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Broken symmetries and the generation of classical observables in large systems

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Summary. Infinite systems are described by an asymptotically abelian action $\bar{\alpha}$ of space translations on a quasilocal C^*-algebra $\mathcal{A}$. Broken symmetries with respect to an action $\bar{\beta}$ of a locally compact separable group G (essentially) commuting with $\bar{\alpha}$ are examined. It is shown that broken symmetries give rise to classical observables based on a transitive system of imprimitivities over G/H, where $H \subseteq G$ denotes the subgroup of preserved symmetries and G/H is the set of left cosets of H in G.

I. Introduction

In textbooks and research papers of quantum mechanics the concept of 'observables' frequently is a heuristic one: Observables are said to correspond to measurable physical quantities. Depending on the particular formalism they are described by (self-adjoint) operators on a Hilbert space or elements (operators) of an algebra, a C^*- or W^*-algebra for example. 'Observable' and 'operator' sometimes are almost synonymous and the set of operators is introduced as 'algebra of observables'.

A more subtle view of this matter involves a distinction between observables and their measurement and regards observables as quantities which characterize a system intrinsically. Observables (such as position and momentum) then correspond to very particular operators of an algebra, thus giving structure to the very abstract mathematical apparatus of quantum mechanics.

One way to introduce observables without use of recipes (e.g. the rules given by the correspondence principle) is based on group-theoretical concepts (cf. [1], [2], [3], [4, 5] a.o.): Observables are defined as operators which transform 'suitably' under the action of a kinematical group, e.g. the Galilei group. This way to look at observables is well elaborated in case of observables such as position and momentum (cf. also [6, 7], [8]) and can be extended to more general situations by giving a proper sense to the word 'suitably' (s. [4, 5] and chapter II).

Within the quantum mechanical formalism a prominent but sometimes underestimated role is played by the *classical* observables (superselection rules). In the algebraic formalism they correspond to elements of the *center* of the respective algebra denoted by $\mathcal{M}$, i.e. by elements from $Z(\mathcal{M}) \stackrel{\text{def}}{=} \{x \in \mathcal{M} \mid xy = yx$ for all $y \in \mathcal{M}\}$. Examples for classical observables are charge and mass of particles, chirality and nuclear frame of molecules, temperature and chemical potential of substances and the time operator in Galilei-relativistic quantum systems (s. [5]: ch. IV).

A particularly interesting class of classical observables arises in systems with broken symmetry. Such observables are sometimes said to be generated 'spontaneously'. As examples consider the spontaneous magnetization (the rotation symmetry is broken, cf. [9]: 3.2, 11(3)), the phase operator (the gauge symmetry is broken, cf. [9]: 3.2, 11(2), [10]) or the momentum operator in an infinite system (in the latter example the 'boost'-symmetries are broken, cf. [11]).

A symmetry $\bar{\beta}$ of a system of algebraic quantum mechanics is called *broken* with respect to a factor representation π of the underlying algebra $\mathcal{A}$ of the system, if there is no automorphism β of the generated von Neumann algebra $\pi(\mathcal{A})''$ such that $\beta \cdot \pi = \pi \cdot \bar{\beta}$. If the representation π is a GNS-representation with respect to a factor state – invariant with respect to an additional (asymptotically abelian) action $\bar{\alpha}$ – then a symmetry $\bar{\beta}$ commuting with $\bar{\alpha}$ is accordingly broken if and only if the state is not invariant under $\bar{\beta}$ (see chapter III, obs. 1; cf. [9]: 3.2, 12(3)).

The generation of classical observables in systems with broken symmetries has not yet been entirely clarified:

- Classical observables are often constructed and introduced not as central operators of an algebraic system but in a degenerated form, namely as $\mathbb{C}$-numbers (expectation values) associated with certain factor states.
- Sometimes classical observables are constructed by considering direct sums of representations, e.g. the direct sum of representations with different momentum parametrized by elements from $\mathbb{R}^3$. This procedure always leads to classical observables with discrete spectrum which is undesirable in many situations.

