Notes on asymptotic perturbation theory for Schrödinger eigenvalue problems

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Notes on asymptotic perturbation theory for Schrödinger eigenvalue problems

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Abstract. These notes are intended as a selfcontained introduction to asymptotic perturbation theory for Schrödinger operators. The emphasis is on geometric methods of spectral analysis, which allow considerable simplifications in proving stability of eigenvalues and asymptotic estimates. Special attention is given to non-selfadjoint problems arising in the theory of resonances. The general theory is developed for a simple class of model operators and illustrated with a number of examples: shape resonances, stable and unstable anharmonic oscillators, the N-body Stark effect and the Zeeman effect in atoms.

1. Introduction

Analytic perturbation theory as developed by Rellich and Kato is a pure Hilbert space theory: it deals with abstract operator families H_κ, assuming only that the resolvent $(z - H_\kappa)^{-1}$ depends analytically on the perturbation parameter κ in some neighbourhood of $\kappa = 0$, and it derives convergent perturbation series for discrete eigenvalues and for the corresponding eigenvectors [1].

For non-analytic perturbations these expansions may still be defined, but it is much harder to see what they describe. In Chapter VIII of [1] Kato discusses the basic problems arising in a theory where analyticity of the resolvent is replaced by the much weaker requirement of strong convergence as $\kappa \to 0$, the foremost being the stability of the spectrum. In this general setting it may indeed happen that the spectrum changes abruptly as soon as $\kappa \neq 0$: discrete eigenvalues of H_0 may be covered immediately by the continuum, and new discrete eigenvalues can suddenly appear. In dealing with a given discrete eigenvalue λ of H_0 the first task is therefore to prove its stability in the sense given by Kato: λ is the limit as $\kappa \to 0$ of a group of discrete perturbed eigenvalues having the same (total) multiplicity as λ. It may then be possible to prove that the formal perturbation expansions are asymptotic to the perturbed eigenvalues up to some finite order or to all orders in κ. However, in this abstract framework there are no general results which are powerful enough to handle the great variety of concrete problems.

In physics, the divergence of formal perturbation series was first discussed by Dyson [22] in quantum electrodynamics. It took some time to realise that even non-relativistic quantum mechanics abounds with examples of this kind, among them the classic problems for which perturbation theory was originally invented: the anharmonic oscillator, the Stark effect and the Zeeman effect. These are precisely the examples which led to the mathematical development of an asymptotic perturbation theory for Schrödinger operators, notably by Bender and Wu [15], Graffi, Grecchi and Simon [24, 40] for the anharmonic oscillator; by Graffi, Grecchi [25, 26], Herbst and Simon [30, 31], for the Stark effect and by Avron, Herbst and Simon [11, 12, 13] for the Zeeman effect. A review of this development with some personal reminiscences has been given by Simon [8]. As a result we now know that the Rayleigh–Schrödinger perturbation series in these cases are divergent, essentially because the unperturbed operator H_0 sits right on the boundary of a region on analyticity in κ (Dyson's argument). Nevertheless it turns out that the divergent series still determine the perturbed eigenvalues uniquely via the Borel summation procedure – provided that the right meaning is given to the notion of "perturbed eigenvalues". In the Stark case (and in the other cases for non-physical values of κ) we are in fact dealing not with eigenvalues of H_κ but with resonances. Here the Balslev–Combes theory [14, 3, 4] or one of its variants [43, 36, 20, 33] comes into play, which define resonances as discrete eigenvalues of some non-selfadjoint operators associated with H_κ. By necessity we are thus getting involved with non-selfadjoint perturbation problems: $H_\kappa^* \neq H_\kappa$ for real κ. A similar situation arises in perturbation theory for eigenvalues embedded in the continuum [41, 3].

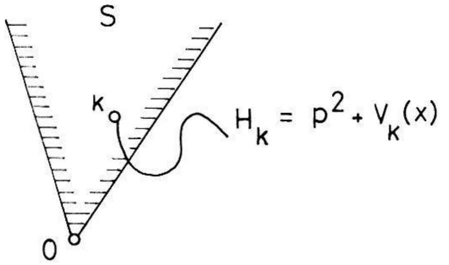
Our goal in these notes is to give a new introduction to this field from a point of view which offers considerable simplifications. Rather than setting up an abstract framework we prefer to explain the main steps for a simple class of model operators, introduced in Section 2. For this class a general theory is developed in Sections 3–9, with emphasis on the stability theorem (Theorem 5.1), stability estimates and asymptotic estimates. The fundamental notions like strong resolvent convergence, stability of eigenvalues, reduction to finite dimension and the Rayleigh–Schrödinger expansion are taken from Kato's book [1]. By going through the proofs it should become clear that only a few basic properties of H_κ are used in an essential way: the local character of H_κ and H_κ^* as partial differential operators, local continuity as $\kappa \to 0$ and, most important of all, the property to which we refer as local compactness (Section 3). In other words, we are taking every advantage of the Schrödinger representation by using configuration space (PDE) methods. Here we were originally guided by ideas of Enss [23] which have since blossomed into "geometric spectral analysis" [42, 35, 4].

