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Helvetica Physica Acta, Vol. **59** (1986) 363–366 0018–0238/86/030363–04$1.50 + 0.20/0

Lyapunov exponent, scale symmetry and chaos in Yang–Mills equations

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(7. X. 1985)

Abstract. The chaotic motion of a Hamilton system which arises in classical Yang–Mills equations is investigated. The one-dimensional Lyapunov exponent is used for characterizing the chaotic states. Since the equations of motion are scale invariant the connection between the scale symmetry and the one-dimensional Lyapunov exponent is given.

Matinyan *et al.* [1] derived from classical Yang–Mills equations in Minkowski space with underlying Lie algebra $su(2)$ the Hamiltonian

$$H(p, x) = (1/2)(p_1^2 + p_2^2 + p_3^2) + x_1^2x_2^2/2 + x_2^2x_3^2/2 + x_1^2x_3^2/2 \tag{1}$$

with the equations of motion

$$\dot{x}_1 = p_1, \qquad \dot{x}_2 = p_2, \qquad \dot{x}_3 = p_3 \tag{2a}$$

$$\dot{p}_1 = -x_1(x_2^2 + x_3^2), \qquad \dot{p}_2 = -x_2(x_1^2 + x_3^2), \qquad \dot{p}_3 = -x_3(x_1^2 + x_2^2). \tag{2b}$$

Special gauge conditions have been imposed so that the resulting equations do not depend on the space coordinate. The quantities x_i and p_i $(i = 1, 2, 3)$ are related to the vector potential (compare Matinyan *et al.* [1] for details).

The simplified system where $x_3(t) = p_3(t) = 0$ has been studied in detail by Matinyan *et al.* [1], Carnegie and Percival [2] Chang [3], Steeb and Kunick [4], and Steeb *et al.* [5].

In the present note we study the chaotic behaviour of system (2). Since system (2) is scale invariant, we also study the connection between scale symmetry and the one-dimensional Lyapunov exponent.

Instead of system (2) we can also investigate the system of second order differential equations

$$\ddot{x}_1 = -x_1(x_2^2 + x_3^2) \tag{3a}$$

$$\ddot{x}_2 = -x_2(x_1^2 + x_3^2) \tag{3b}$$

$$\ddot{x}_3 = -x_3(x_1^2 + x_2^2). \tag{3c}$$

Particular solutions can be found as follows: (a) $x_2(t) = x_3(t) = 0$, $x_1(t) = C_1t + C_2$ (analogously for x_2 and x_3); (b) $x_1(t) = x_2(t) = x_3(t)$ with $\ddot{x} + 2x^3 = 0$. In case (b) we have a periodic solution which can be expressed with the help of elliptic functions. Obviously, $x_1(t) = x_2(t) = -x_3(t)$ (and so on) leads to the same equation.

To characterize chaotic orbits we introduce the one-dimensional Lyapunov exponent (Contopoulos *et al.* [6]). We slightly modify the definition so that we can apply the scale invariance of the system (2). Let $\dot{x}_i = F_i(x)$ be an autonomous system of differential equations ($i = 1, \ldots, n$). In the present case we put $x_4 \equiv p_1$, $x_5 \equiv p_2$, $x_6 \equiv p_3$. Therefore, $n = 6$. The variational system is given by

$$\dot{y}_i = \sum_{i=1}^{n} (\partial F_i / \partial x_k) y_k. \tag{4}$$

In the present case we have

$$\dot{y}_1 = y_4, \qquad \dot{y}_2 = y_5, \qquad \dot{y}_3 = y_6 \tag{5a}$$

$$\dot{y}_4 = -(x_2^2 + x_3^2)y_1 - 2x_1x_2y_2 - 2x_1x_3y_3 \tag{5b}$$

$$\dot{y}_5 = -2x_1x_2y_1 - (x_1^2 + x_3^2)y_2 - 2x_2x_3y_3 \tag{5c}$$

$$\dot{y}_6 = -2x_1x_3y_1 - 2x_2x_3y_2 - (x_1^2 + x_2^2)y_3; \tag{5d}$$

Any set of initial data $x_{10}, \ldots, x_{n0}, y_{10}, \ldots, y_{n0}$ gives a solution $x_i(t, x_{10}, \ldots, x_{n0})$, $y_i(t, x_{10}, \ldots, x_{n0}, y_{10}, \ldots, y_{n0})$. If the system $\dot{x}_i = F_i(x)$ is defined on a compact manifold and preserves a measure for almost all $x_{10}, \ldots, x_{n0}$ and for all $y_{10}, \ldots, y_{n0}$ the limit

$$\lim_{t \to \infty} \frac{1}{t} \ln \|y(t)\| = \lambda(x_{10}, \ldots, x_{n0}, y_{10}, \ldots, y_{n0}) \tag{6}$$

exists, where $\|y(t)\|$ denotes the norm of $y_1(t), \ldots, y_n(t)$. The quantity λ is called one-dimensional Lyapunov exponent. In the following we use the Euclidean metric. For studying the connection with the scale invariance we slightly modify the definition, namely

$$\lambda(x_{10}, \ldots, x_{n0}, y_{10}, \ldots, y_{n0}) := \lim_{t \to \infty} \frac{1}{t} \ln ((y_1/y_{10})^2 + \cdots + (y_n/y_{n0})^2)^{1/2}. \tag{7}$$

For our numerical computation of λ, we integrate directly system (2) and system (5). Our numerical results strongly suggest that there is no regular region and that the motion is always irregular except for special orbits, like the periodic orbits that form a set of measure zero. Besides the "trivial" periodic orbits discussed above ($x_1 = \pm x_2 = \pm x_3$) (in this case the problem is one-dimensional) there are genuine two-dimensional and three-dimensional periodic orbits. For the case $x_3(t) = 0$ Steeb *et al.* [5] proved that the trivial periodic orbits are unstable. They have calculated the index of stability [7]. In the present case it can also be proved that the trivial periodic solution are not stable.