It is the goal of the present paper to overcome difficulties such as those sketched above. The formalism of C^*- and W^*-algebraic quantum mechanics will be used in the sequel. A comprehensive review of this formalism from the physical point of view is given by Primas ([12]). For a shortcut presentation of group-theoretical aspects in it see e.g. ([5]: Ch. I). Infinite systems and quantum statistical mechanics on an operator algebraic level are discussed by Bratteli and Robinson ([13, 14]). The purely mathematical aspects of operator theory are treated in the monographs of Dixmier ([15, 16]), Pedersen ([17]), Sakai ([18]) and Takesaki ([19]).

II. Observables based on systems of imprimitivities and induced representations

C^*- and W^*-systems

A *C^*-system* $(\mathcal{A}, G, \beta)$ consists of

- a C^*-algebra $\mathcal{A}$

– a locally compact separable group G
– a mapping $\beta: G \to \mathrm{Aut}\,\mathscr{A}$ of G into the automorphism group $\mathrm{Aut}\,\mathscr{A}$ of the *-algebra $\mathscr{A}$ with the properties
  (i) $\beta_{g_1} \circ \beta_{g_2} = \beta_{g_1 g_2}$, $g_1, g_2 \in G$ (β is a representation),
  (ii) For every operator $x \in \mathscr{A}$, the function $\{G \ni g \to \beta_g(x)\}$ is continuous with respect to the norm-topology on $\mathscr{A}$ (β is pointwise norm-continuous).

A *W^*-system* $(\mathscr{M}, G, \beta)$ is defined similarly: $\mathscr{M}$ is then a W^*-algebra and the norm-topology in (ii) is replaced by the σ-weak topology.

W^*-systems $(\mathscr{M}_1, G, \beta_1)$ and $(\mathscr{M}_2, G, \beta_2)$ are said to be *conjugate* if there exists a *-isomorphism κ: $\mathscr{M}_1 \to \mathscr{M}_2$ such that $\beta_{2g} \circ \kappa = \kappa \circ \beta_{1g}$, $\forall g \in G$, holds.

Commutative W^*-systems

To each closed subgroup H of G one can associate in a natural way a commutative W^*-system $(\mathscr{L}_\infty(G/H), G, \mathrm{Ad}\,\lambda_{G/H})$: G/H denotes the set $\{gH \mid g \in G\}$ of left cosets of H in G (with the quotient topology). On G/H there exists a probability measure $\mu_{G/H}$, quasiinvariant under the transitive representation

$$s_{g_0}: G/H \to G/H, \qquad g_0 \in G,$$
$$s_{g_0}: gH \to g_0 gH, \qquad g \in G,$$

(cf. [20]: V.3). Every probability measure μ with the same property is quasiequivalent to $\mu_{G/H}$.

$\mathscr{L}_\infty(G/H)$ is the commutative W^*-algebra consisting of equivalence classes (modulo $\mu_{G/H}$-null sets) of essentially bounded complex-valued $\mu_{G/H}$-measurable functions on G/H. The representation

$$\mathrm{Ad}\,\lambda_{G/H}(g): \mathscr{L}_\infty(G/H) \to \mathscr{L}_\infty(G/H), \qquad g \in G,$$

$$(\mathrm{Ad}\,\lambda_{G/H}(g)f)(x) \overset{\mathrm{def}}{=} f(s_{g^{-1}}x), \qquad x \in G/H, \qquad f \in \mathscr{L}_\infty(G/H),$$

is well-defined and σ-weakly continuous.

Observables

Observables in a W^*-system $(\mathscr{M}, G, \beta)$ are described here by a faithful normal (i.e. σ-weakly continuous) covariant *-isomorphism

$$\tau: \mathscr{L}_\infty(G/H) \to \mathscr{M}$$
$$\beta_g \circ \tau = \tau \circ \mathrm{Ad}\,\lambda_{G/H}(g), \qquad g \in G, \text{ (covariance)},$$

of $\mathscr{L}_\infty(G/H)$ into $\mathscr{M}$, where H is a closed subgroup of G. This description is equivalent to a transitive system of imprimitivity in the algebra $\mathscr{M}$ (cf. [2], [3], [21]).