Sections 10 and 11 are of a different character and can be read independently. First we recall the notion of Borel summability in a slightly generalized form needed to treat degenerate eigenvalues (a case largely neglected in the literature). Section 11 then deals with the finite dimensional problems arising in this context. Here we confine ourselves to an exposition of the results given in [34] since we have nothing new to offer in the way of proofs.

The examples of Section 12 form in some sense the core of these notes. They are collected at the end to keep the general part brief. However, since the whole set-up is abstracted from concrete cases, the reader will be advised to look for illustrations in Section 12 while working through the previous parts. The most serious omission is that we have not included a systematic discussion of Schrödinger operators with magnetic fields – apart from the atomic Zeeman effect. Here we refer to [11, 6, 7] as a starting point.

2. Schrödinger operators

Our examples are Schrödinger operators H_κ on $L^2(R^\nu)$ of the form:



(2.1)

Figure 1

Here κ is a complex perturbation parameter living in a sector S near $\kappa = 0$: H_0 is the unperturbed operator. We will use the language of quantum mechanics:

$$p_k = -i\,\partial/\partial x_k; \qquad k = 1 \cdots \nu$$

are the momentum operators. States are unit vectors $u \in L^2$ and

$$\langle H \rangle : u \to (u, Hu) \tag{2.2}$$

is the expectation value of the operator H, defined for all states $u \in D(H)$. The range of (2.2) is called the numerical range of H. A fundamental estimate is

$$\begin{aligned} \|(z - H)u\| &\geqq |z - (u, Hu)| \\ &\geqq \text{distance of } z \text{ from the numerical range of } H. \end{aligned} \tag{2.3}$$

The following result is well known from the Hille–Yosida theory of contraction semigroups and will serve to construct H_κ.

Lemma 2.1. *Let H be a densely defined operator on a Hilbert space with numerical range in a closed complex halfplane A. If the range $R(z - H)$ of $z - H$ is dense for some $z \notin A$, then it is dense for all $z \notin A$ and H has a closure with spectrum in A.*

Conditions on the potential

The complex potential $V_\kappa(x)$ is assumed to be locally square integrable so that H_κ is *a priori* defined on $C_0^\infty(R^\nu)$. V_κ is a sum

$$V_\kappa = V_\kappa^1 + V_\kappa^2 \tag{2.4}$$

in L^2_{loc} of two parts with the following properties. V^1_κ is uniformly small with respect to p^2, i.e.

$$\|V^1_\kappa u\| \leqq \alpha \, \|p^2 u\| + \beta(\alpha) \, \|u\| \tag{2.5}$$

holds uniformly in κ for any $\alpha > 0$ and all $u \in C_0^\infty$. V^2_κ is locally bounded (L^∞_{loc}) and takes values in a κ-dependent halfplane away from the negative reals:

$$0 \leqq \mathrm{Re}\, e^{-i\gamma_\kappa} V^2_\kappa(x); \qquad |\gamma_\kappa| \leqq \pi/2 - \varepsilon \tag{2.6}$$

with $\varepsilon > 0$ independent of κ. As $\kappa \to 0$ in S we require that

$$\|(V_\kappa - V_0)u\| \to 0 \tag{2.7}$$

for all $u \in C_0^\infty$. A first look at some of the examples in Section 12 will help to understand these conditions.

Lemma 2.2. *Under the hypotheses* (2.4)–(2.6) *we have*:

(i) $$\langle p^2 \rangle \leqq a \,\mathrm{Re}\, e^{-i\gamma_\kappa} \langle H_\kappa \rangle + b \tag{2.8}$$

for all states $u \in C_0^\infty$, with a and b independent of κ. In particular H_κ has numerical range in a halfplane A_κ:

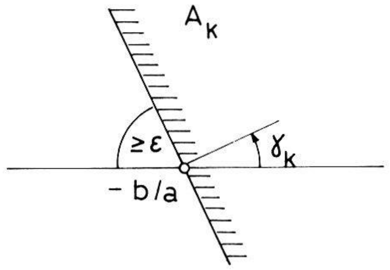


Figure 2

(ii) *$(z - H_\kappa)C_0^\infty$ is dense for $z \notin A_\kappa$. Therefore H_κ has a closure with spectrum contained in A_κ.*

Proof. (i) By the Kato–Rellich theorem ([2], Section X.2) the operator $p^2 - |V^1_\kappa(x)|$ is bounded from below uniformly in κ, namely by $-\beta(\alpha)(1-\alpha)^{-1}$ for $\alpha < 1$. We choose $a > (\sin \varepsilon)^{-1}$ with ε given in (2.6). By the remark above we then have

$$-b \leqq (a \sin \varepsilon - 1)\langle p^2 \rangle - a \langle |V^2_\kappa| \rangle$$

for some b independent of κ. This implies

$$\langle p^2 \rangle \leqq a \,\mathrm{Re}\, e^{-i\gamma_\kappa} \langle p^2 + V^1_\kappa \rangle + b.$$

Adding to this

$$0 \leqq a \,\mathrm{Re}\, e^{-i\gamma_\kappa} \langle V^2_\kappa \rangle$$

we arrive at (2.8). The estimate (2.3) then gives

$$\|(z - H_\kappa)u\| \geqq \mathrm{dist}\,(z, A_\kappa) \tag{2.9}$$

for all states $u \in C_0^\infty$ and with A_κ given in Fig. 2.