Let us now describe the connection between the scale invariance and the one-dimensional Lyapunov exponent. Let

$$t \to \alpha^{-1} t, \qquad x_i \to \alpha x_i, \qquad p_i \to \alpha^2 p_i \tag{8a}$$

and

$$y_i \to \alpha y_i \quad (i = 1, 2, 3), \qquad y_i \to \alpha^2 y_i \quad (i = 4, 5, 6). \tag{8b}$$

The systems (2) and (5) remain unchanged (scale invariance). For the Hamiltonian (1) we obtain $H(\alpha x, \alpha^2 p) = \alpha^4 H(x, p)$.

We notice that in the language of singular point analysis $r = 4$ is a resonance (Kowalewski exponent). Notice that system (2) does not pass the Painlevé test. We find a logarithmic psi-series for the main branch. The resonances are $-1, 1, 1, 2, 2$ and 4. If we put $x_3(t) = 0$, then we find a psi-series with complex resonances. Due to a theorem of Yoshida [8, 9] the equations of motion cannot be algebraically integrable.

Now the one-dimensional Lyapunov exponent given by equation (7) is not scale invariant but

$$\lambda \to \alpha\lambda. \tag{9}$$

This means the following: Given two sets of initial values

$$\{x_{10}, \ldots, x_{60}, y_{10}, \ldots, y_{60}\} \tag{10}$$

and

$$\{\bar{x}_{10}, \ldots, \bar{x}_{60}, \bar{y}_{10}, \ldots, \bar{y}_{60}\} \tag{11}$$

where

$$\bar{x}_{i0} = \alpha x_{i0} \quad (i = 1, 2, 3) \tag{12a}$$

$$\bar{x}_{i0} = \alpha^2 x_{i0} \quad (i = 4, 5, 6) \tag{12b}$$

$$\bar{y}_{i0} = \alpha y_{i0} \quad (i = 1, 2, 3) \tag{12c}$$

$$\bar{y}_{i0} = \alpha^2 y_{i0} \quad (i = 4, 5, 6). \tag{12d}$$

Then it follows that $\bar{E} = \alpha^4 E$ and $\bar{\lambda} = \alpha\lambda$, where $E = H(x_{i0}, p_{i0})$,

Finally let us give a numerical example. Let

$$\bar{x}_{10} = \bar{x}_{20} = \bar{x}_{30} = 0 \tag{13a}$$

$$\bar{x}_{40} = (0.3)^{1/2}, \qquad \bar{x}_{50} = (0.4)^{1/2}, \qquad \bar{x}_{60} = (0.1)^{1/2} \tag{13b}$$

$$\bar{y}_{10} = \bar{y}_{40} = 0.1, \qquad \bar{y}_{20} = \bar{y}_{50} = 0.2, \qquad \bar{y}_{30} = \bar{y}_{50} = 0.15. \tag{13c}$$

Consequently, $\bar{E} = 0.4$. Let $\alpha^2 = 2$. Then $E = 0.1$. From numerical studies we find that $\bar{\lambda} = 0.32$ and $\lambda = 0.225$. It follows that $\bar{\lambda}/\lambda = 1.42$. This coincides with the "theoretical" value $\alpha = 2^{1/2}$.

The results discussed above can be extended to higher dimensions. Let us consider the Hamiltonian

$$H(x, p) = (1/2) \sum_{i=1}^{n} p_i^2 + (1/2) \sum_{i<j}^{n} x_i^2 x_j^2 \tag{14}$$

with the equations of motion

$$\dot{x}_i = p_i, \qquad \dot{p}_i = -x_i \sum_{\substack{j=1 \\ i \neq j}}^{n} x_j^2. \tag{15}$$

For arbitrary n the equations of motion do not pass the Painlevé test. The equations of motion are invariant under $t \to \alpha^{-1} t$, $x_i \to \alpha x_i$, $p_i \to \alpha^2 p_i$. Then for the Hamiltonian we find again $H(\alpha x, \alpha^2 p) = \alpha^4 H(x, p)$. Our numerical investigations for $n = 4, 5, 6$ (we have calculated again the one-dimensional Lyapunov exponents) strongly suggest that there is no regular region and that the motion is always irregular except for special orbits, like the periodic orbits. Due to the scale invariance we have to do our calculation only for one energy shell.

REFERENCES

[1] S. G. MATINYAN, G. K. SAVVIDI and N. G. TER-ARUTYUNYAN-SAVVIDI, Sov. Phys. JETP *53*, 421 (1981).
[2] A. CARNEGIE and I. C. PERCIVAL, J. Phys. A: Math. Gen. *17*, 801 (1984).
[3] S.-J. CHANG, Phys. Rev. *D29*, 259 (1984).
[4] W.-H. STEEB and A. KUNICK, Lett. Nuovo Cim. *42*, 83 (1985).
[5] W.-H. STEEB, C. M. VILLET, and A. KUNICK, J. Phys. A: Math. Gen. *18*, 3269 (1985).
[6] G. CONTOPOULOS, L. GALGANI, and A. GIORGILLI, Phys. Rev. A *18*, 1183 (1978).
[7] H. YOSHIDA, Celest. Mech. *32*, 73 (1984).
[8] H. YOSHIDA, Celest. Mech. *31*, 363 (1983).
[9] H. YOSHIDA, Celest Mech. *31*, 381 (1983).