This definition is adapted to the particular situation of commuting (central) observables treated here. More general situations require a covariant (positive, linear, normalized but not necessarily multiplicative) mapping $\varphi:\mathcal{L}_\infty(G)\to\mathcal{M}$ ($H=\{e\}$ is trivial) (s. [5]). If $(\mathcal{M}, G, \beta)$ is ergodic and admits such a mapping φ, a closed subgroup H of G can be shown to exist such that the central system $(\mathcal{Z}(\mathcal{M}), G, \beta|_{\mathcal{Z}(\mathcal{M})})$ is conjugate to $(\mathcal{L}_\infty(G/H), G, \operatorname{Ad}\lambda_{G/H})$ (s. [5]: Theorem III.4). Thus the consideration of central observables alone just leads to the definition given above.

As an example take the 'Weyl group' $G=\mathbb{R}^6=\{(\vec{q},\vec{p})\mid \vec{q},\vec{p}\in\mathbb{R}^3\}$, where $\vec{q}$ describes a space translation and $\vec{p}$ describes a shift of momentum. Then the subgroup $H=\{(\vec{0},\vec{p})\mid \vec{p}\in\mathbb{R}^3\}$ corresponds to a position observable whereas the subgroup $\{(\vec{q},\vec{0})\mid \vec{q}\in\mathbb{R}^3\}$ corresponds to a momentum observable. The former case does not play a role in infinite systems, whereas the latter has been investigated in ([11]).

Induced representations

Let $\{\operatorname{Ad}\rho(g)\mid g\in G\}$ denote the representation

$$\operatorname{Ad}\rho(g):\mathcal{L}_\infty(G)\to\mathcal{L}_\infty(G),\qquad g\in G,$$

$$(\operatorname{Ad}\rho(g)f)(s)\overset{\text{def}}{=}f(sg),\qquad s\in G,\ g\in G,$$

of G on $\mathcal{L}_\infty(G)$. $\operatorname{Ad}\rho$ acts 'from the right' just as $\operatorname{Ad}\lambda\overset{\text{def}}{=}\operatorname{Ad}\lambda_{G/\{e\}}$ acts 'from the left' on $\mathcal{L}_\infty(G)=\mathcal{L}_\infty(G/\{e\})$.

Consider a W^*-system $(\mathcal{F}, H, \gamma)$ of the closed subgroup H of G on the W^*-algebra $\mathcal{F}$. The W^*-system $(\mathcal{N}, G, \varepsilon)=\operatorname{Ind}_H^G\{\mathcal{F},\gamma\}$ *induced from* $\{\mathcal{F}, H, \gamma\}$ is defined by (cf. [21])

$$\mathcal{N}\overset{\text{def}}{=}\{y\in\mathcal{L}_\infty(G)\mathbin{\bar{\otimes}}\mathcal{F}\mid(\operatorname{Ad}\rho(h)\otimes\gamma_h)(y)=y,\qquad \forall h\in H\}$$

$$\varepsilon_k(y)\overset{\text{def}}{=}(\operatorname{Ad}\lambda(k)\otimes\operatorname{Id})(y),\qquad y\in\mathcal{N},\ k\in G.$$

Here $\bar{\otimes}$ denotes the W^*-tensor product, $\{\operatorname{Ad}\rho(h)\otimes\gamma_h\}$ the tensor product automorphism of the automorphisms $\operatorname{Ad}\rho(h)$ and γ_h, and Id the identity automorphism of $\mathcal{F}$. $\mathcal{F}$ is supposed to have a separable predual $\mathcal{F}_*$.

The W^*-algebra $\mathcal{L}_\infty(G)\mathbin{\bar{\otimes}}\mathcal{F}$ can be considered as the set of (equivalence classes of) essentially bounded μ_G-measurable (in the sense of [19]: Chapter IV.7) functions $y:G\to\mathcal{F}$ from G into $\mathcal{F}$. If $y:G\to\mathcal{F}$ is an element of $\mathcal{N}$, it can be assumed (by eventually changing y on a null set) that

$$y(gh)=\gamma_h^{-1}(y(g)),\qquad g\in G,\qquad h\in H,\tag{1}$$

holds. Conversely every $y\in\mathcal{L}_\infty(G)\mathbin{\bar{\otimes}}\mathcal{F}$ fulfilling the relation (1) constitutes an element of $\mathcal{N}$.