(ii) For the rest of the proof we omit the parameter κ which is fixed. For any $z \notin A$, $f \in C_0^\infty$ and $\varepsilon > 0$ we will construct $g \in C_0^\infty$ such that

$$\|(z - H)g - f\| < 2\varepsilon. \tag{2.10}$$

Let $F \in C_0^\infty$, $0 \leqq F \leqq 1$ and $H_F = p^2 + VF$. The potential VF satisfies (2.4)–(2.6) with the same constants, therefore (2.8) and (2.9) also hold for H_F with a, b independent of F. Since VF is small relative to p^2, $(z - H_F)C_0^\infty$ is dense for large negative z. It follows from Lemma 2.1 that $(z - H_F)C_0^\infty$ is dense for all $z \notin A$. Therefore we can pick $h \in C_0^\infty$ so that

$$\|(z - H_F)h - f\| < \varepsilon. \tag{2.11}$$

Using the estimates (2.8) and (2.9) for H_F we obtain

$$\|h\| \leqq (\|f\| + \varepsilon)\{\operatorname{dist}(z, A)\}^{-1}$$

and

$$\|(1 + p^2)^{1/2}h\| < c \tag{2.12}$$

with c independent of the choice of F and h. Now let $G \in C_0^\infty$ such that $0 \leqq G \leqq 1$, $Gf = f$ and

$$\|[G, p^2](1 + p^2)^{-1/2}\| \leqq \varepsilon/c. \tag{2.13}$$

This is possible since the derivatives of G can be taken arbitrary small. Adjusting F (and h) such that $GF = G$ we then find from (2.11)–(2.13):

$$\begin{aligned}\varepsilon &\geqq \|G(z - H_F)h - f\| \\ &\geqq \|(z - H_F)Gh - f\| \\ &\quad - \|[G, p^2](1 + p^2)^{-1/2}\| \, \|(1 + p^2)^{1/2}h\| \\ &\geqq \|(z - H)Gh - f\| - \varepsilon.\end{aligned}$$

This proves (2.10) for $g = Gh$. □

Schrödinger operators (Definition). We denote with H_κ the closure of the operator given by $p^2 + V_\kappa(x)$ on C_0^∞, and we refer to this definition simply by saying that H_κ is a family of Schrödinger operators. For future reference we spell out the essential properties of H_κ:

(i) C_0^∞ is a core of H_κ.
(ii) The estimates (2.8) and (2.9) hold for all states $u \in D(H_\kappa)$
(iii) H_κ has numerical range and spectrum in the halfplane A_κ of Fig. 2. In particular,

$$\|(z - H_\kappa)^{-1}\| \leqq \{\operatorname{dist}(z, A_\kappa)\}^{-1}. \tag{2.14}$$

(iv) As $\kappa \to 0$ in S, H_κ is continuous in the sense that

$$H_\kappa u \to H_0 u \qquad \text{for all } u \in C_0^\infty. \tag{2.15}$$

(v) The adjoint H_κ^* is the complex conjugate of H_κ:

$$H_\kappa^* = JH_\kappa J, \tag{2.16}$$

where J is the complex conjugation. This follows from the observation that the complex conjugate potential also satisfies (2.4)–(2.6). Therefore H_κ^* is again a family of Schrödinger operators.

(vi) H_κ is an analytic family ([3], Section XII.2) in an open region $G \subset S$ if $\|V_\kappa^2 F\|_\infty$ is bounded on G for any $F \in C_0^\infty$ and if the functions $\kappa \to (u, H_\kappa v)$ are analytic in G for all $u, v \in C_0^\infty$.

