_n to F on the straight line path of integration, in the first instance, and at its endpoints in the second. We will thus pick δ so that the expression $G_n(\lambda_1,\lambda_2)-G(\lambda_1,\lambda_2)$ is also small for large n.

Since λ_1 and λ_2 are in either the continuous spectrum or resolvent set, we have $\delta F(\lambda_j+i\delta)\to 0$, as $\delta\to 0$, by the result in [11] quoted below (3.2). Consequently, $H(\lambda_1,\lambda_2)$ can be made small by making δ suitably small, and the same must hold for $G(\lambda_1,\lambda_2)$ by (4.10). Turning then to the remaining term $G_n(\lambda_1,\lambda_2)$ in (5.1), one has

$$\|G_n(\lambda_1,\lambda_2)\|\leq\|H_n(\lambda_1,\lambda_2)\|\leq\|H_n(\lambda_1,\lambda_2)-H(\lambda_1,\lambda_2)\|+\|H(\lambda_1,\lambda_2)\| \tag{5.2}$$

by the triangle inequality. As noted above, the term $\|H(\lambda_1,\lambda_2)\|$ on the right of (5.2) has been made small by choosing δ near 0. For this fixed δ we drive the term adjacent to it to 0 by making $n\to\infty$. This completes the proof of the theorem.

6. Examples, extensions and applications

We close the paper with some examples illustrating Theorem 1, a discussion of two singular endpoint problems and some remarks on convergence of spectral functions.

(1) Consider the Sturm–Liouville equation $-y''+r(x)y-\varepsilon n(x)y=\lambda\omega(x)y$ on $a\leq x<\infty$, where the coefficients are differentiable, $n(x)>0$ and $\omega(x)>0$. Writing the equation as a system, we may apply the methods of [17] and [15] to give conditions where the unperturbed ($\varepsilon=0$) spectrum is purely discrete and the perturbed spectrum ($\varepsilon>0$) is purely continuously differentiable. Briefly, we need

$r(x)$ to dominate $\omega(x)$ for discreteness and $n(x)$ to dominate both $r(x)$ and $\omega(x)$ for continuous perturbed spectrum; see [17] and [15] for technicalities. For example, when $\omega(x)=x^{\alpha}$, $r(x)=x^{k}$ and $n(x)=x^{m}$ the desired situation prevails when $-2<\alpha<k<m<(\alpha+1)$.

(2) Similarly, there are criteria given discrete spectrum and continuous spectrum for the Dirac system

$$y'=\begin{pmatrix} 0 & \lambda(\alpha(x)+\varepsilon\hat{\alpha}(x))-p(x) \\ -\lambda(\alpha(x)+\varepsilon\hat{\alpha}(x))-p(x) & 0 \end{pmatrix}y, \qquad a\leqslant x<\infty,$$

in the unperturbed and perturbed cases, respectively; see [17] and [16] for details. Requirements on the coefficients are similar to Example (1) above; $p(x)$ must dominate $\alpha(x)$ and $\hat{\alpha}(x)$ must dominate $p(x)$. If $\alpha(x)=x^{\gamma}$, $p(x)=x^{k}$ and $\hat{\alpha}(x)=x^{\delta}$, $1\leqslant x<\infty$, then $(-\frac{1}{2})\leqslant\gamma<k<\delta$ will be sufficient.

(3) Our results apply to perturbations of the type $-y''+(\ln x-x^{\varepsilon})y=\lambda y$, $1\leqslant x<\infty$, $2\geqslant\varepsilon\geqslant 0$. As in Example (1), [15] shows there to be purely continuous spectrum over $(-\infty,\infty)$ when $\varepsilon>0$. For $\varepsilon=0$ the spectrum is discrete because $\ln x\to\infty$ as $x\to\infty$; see [3].

(4) Suppose now that (2.1) and $(2.1)_n$ are singular and of limit point type at both ends of (a,b). Selfadjoint operators T and T_n are defined as in (2.6) except that no boundary condition of the type (iii), above (2.6), is necessary. The characteristic function for T is given by (see [14])

$$F(\lambda)=\begin{pmatrix} (M_a-M_b)^{-1} & (\frac{1}{2})(M_a-M_b)^{-1}(M_a+M_b) \\ (\frac{1}{2})(M_a+M_b)(M_a-M_b)^{-1} & M_b(M_a-M_b)^{-1}M_a \end{pmatrix}, \tag{6.1}$$

where $M_a(\lambda)$ and $M_b(\lambda)$ are the Titchmarsh–Weyl functions for T at $x=a$ and $x=b$, respectively. The spectral function $\tau(\lambda)$ is linked to $F(\lambda)$ by precisely the same formulas (3.4) and (3.5), and the spectrum $\sigma(T)$ may be characterized by the regular, pole and singular structure of F just as we described below (3.2) for the one endpoint problem. All this was done in [14]. If we assume that (2.8) holds, then it is true that $F_n(\lambda)\to F(\lambda)$ on compact subsets of $\operatorname{Im}(\lambda)>0$ by (6.1). So the proof of Theorem 1 would be valid for (6.1). The conclusion is therefore valid for two singular endpoint problems, including (1.1).