Let $p:G\to G/H$, $p(g)\overset{\text{def}}{=}gH$, $g\in G$, denote the projection mapping of G onto G/H. There exists a Borel mapping $r:G/H\to G$ with the property $p\circ r=\operatorname{Id}$ (use e.g. [17]: 4.2.13, and of course the separability assumptions made here). Id is the identity mapping on G/H, and r is Borel means that it is measurable with respect to the σ-algebras generated by the open sets on G/H and G, respectively.

In the following elements of G/H will be denoted by $\dot{g}$. The *-isomorphism

$$J^{-1}: \mathcal{N} \to \mathcal{L}_\infty(G/H) \bar{\otimes} \mathcal{F}$$

$$J^{-1}\{G \ni g \to y(g) \in \mathcal{F}\} \stackrel{\text{def}}{=} \{g/H \ni \dot{g} \to y(r(\dot{g})) \in \mathcal{F}\}$$

is well-defined if every $y \in \mathcal{N}$ is represented by a mapping $\{G \ni g \to y(g) \in \mathcal{F}\}$ fulfilling relation (1). It's inverse is given by

$$J: \mathcal{L}_\infty(G/H) \bar{\otimes} \mathcal{F} \to \mathcal{N}$$

$$J\{G/H \ni \dot{g} \to x(\dot{g}) \in \mathcal{F}\} \stackrel{\text{def}}{=} \{G \ni g \to \gamma_{g^{-1}r(p(g))}(x(p(g)))\}.$$

The representation $\{J^{-1}\varepsilon_k J \mid k \in G\}$ on $\{\mathcal{L}_\infty(G/H) \bar{\otimes} \mathcal{F}\}$ has the property

$$J^{-1}\varepsilon_k J(f \otimes 1) = \operatorname{Ad} \lambda_{G/H}(f) \otimes 1 \tag{2}$$
$$f \in \mathcal{L}_\infty(G/H), \qquad k \in G.$$

III. Observations and results

We consider a *separable* unital C^*-algebra $\mathcal{B}$, describing an infinite system, and two actions $\bar{\alpha}: \mathbb{R}^3 \to \operatorname{Aut} \mathcal{B}$, $\bar{\beta}: G \to \operatorname{Aut} \mathcal{B}$, where G is an arbitrary locally compact second countable group (e.g. the Galilei group). $\bar{\alpha}$ corresponds to the space translations of the system. The following properties are assumed to hold:

(i) $\lim_{|\vec{q}| \to \infty} \|[\bar{\alpha}_{\vec{q}}(x), y]\| = 0 \qquad \forall x, y \in \mathcal{B}$,
(ii) for arbitrary $g \in G$ and $\vec{q} \in \mathbb{R}^3$ there exists an element $g[\vec{q}] \in \mathbb{R}^3$ such that

$$\bar{\beta}_g \circ \bar{\alpha}_{\vec{q}} \circ \bar{\beta}_{g^{-1}} = \bar{\alpha}_{g[\vec{q}]}.$$

Asymptotic abelianess (i) with respect to space translations is a property typical for physically relevant infinite systems. It is a very strong assumption since it implies that Einstein–Podolsky–Rosen correlations vanish for large spatial distances.

Observation 1

Two extremal $\bar{\alpha}$-invariant states on $\mathcal{B}$ are either equal or disjoint (use [13]: 4.3.19 and [17]: 7.12.8). Furthermore each $\bar{\alpha}$-invariant *factor* state Ψ on $\mathcal{B}$ is extremal invariant (this follows from [17]: 7.12.8). If such a factor state Ψ is quasiinvariant under an automorphism $\bar{\beta}_g$ it follows that Ψ is $\bar{\beta}_g$-invariant. Thus $\bar{\beta}_g$ can be implemented on the von Neumann algebra $\pi_\Psi(\mathcal{B})''$ generated by the GNS-representation of $\mathcal{B}$ with respect to Ψ if and only if Ψ is $\bar{\beta}_g$-invariant. Accordingly the symmetry $\bar{\beta}_g$ is broken with respect to the factor state Ψ if and only if $\Psi \neq \Psi \circ \bar{\beta}_g$. For this observation the separability of $\mathcal{B}$ is unneccessary.