Proof of (vi). We choose a sequence $F^n \in C_0^\infty$ with $0 \leqq F^n \leqq 1$ and $F^n(x) = 1$ for $|x| < n$. As in the proof of Lemma 2.2 we use that $H_\kappa^n = p^2 + V_\kappa F^n$ has constant domain $D(p^2)$ and a resolvent $R_\kappa^n(z)$ satisfying

$$\|R_\kappa^n(z)\| \leqq \{\text{dist}\,(z, A_\kappa)\}^{-1} \tag{2.17}$$

uniformly in n. By hypothesis, H_κ^n is bounded relative to p^2 uniformly in $\kappa \in G$ for fixed n. Therefore the functions

$$\kappa \to (u, H_\kappa^n v)$$

are analytic in G for any $u \in L^2$ and $v \in D(p^2)$. This implies that the vector function $\kappa \to H_\kappa^n v$ is analytic in G for any $v \in D(p^2)$, i.e. that H_k^n is an analytic family of type A ([3], Section XII.2)

Now we fix a real $z < -b/a$ (Fig. 2) and note that

$$\lim_{n\to\infty} H_\kappa^n u = H_\kappa u \tag{2.18}$$

for all $u \in C_0^\infty$. Since C_0^∞ is a core of H_κ it follows from (2.17) and (2.18) that

$$s - \lim_{n\to\infty} R_\kappa^n(z) = R_\kappa(z)$$

(The argument is given in the proof of Lemma 4.1). Therefore

$$(u, R_\kappa(z)v) = \lim_{n\to\infty} (u, R_\kappa^n(z)v)$$

for all $u, v \in L^2$. Since the functions on the r.h.s. are analytic in $\kappa \in G$ and uniformly bounded we conclude from Vitali's theorem that $(u, R_\kappa(z)v)$ is analytic in G, which in turn implies analyticity of the operator function $\kappa \to R_\kappa(z)$. □

3. Local compactness

As a consequence of (2.8) any multiplier $F \in C_0^\infty$ maps the κ-dependent balls $\|u\| + \|H_\kappa u\| \leqq 1$ into a compact set independent of κ: the image under F of some ball $\|(1 + p^2)^{1/2} u\| \leqq \text{const}$. This proves the first part of

Lemma 3.1. *Let H_κ be a family of Schrödinger operators. Suppose that $\{\kappa\}$ is* *a sequence in S and* $u_\kappa \in D(H_\kappa)$ *a corresponding sequence with* $\|u_\kappa\| + \|H_\kappa u_\kappa\| <$ const. *If* $u_\kappa \underset{w}{\to} 0$, *then*

$$\|F\ u_\kappa\| \to 0 \text{ and} \tag{3.1}$$

$$\|[H_\kappa, F]u_\kappa\| \to 0 \tag{3.2}$$

for any $F \in C_0^\infty$.

Proof. The estimate

$$\|p_k u\| \leqq ab^{-1/2} \|H_k u\| + b^{1/2} \|u\| \tag{3.3}$$

follows from (2.8) and shows that p_k is small relative to H_κ since b is arbitrary large. Therefore p_k and

$$[H_\kappa, F] = [p^2, F] = 2iF_{,k}p_k - \Delta F \tag{3.4}$$

are defined on $D(H_\kappa)$, where

$$\|[H_\kappa, F]u_\kappa\| \leqq \alpha \|H_\kappa u_\kappa\| + \beta(\alpha) \|u_\kappa\| \tag{3.5}$$

for any $\alpha > 0$, with $\beta(\alpha)$ independent of κ. In particular, $\|H_\kappa F u_\kappa\|$ is bounded under the hypothesis of the lemma. Now let $G \in C_0$ such that $GF = F$. Then

$$\begin{aligned}\|[H_\kappa, F]u_\kappa\| &= \|[H_\kappa, F]Gu_\kappa\| \\ &\leqq \alpha \|H_\kappa G u_\kappa\| + \beta(\alpha) \|Gu_\kappa\|.\end{aligned}$$

For small α the first term is arbitrary small uniformly in κ, while the second term converges to zero by (3.1). □

Local compactness in the sense of Lemma 3.1 will play the key role in proving stability of eigenvalues and asymptotic estimates. Here we deviate briefly from this course. Following Enss [23] we show the use of local compactness for locating the essential spectrum. This will not be needed in Sections 4–11 but will be convenient to describe the spectral properties of the examples.

The essential spectrum

We define the essential spectrum of an operator H by

$$\sigma_{\text{ess}}(H) = \sigma(H) \setminus \sigma_{\text{disc}}(H)$$

where the discrete spectrum $\sigma_{\text{disc}}(H)$ is the set of all isolated eigenvalues with finite multiplicities.

We will use the following abstract result, valid for any closed operator on a Hilbert space:

Lemma 3.2. *Suppose that*

$$\|(\lambda - H)v_m\| \to 0 \text{ for a sequence of states } v_m \in D(H) \text{ with } v_m \underset{w}{\to} 0. \tag{3.6}$$

Then $\lambda \in \sigma_{ess}(H)$. *Conversely (and this is the deep part)* (3.6) *holds if* λ *is a boundary point of* $\sigma_{ess}(H)$.

Remarks

If $\sigma_{ess}(H)$ contains no interior points then (3.6) is evidently equivalent to $\lambda \in \sigma_{ess}(H)$. An example is the case $H = H^*$ where (3.6) is known as Weyl's criterion and has a simple proof by the spectral representation. The proof for the general case is implicit in [1] (see [44]).