(5) We apply our results now to spectral families of scalar (limit point) operators $Ly=-y''+q(x)y$, $0\leqslant x<\infty$, where with $\omega(x)=p(x)=1$ in (2.4), Ly appears as the first component in (2.6). The domain of L is defined in the usual way by assigning a boundary condition $\cos\alpha y(0)+\sin\alpha y'(0)=0$. Letting $\varphi(x,\lambda)$ be the solution of $-y''+q(x)y=\lambda y$ with $\varphi(0,\lambda)=-\sin\alpha$ and $\varphi'(0,\lambda)=\cos\alpha$, the "transform" with respect to L of a function $f\in L^2[0,\infty)$ is defined as ([3])

$$g_0(\lambda)=\int_0^{\infty} f(t)\varphi(t,\lambda)\,dt.$$

If $\rho(\lambda)$ is the spectral function for L, then the eigenfunction expansion

$$f(t)=\int_{-\infty}^{\infty} g_0(\lambda)\varphi(t,\lambda)\,d\rho(\lambda)$$

converges in the mean to $f\in L^2[0,\infty)$; see [3]. The spectral family ([25]) of L

consists of the projection operators on $L^2[0,\infty)$ given by

$$[S(\mu)f](t)=\int_{-\infty}^{\mu} g_0(\lambda)\varphi(t,\lambda)\,d\rho(\lambda).$$

Now let $\{q(x)\}$ be a sequence of potentials such that $q_n(x)\to q(x)$ uniformly on compact subsets of $[0,\infty)$. Then it follows from [25, pp. 283–286] that for the corresponding operators L_n (with same boundary condition at $x=0$ as L), $L_n\to L$ in the strong resolvent sense, and therefore that $S_n(\mu)\to S(\mu)$ (i.e., $S_n(\mu)f\to S(\mu)f$ in L^2 norm for all $f\in L^2[0,\infty)$) where $S_n(\mu)$ is the spectral family for L_n.

We now use Theorem 1 to prove *pointwise* convergence of the projection operators $[S_n(\mu_2)-S_n(\mu_1)]f$ to $[S(\mu_2)-S(\mu_1)]f$, where ρ is continuous at μ_1 and μ_2, for functions f of compact support. For if f is supported on $[c,d]$, then

$$\begin{aligned}[S_n(\mu_2)-S_n(\mu_1)]f(x)&=\int_{\mu_1}^{\mu_2}\left[\int_c^d f(t)\varphi_n(t,\lambda)\,dt\right]\varphi_n(x,\lambda)\,d\rho_n(\lambda)\\&=\int_c^d f(t)\left[\int_{\mu_1}^{\mu_2}\varphi_n(t,\lambda)\varphi_n(x,\lambda)\,d\rho_n(\lambda)\right]dt. \qquad (6.2)\end{aligned}$$

By an extension of the proof of the Helly–Bray Theorem in [26, p. 31], $\int_{\mu_1}^{\mu_2}\varphi_n(t,\lambda)\varphi_n(x,\lambda)\,d\rho_n(\lambda)\to\int_{\mu_1}^{\mu_2}\varphi(t,\lambda)\varphi(x,\lambda)\,d\rho(x)$, $n\to\infty$, uniformly for $c\leqslant t\leqslant d$, in view of Theorem 1. Hence (6.2) approaches $[S(\mu_2)-S(\mu_1)]f(x)$, as $n\to\infty$, and this is the desired result.

The above proof shows that $(S_n(\mu)f)(x)\to(S(\mu)f)(x)$, pointwise for functions f of compact support, if the spectra of L and L_n are bounded below by a single number, say $d\rho_n(\lambda)=d\rho(\lambda)=0$ on $-\infty<\lambda<\lambda_0$. For the integrals from μ_1 to μ_2 in (6.2) could then be replaced by the integrals from $-\infty$ to μ. To prove the analogous result for spectra unbounded from below would require uniform smallness of the integrals $\int_{-\infty}^{\lambda_0}\varphi_n(t,\lambda)\varphi_n(x,\lambda)\,d\rho_n(\lambda)$ for large negative λ_0 and $n=1,2,3,\ldots$. While we do know that each $\rho_n(\lambda)$ is exponentially small as $\lambda\to-\infty$ ([2, p. 18]) and that the terms $\varphi_n(t,\lambda)\varphi_n(x,\lambda)$ are exponentially bounded as $\lambda\to-\infty$, it is not clear that the resulting bounds are independent of n.

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