Observation 2

Let ω be a $\bar{\beta}$-quasiinvariant state on $\mathcal{B}$ and let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ denote the corresponding GNS-representation. Then $\mathcal{H}_\omega$ is a separable Hilbert space and the extension β of $\bar{\beta}$ to $\pi_\omega(\mathcal{B})''$ is pointwise σ-weakly continuous (cf. [11]: chapter III), i.e. $(\pi_\omega(\mathcal{B})'', G, \beta)$ is a W^*-system.

Theorem. *Let ϕ be an extremal $\bar{\alpha}$-invariant state on the separable C^*-algebra $\mathcal{B}$ and define $H \stackrel{\text{def}}{=} \{g \in G \mid \phi \circ \beta_g = \phi\}$. Then H is a closed subgroup of G and the action $\{\bar{\beta}_h \mid h \in H\}$ can be extended to an action $\{\gamma_h \mid h \in H\}$ on $\pi_\phi(\mathcal{B})''$ where π_ϕ is the GNS-representation of $\mathcal{B}$ with respect to ϕ. There exist*

(i) *an $\bar{\alpha}$-invariant and $\bar{\beta}$-quasiinvariant state ω on $\mathcal{B}$ (the corresponding extensions of $\bar{\alpha}$ and $\bar{\beta}$ to $\mathcal{M} \stackrel{\text{def}}{=} \pi_\omega(\mathcal{B})''$ are denoted by α and β)*
(ii) *and a faithful *-isomorphism $\tau : \mathcal{L}_\infty(G/H) \to \mathcal{Z}(\mathcal{M})$ of $\mathcal{L}_\infty(G/H)$ into the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$ with the properties*

$$\beta_g \circ \tau = \tau \circ \operatorname{Ad} \lambda_{G/H}(g), \qquad g \in G,$$
$$\alpha_{\vec{q}} \circ \tau = \tau, \qquad \vec{q} \in \mathbb{R}^3.$$

The state ω can be chosen such that the W^*-system $(\mathcal{M}, G, \beta)$ is conjugate to the induced W^*-system $\operatorname{Ind}_H^G \{\pi_\phi(\mathcal{B})'', H, \gamma\}$. In particular $\mathcal{M}$ is then of the form $\mathcal{M} \cong \mathcal{L}_\infty(G/H) \bar{\otimes} \pi_\phi(\mathcal{B})''$.

Proof. The subgroup H as defined in the theorem is clearly closed. Let $r : G/H \to G$ be a Borel cross section and define a state ω on $\mathcal{B}$ (not depending on the particular choice for r) by

$$\omega(x) \stackrel{\text{def}}{=} \int_{G/H} \phi(\bar{\beta}_{r(\dot{g})^{-1}}(x))\, d\mu_{G/H}(\dot{g}), \qquad x \in \mathcal{B}.$$

It follows from the commutation properties (ii) that ω is $\bar{\alpha}$-invariant. Define

$$\mathcal{H} \stackrel{\text{def}}{=} \int_{G/H}^{\oplus} \mathcal{H}_{\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}}\, d\mu_{G/H}(\dot{g}),$$

$$\pi(x) \stackrel{\text{def}}{=} \int_{G/H}^{\oplus} \pi_{\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}}(x)\, d\mu_{G/H}(\dot{g}), \qquad x \in \mathcal{B},$$

$$D \stackrel{\text{def}}{=} \int_{G/H}^{\oplus} \{\mathbb{C} \cdot 1_{\mathcal{H}_{\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}}}\}\, d\mu_{G/H}(\dot{g}) \cong \mathcal{L}_\infty(G/H)$$

$$\tilde{\tau}(f) \stackrel{\text{def}}{=} \int_{G/H}^{\oplus} \{f(\dot{g}) 1_{\mathcal{H}_{\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}}}\}\, d\mu_{G/H}(\dot{g}), \qquad f \in \mathcal{L}_\infty(G/H).$$