We now consider a single Schrödinger operator (depending trivially on κ):

Lemma 3.3. *Let H be a Schrödinger operator and suppose that*

$$\|(\lambda - H)u_n\| \to 0 \text{ for a sequence of states } u_n \in C_0^\infty(|x| > n),\ n \to \infty. \tag{3.7}$$

Then $\lambda \in \sigma_{ess}(H)$. *Conversely,* (3.7) *holds if* λ *is a boundary point of* $\sigma_{ess}(H)$.

Proof. $u_n \in C_0^\infty(|x| > n)$ means that $u_n \in C_0^\infty$ with supp (u_n) in $|x| > n$. Then $u_n \xrightarrow[w]{} 0$ so that the first part is obvious from Lemma 3.2. Now let λ be boundary point of $\sigma_{ess}(H)$ so that (3.6) holds. Since C_0^∞ is a core of H we can assume that $v_m \in C_0^\infty$. Let $F_n \in C_0^\infty$ with $F_n(x) = 1$ for $|x| < n$. By Lemma 3.1 we can then choose $m = m(n)$ such that

$$\|(1 - F_n)v_m\| > \tfrac{1}{2} \quad \text{and} \quad \|(\lambda - H)(1 - F_n)v_m\| < 1/n.$$

Then (3.7) holds for the sequence

$$u_n = (1 - F_n)v_m \, \|(1 - F_n)v_m\|^{-1}. \qquad \square$$

4. Strong resolvent convergence

Strong convergence of the resolvent

$$R_\kappa(z) = (z - H_\kappa)^{-1}$$

as $\kappa \to 0$ implies that $\|R_\kappa(z)\|$ is bounded for small κ in S, by the uniform boundedness theorem. We therefore introduce the sets

$$\begin{aligned} P(H_\kappa) &= \{z \mid R_\kappa(z) \text{ exists and is uniformly bounded for small } \kappa\} \\ \sum(H_\kappa) &= C \setminus P(H_\kappa). \end{aligned} \tag{4.1}$$

These sets reflect the behaviour of H_κ as $\kappa \to 0$. In many respects they will play a role similar to the resolvent set and spectrum of a single operator, to which they reduce if H_κ is independent of κ. We remark, however, that $\sum(H_\kappa)$ is not determined by the spectrum $\sigma(H_\kappa)$ for small κ (see Example (a)). The standard resolvent estimate

$$\|R_\kappa(z)\| \leqq \|R_\kappa(z_0)\| \{1 - |z - z_0| \|R_\kappa(z_0)\|\}^{-1} \tag{4.2}$$

shows that $P(H_\kappa)$ is an open subset of the unperturbed resolvent set $\rho(H_0)$. Moreover, given any compact $\Gamma \subset P(H_k)$, there exists a neighbourhood U (in S) of $\kappa = 0$ so that

$$\|R_\kappa(z)\| \leqq \text{const. for } (z, \kappa) \in \Gamma \times U. \tag{4.3}$$

This follows from (4.2) by a simple covering argument.

Lemma 4.1. *For a family of Schrödinger operators, $R_\kappa(z)$ and $R_\kappa^*(z)$ are strongly continuous at $\kappa = 0$ if and only if $z \in P(H_\kappa)$.*

Proof. Let $z \in P(H_\kappa)$. Since $\|R_\kappa(z)\| < c$ for small κ it suffices to prove $R_\kappa(z)u \to R_0(z)u$ as $\kappa \to 0$ for the set of vectors $u = (z - H_0)v$, $v \in C_0^\infty$. This set is dense because C_0^∞ is a core of H_0. Then

$$\begin{aligned} \|(R_\kappa(z) - R_0(z))u\| &= \|R_\kappa(z)(H_\kappa - H_0)v\| \\ &\leqq c\, \|(H_\kappa - H_0)v\| \to 0 \end{aligned}$$

by (2.15). The same argument applies to H_κ^*. □

Remark

There is a sharper version of Lemma 4.1 which will be needed in the proof of Theorem 9.2. Suppose that $\|R_\kappa(z)\| < c$ for some fixed z and for a sequence $\{\kappa\} \to 0$. Then $z \in \rho(H_0)$ and $R_\kappa(z) \underset{s}{\to} R_0(z)$. Proof: Given $u \in L^2$ we have $R_\kappa(z)u \underset{w}{\to} v$ for a subsequence $\{\kappa\} \to 0$ and therefore

$$\begin{aligned} (f, u) &= \lim ((\bar{z} - H_\kappa^*)f, R_\kappa(z)u) \\ &= ((\bar{z} - H_0^*)f, v) \end{aligned}$$

for any $f \in C_0^\infty$. Since C_0^∞ is a core of H_0^* this extends to all $f \in D(H_0^*)$ which proves that $u = (z - H_0)v$, i.e. that $R(z - H_0) = L^2$. By the same argument, $R(\bar{z} - H_0^*) = L^2$, which shows that $z - H_0$ is injective. Therefore $(z - H_0)^{-1}$ exists and is bounded by the closed graph theorem. Strong resolvent convergence now follows from Lemma 4.1. □