Here $(\mathcal{H}_{\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}}, \pi_{\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}}, \Omega_{\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}})$ is the GNS-representation of $\mathcal{B}$ with respect to the state $\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}$, $\dot{g} \in G/H$. Recall that $\pi_{\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}}(x) = \pi_\phi(\bar{\beta}_{r(\dot{g})^{-1}}(x))$, $x \in \mathcal{B}$, $\dot{g} \in G/H$, and $\Omega_{\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}} = \Omega_\phi$ hold in a natural identification of the GNS-Hilbert spaces $\mathscr{H}_{\phi\circ\bar\beta_{r(\dot g)^{-1}}}$ and $\mathscr{H}_\phi$. The direct integral operators

$$\int_{G/H}^{\oplus} \pi_{\phi\circ\bar\beta_{r(\dot g)^{-1}}}(x)\, d\mu_{G/H}(\dot g), \qquad x \in \mathscr{B},$$

can therefore be regarded as bounded $\mu_{G/H}$-measurable mappings (in the sense of [19]: Chapter IV.7) from G/H into $\pi_\phi(\mathscr{B})''$. The algebra of all essentially bounded $\mu_{G/H}$-measurable mappings from G/H to $\pi_\phi(\mathscr{B})''$ (modulo null sets) is naturally isomorphic to $\mathscr{L}_\infty(G/H)\,\bar\otimes\,\pi_\phi(\mathscr{B})''$, where D corresponds to $\mathscr{L}_\infty(G/H)\otimes\mathbb{C}$. Using ([13]: 4.3.14, 4.4.9) one proves that the diagonal algebra D is part of the center of $\pi(\mathscr{B})''$ and that there exists a unitary $U:\mathscr{H}_\omega\to\mathscr{H}$ such that $U\pi_\omega(x)U^* = \pi(x)$, $x\in\mathscr{B}$, holds (cf. the proof of ([11]: lemma in section II)).

Set $(\mathscr{N}, G, \varepsilon) \overset{\text{def}}{=} \operatorname{Ind}_H^G\{\pi_\phi(\mathscr{B})'', \gamma\}$. Elements of $\mathscr{N}$ are regarded as mappings

$$y: G \to \pi_\phi(\mathscr{B})''$$

fulfilling relation (1) (s. chapter II). One has

$$\begin{aligned} &J\circ\pi:\mathscr{B}\to\mathscr{N}\\ &J(\pi(x)): g\to \gamma_{g^{-1}r(p(g))}\{\pi_\phi(\bar\beta_{r(p(g))^{-1}}(x))\}\\ &\quad = \pi_\phi\{\bar\beta_{g^{-1}r(p(g))}\bar\beta_{r(p(g))^{-1}}(x)\} = \pi_\phi\{\bar\beta_{g^{-1}}(x)\}, \qquad g\in G,\, x\in\mathscr{B}. \end{aligned}$$

It is now a matter of simple verification to show that

$$\varepsilon_k(J\circ\pi(x)) = J\circ\pi(\bar\beta_k(x)), \qquad x\in\mathscr{B},\, k\in G,$$

is true. From this relation it is immediately inferred that ω is $\bar\beta$-quasiinvariant. The corresponding action β on $\mathscr{M}$ is given by

$$\beta_k(x) \overset{\text{def}}{=} U^*(J^{-1}(\varepsilon_k(J(UxU^*))))U, \qquad x\in\pi_\omega(\mathscr{B})'',\, k\in G.$$

Defining $\tau:\mathscr{L}_\infty(G/H)\to\pi_\omega(\mathscr{B})''$ by $\tau(f)\overset{\text{def}}{=} U^*\tilde\tau(f)U$, $f\in\mathscr{L}_\infty(G/H)$, one gets

$$\beta_g\circ\tau = \tau\circ\operatorname{Ad}\lambda_{G/H}(g), \qquad g\in G. \tag{3}$$

(3) is an immediate consequence of relation (2).

The next point is to show that $\mathscr{N} = J(\pi(\mathscr{B})'')$ holds: Since $J(\pi(\mathscr{B})'')$ contains $\mathscr{L}_\infty(G/H)$, that is, all the elements $\{J(\tilde\tau(f))\mid f\in\mathscr{L}_\infty(G/H)\}$ and since $J(\pi(\mathscr{B})'')$ is globally ε-invariant, Proposition 10.4 from ([21]) can be used: Therefore there exists a globally γ-invariant W^*-subalgebra $\mathscr{F}$ of $\pi_\phi(\mathscr{B})''$ such that $J(\pi(\mathscr{B})'')\subseteq\mathscr{N}$ consists of all the mappings $y: g\to y(g)\in\mathscr{F}$, $g\in G$, fulfilling relation (1). If x is an arbitrary element of $\mathscr{B}$, the values $\pi_\phi(\bar\beta_{g^{-1}}(x))$, $g\in G$, of $J\circ\pi(x)$ are thus in $\mathscr{F}$ for $g\notin N$, where N is a null set. By continuity N is empty and i.p. $\pi_\phi(x)\in\mathscr{F}$. Therefore $\pi_\phi(\mathscr{B})\subseteq\mathscr{F}\subseteq\pi_\phi(\mathscr{B})''$ implies $\pi_\phi(\mathscr{B})''=\mathscr{F}$ and furthermore $\mathscr{N} = J(\pi(\mathscr{B})'')$.

Let $V_\phi(\vec q)$ denote the unitary operator implementing $\bar\alpha_{\vec q}$ on $\mathscr{H}_\phi$ (cf. [13]: 2.3.17). The mapping $\dot g\to V_\phi(r(\dot g)^{-1}[\vec q])$ is $\mu_{G/H}$-measurable: To prove this consider the W^*-algebra $\mathcal{M} = \pi_\omega(\mathcal{B})''$ (with separable predual) and the σ-weakly continuous actions α and β. In the standard representation of $\mathcal{M}$ α and β are implemented by strongly continuous unitary representations $\vec{q} \to U_{\vec{q}}$, $\vec{q} \in \mathbb{R}^3$, and $g \to W_g$, $g \in G$ ([13]: 2.5.32). Since the unitaries on a Hilbert space form a topological group with respect to the strong (or weak or strong*) topology, the mapping

$$G \ni g \to W_g U_{\vec{q}} W_g^* = U_{g[\vec{q}]}$$

is strongly continuous. That $G \ni g \to g[\vec{q}]$ is Borel now follows from ([22]: 8.3.5, 8.3.7) and the fact that the unitaries on a separable Hilbert space form a Polish space. Thus $G/H \ni \dot{g} \to r(\dot{g})^{-1}[\vec{q}]$ is Borel and therefore $G/H \ni \dot{g} \to V_\phi(r(\dot{g})^{-1}[\vec{q}]) \in \mathcal{B}(\mathcal{H}_\phi)$ is $\mu_{G/H}$-measurable.

One has

$$\begin{aligned}&V_\phi(r(\dot{g})^{-1}[\vec{q}])\pi_\phi(\bar{\beta}_{r(\dot{g})^{-1}}(x))V_\phi^*(r(\dot{g})^{-1}[\vec{q}])\\ &\quad = \pi_\phi(\bar{\alpha}_{r(\dot{g})^{-1}[\vec{q}]}(\bar{\beta}_{r(\dot{g})^{-1}}(x))) = \pi_\phi \circ \bar{\beta}_{r(\dot{g})^{-1}}\{\bar{\alpha}_{\vec{q}}(x)\}, \qquad x \in \mathcal{B}, \dot{g} \in G/H.\end{aligned}$$

Therefore the unitary operator $V(\vec{q}) \in \{\mathcal{L}_\infty(G/H) \bar{\otimes} \mathcal{B}(\mathcal{H}_\phi)\}$ corresponding to $\{\dot{g} \to V_\phi(r(\dot{g})^{-1}[\vec{q}]), \dot{g} \in G/H\}$ implements $\alpha_{\vec{q}}$ on $\pi(\mathcal{A})''$ and leaves D pointwise invariant. I.p. $\alpha_{\vec{q}} \circ \tau = \tau$, $\vec{q} \in \mathbb{R}^3$, follows. q.e.d.