Lemma 4.2. *For a family of Schrödinger operators the following are equivalent:*

(i) $z \in P(H_\kappa)$

(ii) $\|(z - H_\kappa)u\| \geqq \varepsilon > 0$ (4.4)
for small κ and all states $u \in C_0^\infty$

(iii) $z \in \rho(H_0)$ *and*
$\|(z - H_\kappa)u\| \geqq \varepsilon > 0$ (4.5)
for small κ and all states $u \in C_0^\infty(|x| > n)$,
n fixed but arbitrary

Proof. (ii) and (iii) evidently follow from (i). To show (ii)$\to$(i) we note that

$$\|(\bar{z} - H_\kappa^*)u\| \geqq \varepsilon > 0 \tag{4.6}$$

follows from (4.4) by complex conjugation. Since C_0^∞ is a core of H_κ and of H_κ^*, (4.4) and (4.6) hold for all u in the domain of H_κ and H_κ^*. The two inequalities imply that $R(z - H_\kappa) = L^2$ and $\|R_\kappa(z)\| \leqq 1/\varepsilon$ for small κ.

(iii)$\to$(i). We assume that $z \notin P(H_\kappa)$ and derive a contradiction to (iii). By (ii) we have

$$\|(z - H_\kappa)u_\kappa\| \to 0 \tag{4.7}$$

for some sequence $\{\kappa\} \to 0$ and corresponding states $u_\kappa \in C_0^\infty$. By passing to a subsequence we may assume that $u_k \underset{w}{\to} u$. For any $v \in C_0^\infty$ we then find

$$0 = \lim ((\bar{z} - H_\kappa^*)v, u_\kappa) = ((\bar{z} - H_0^*)v, u).$$

This extends to all $v \in D(H_0^*)$, which proves that $(z - H_0)u = 0$. Since $z \in \rho(H_0)$ it follows that $u = 0$, i.e. $u_\kappa \underset{w}{\to} 0$. From (4.7) and Lemma 3.1 we now deduce

$$\|(1 - F)u_\kappa\| \to 1 \text{ and } \|(z - H_\kappa)(1 - F)u_\kappa\| \to 0$$

for any $F \in C_0^\infty$. This is in contradiction to (4.5) if $F(x) = 1$ for $|x| \leqq n$. □

5. Stable eigenvalues

A discrete eigenvalue λ of H_0 is said to be stable (with respect to the family H_κ) if the following two conditions are satisfied:

(i) λ is embedded in the set $P(H_\kappa)$, i.e.

$$\{z \mid 0 < |z - \lambda| < \delta\} \subset P(H_\kappa) \tag{5.1}$$

for some δ. By (4.3) the spectral projection

$$P_\kappa = (2\pi i)^{-1} \oint_\Gamma dz R_\kappa(z) \tag{5.2}$$

is then defined for any circle Γ of radius $r < \delta$ around λ if κ is sufficiently small.

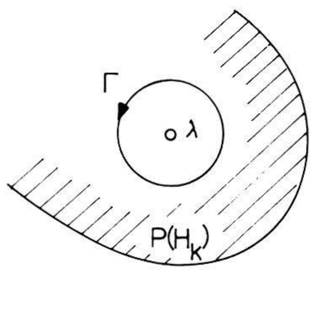


Figure 3

(ii) $$\lim_{\kappa \to 0} \|P_\kappa - P_0\| = 0. \tag{5.3}$$

Remarks

The second condition implies

$$\dim P_\kappa = \dim P_0 \tag{5.4}$$

for small κ (as soon as $\|P_\kappa - P_0\| < 1$). Then the part of $\sigma(H_\kappa)$ enclosed by Γ consists only of discrete eigenvalues with total algebraic multiplicity given by $\dim P_0$. Since the radius r of Γ is arbitrary small, this group of perturbed eigenvalues (called the λ-group) converges to λ as $\kappa \to 0$.

It is important to understand the difference between strong convergence and norm convergence of P_κ as $\kappa \to 0$. (5.1) together with Lemma 4.1 implies

$$P_\kappa \underset{s}{\to} P_0 \quad \text{and} \quad P_\kappa^* \underset{s}{\to} P_0^* \tag{5.5}$$

and therefore

$$\dim P_\kappa \geqq \dim P_0$$

for small κ. Thus $\dim P_\kappa$ need not be finite, and even if it is finite, perturbed eigenvalues may disappear at $\kappa = 0$ (Example (b)). On the other hand it is known that (5.4) together with (5.5) implies (5.3) ([1] chap. VIII, §.1. An error in Lemma 1.21 has been corrected in the 2nd edition).