The above theorem can be generalized to the case where $\mathcal{B}$ is not necessarily separable. Nevertheless, it is important to assume that the GNS-Hilbert space $\mathcal{H}_\phi$ with respect to the state ϕ is *separable*.

For such a generalization it is not even necessary to consider the formalism of Wils ([23, 24], cf. also [25]: Propos. 4.3). The proofs of the relevant theorems in the separable case (in particular [13]: 4.4.9) can be adjusted without great difficulties.

In the following we consider a *quasilocal C^*-algebra $\mathcal{A}$*, obtained as an inductive limit of an increasing sequence of local C^*-algebras $\mathcal{A}_n$, $n = 1, 2, \ldots$ (the local C^*-algebras $(\mathcal{A}_n)_{n\in\mathbb{N}}$ may, for example, correspond to bounded volumes in $\mathbb{R}^3$). For every algebra $\mathcal{A}_n$ a faithful representation $\pi_n : \mathcal{A}_n \to \mathcal{B}(\mathcal{H}_n)$ on a *separable* Hilbert space $\mathcal{H}_n$ is supposed to exist.

A state Ψ on $\mathcal{A}$ is called *locally normal*, if the restriction of Ψ to $\mathcal{A}_n$ extends to a normal state on the von Neumann algebra $\{\pi_n(\mathcal{A}_n)\}''$ for every $n \in \mathbb{N}$. Generalizing ([26]: Prop. 8) one can show that the GNS-Hilbert space $\mathcal{H}_\Psi$ of a locally normal state Ψ is separable.

Therefore, the above theorem holds true if the ($\bar{\alpha}$-invariant) state ϕ on the separable C^*-algebra $\mathcal{B}$ is replaced by an ($\bar{\alpha}$-invariant) locally normal state Ψ on the quasilocal C^*-algebra $\mathcal{A}$.

Examples:

- Let G be the Galilei group and H be the closed subgroup of G generated by rotations, time translations and space translations. Then the corresponding classical observable based on G/H is a momentum operator (cf. [11]).

- Let G be the Galilei group and H be the closed subgroup generated by rotations, space translations and pure Galilean transformations ('boosts'). Then the corresponding classical observable based on G/H is a time operator (cf. [5]: Chapter IV).
- Consider a lattice system and replace $\{\bar{\alpha}_{\vec{q}} \mid \vec{q} \in \mathbb{R}^3\}$ by the permutation group of the lattice. Let $G \stackrel{\text{def}}{=} \{\theta \in \mathbb{C} \mid |\theta| = 1\}$ be a group of gauge transformations represented by automorphisms $\{\bar{\beta}_\theta \mid \theta \in G\}$ commuting with the represented elements of the permutation group. Then the classical observable corresponding to $H = \{e\}$ is a phase operator (cf. [10]).

IV. Concluding remarks

Observables based on a system of imprimitivity over G/H are constructed in the theorem by integrating disjoint representations over G/H. An important point in the proof is to show that the respective diagonal operators are elements of the von Neumann algebra generated by such a direct integral representation. It is interesting to note that for this the asymptotic abelianess of the representation $\bar{\alpha}$ of space translations is essential: Examples are known where a direct integral representation $\pi(\cdot) = \int_X^\oplus \pi_\gamma(\cdot)\, d\gamma$ of mutually disjoint representations π_γ, $\gamma \in X$, of a separable C^*-algebra $\mathscr{B}$ over a standard Borel space X leads to a factor representation (s. [27]: Example 2.3). In particular in such a situation the diagonal operators (which commute with all operators $\pi(y)$, $y \in \mathscr{B}$) cannot be elements of the von Neumann algebra $\pi(\mathscr{B})''$. On the other hand asymptotically abelian space translations need not be considered if $G/H(=X)$ is discrete: The direct integral then degenerates into a direct sum $\pi(\cdot) = \bigoplus_{\dot{g} \in G/H} \pi_{\dot{g}}(\cdot)$ of mutually disjoint representations and the diagonal algebra $\mathscr{L}_\infty(G/H)$ is part of $\pi(\mathscr{B})''$ even if $\mathscr{B}$ or G/H do not fulfill separability requirements (s. [28]: Lemma 3).

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