Theorem 5.1 [44]. *Let H_κ be a family of Schrödinger operators. Then a discrete eigenvalue λ of H_0 is stable if*

$$\|(\lambda - H_\kappa)u\| \geqq \varepsilon > 0 \tag{5.6}$$

for small κ and all states $u \in C_0^\infty(|x| > n)$, n fixed but arbitrary.

Proof. (i) Since $\lambda \in \sigma_{\text{disc}}(H_0)$ we have

$$\{z \mid 0 < |z - \lambda| < \delta\} \subset \rho(H_0) \tag{5.7}$$

for some $\delta < \varepsilon/2$. In this disc (5.6) gives

$$\|(z - H_\kappa)u\| \geqq \varepsilon/2 \tag{5.8}$$

for small κ and all states $u \in C_0^\infty(|x| > n)$. By Lemma 4.2 the disc (5.7) is therefore contained in $P(H_\kappa)$.

(ii) By the remark before Theorem 5.1 we need only prove that

$$\dim P_\kappa \leqq \dim P_0 \tag{5.9}$$

for small κ. Suppose this is false. then we have

$$P_\kappa u_\kappa = u_\kappa \quad \text{and} \quad P_0 u_\kappa = 0 \tag{5.10}$$

for some sequence $\{\kappa\} \to 0$ and corresponding states u_κ. By passing to a subsequence we may assume that $u_k \underset{w}{\to} u$. By using (5.5) we then find

$$P_0 u = u \quad \text{and} \quad P_0 u = 0$$

as the weak limit of (5.10), i.e. $u_\kappa \underset{w}{\to} 0$. On the other hand it follows from (4.3) and (5.2) that $\|H_\kappa u_\kappa\| \leqq \|H_\kappa P_\kappa\| <$ const. for small κ, so that

$$\|(1-F)u_\kappa\| \to 1 \tag{5.11}$$

for any $F \in C_0^\infty$ by Lemma 3.1. Choosing $F(x) = 1$ for $|x| < n$ we deduce from (5.8) that

$$\|(z-H_\kappa)(1-F)u\| \geqq \frac{\varepsilon}{2}\|(1-F)u\|$$

for small κ, all $z \in \Gamma$ and all $u \in C_0^\infty$. By (3.5) this extends to all $u \in D(H_\kappa)$. In particular,

$$\begin{aligned}\frac{\varepsilon}{2}\|(1-F)R_\kappa(z)u_\kappa\| &\leqq (z-H_\kappa)(1-F)R_\kappa(z)u_\kappa\| \\ &\leqq \|(1-F)u_\kappa\| + \|[H_\kappa, F]R_\kappa(z)u_\kappa\|\end{aligned}$$

for small κ and all $z \in \Gamma$. Therefore

$$\begin{aligned}\frac{\varepsilon}{2}\|(1-F)u_\kappa\| &= \frac{\varepsilon}{2}\|(1-F)P_\kappa u_\kappa\| \\ &\leqq (2\pi)^{-1}\oint |dz|\frac{\varepsilon}{2}\|(1-F)R_\kappa(z)u_\kappa\| \\ &\leqq r\|(1-F)u_\kappa\| + (2\pi)^{-1}\oint |dz|\, \|[H_\kappa, F]R_\kappa(z)u_\kappa\|,\end{aligned}$$

where $r < \delta < \varepsilon/2$ is the radius of Γ. By (3.5) (4.3) and part (i) of the proof, the integrand is bounded uniformly for small κ and $z \in \Gamma$. By (3.2) it also vanishes pointwise as $\kappa \to 0$ since $R_\kappa(z)u_k \underset{w}{\to} 0$. Therefore the integral vanishes in the limit $\kappa \to 0$, and since $r < \varepsilon/2$ we conclude that $\|(1-F)u_\kappa\| \to 0$, in contradiction to (5.11). This proves (5.9). □

6. Stability estimates

In this section we discuss ways and means to prove the stability condition (5.6).

(i) *Numerical range*

(5.6) holds if

$$|\lambda - (u, H_\kappa u)| \geqq \varepsilon > 0 \tag{6.1}$$

for small κ and all states $u \in C_0^\infty(|x| > n)$. This is sufficient to prove stability in simple one-body systems like the anharmonic oscillator (Examples (d), (f)) or the one-body Stark effect (Example (e)).

(ii) *Localization*

In some situations where (i) fails we can prove (5.6) by localization. The idea is to divide the region $|x|>n$ into subregions where H_κ reduces to simpler form. This is achieved by using appropriate partitions of unity on R^ν depending on the perturbation parameter κ:

Definition. A partition of unity on $|x|>n$ is a finite set of C^∞-functions $J_\kappa^\alpha(x)$, $\alpha=1\cdots\beta$, on $|x|>n$ with the properties

$$0\leqq J_\kappa^\alpha(x)\leqq 1;\qquad \sum_\alpha J_\kappa^\alpha(x)=1 \tag{6.2}$$

$$\lim_{(\kappa,x)\to(0,\infty)} DJ_\kappa^\alpha(x)=0.$$

where D is any first or second order derivative with respect to x.

Lemma 6.1. *Let H_κ be a family of Schrödinger operators and $\{J_\kappa^\alpha\}$ a partition of unity on $|x|>n$. Then* (5.6) *holds if and only if*

$$\|(\lambda-H_\kappa)J_\kappa^\alpha u\|\geqq\varepsilon\,\|J_\kappa^\alpha u\|,\qquad \alpha=1\cdots\beta, \tag{6.3}$$

for some positive ε, n, small κ and all $u\in C_0^\infty(|x|>n)$.

Proof. (6.3) evidently follows from (5.6). To show the converse suppose that (5.6) is false for any positive ε, n. Then

$$\|(\lambda-H_\kappa)u_{\kappa^n}\|\to 0$$

for some sequence $(\kappa,n)\to(0,\infty)$ and corresponding states $u_{\kappa n}\in C_0(|x|>n)$. It follows from (3.3) and (6.2) that

$$\|[H_\kappa, J_\kappa^\alpha]u_{\kappa n}\|\to 0$$

and therefore

$$\|(\lambda-H_\kappa)J_\kappa^\alpha u_{\kappa n}\|\to 0. \tag{6.4}$$

On the other hand,

$$\sum_{\alpha=1}^{\beta} J_\kappa^\alpha u_{\kappa n}\|=1$$

implies that

$$\|J_\kappa^\alpha u_{\kappa n}\|\geqq\beta^{-1} \tag{6.5}$$

for some fixed α and an infinite subsequence of $\{\kappa,n\}$. (6.4) and (6.5) are in contradiction to (6.3). □

Corollary. *Suppose that on the support of each J_κ^α the operator H_κ reduces to* *an operator H_κ^α in the sense that*

$$\|(H_\kappa - H_\kappa^\alpha)J_\kappa^\alpha u\| \leqq o(\kappa, n)(\|H_\kappa J_\kappa^\alpha u\| + \|J_\kappa^\alpha u\|) \tag{6.6}$$

for all $u \in C_0(|x| > n)$, where $o(\kappa, n) \to 0$ as $(\kappa, n) \to (0, \infty)$. Then H_κ can be replaced by H_κ^α in (6.3). *In particular the stability condition* (5.6) *holds if*

$$\lambda \in P(H_\kappa^\alpha), \qquad \alpha = 1 \cdots \beta. \tag{6.7}$$

This technique is used for the one-body shape resonance (Example (c)) and for N-body problems (Examples (g), (h)) in combination with the result described next:

(iii) *Decomposition into subsystems*

For N-body systems we normally use partitions of unity which decompose the system into noninteracting parts [23, 35, 42]. This means that the local operators H_κ^α are of the form

$$H_\kappa^\alpha = H_\kappa^1 \otimes 1 + 1 \otimes H_\kappa^2 \tag{6.8}$$

with respect to some factorization of L^2. (6.7) then poses the problem to find $P(H_\kappa)$ from $P(H_\kappa^1)$ and $P(H_\kappa^2)$. We will study this question more generally for abstract families of the form

$$H_\kappa = H_\kappa^1 + H_\kappa^2 \quad \text{with} \quad [H_\kappa^1, H_\kappa^2] = 0. \tag{6.9}$$

Given the right sectoriality properties of H_κ^1 and H_κ^2 we can then construct and estimate the resolvent R_κ as the convolution of the resolvents R_κ^1, R_κ^2.

Definition. We say that a family of operators H_κ ($\kappa \in S$) on a Hilbert space $\mathscr{H}$ is uniformly m-sectorial with sector $\mathscr{S}$ if H_κ is densely defined and if both the numerical range and the spectrum of H_κ are contained in some κ-independent complex sector $\mathscr{S}$ with opening angle $< \pi$. This implies in particular

$$\|R_\kappa(z)| \leqq \{\operatorname{dist}(z, \mathscr{S})\}^{-1} \tag{6.10}$$

for all $z \notin \mathscr{S}$, and therefore

$$\sum(H_\kappa) \subset \mathscr{S}. \tag{6.11}$$

Theorem 6.2. *Suppose that H_κ^1 and H_κ^2 are uniformly m-sectorial with sectors $\mathscr{S}^1$ and $\mathscr{S}^2$. If the resolvents $R_\kappa^1(z_1)$ and $R_\kappa^2(z_2)$ commute (wherever they exist) and if $\mathscr{S} = \mathscr{S}^1 + \mathscr{S}^2$ is a sector of opening angle $< \pi$, then $H_\kappa = H_\kappa^1 + H_\kappa^2$ has a closure $\bar{H}_\kappa$ with*

$$\sum(\bar{H}_\kappa) \subset \sum(H_\kappa^1) + \sum(H_k^2). \tag{6.12}$$

Moreover, $\bar{H}_\kappa$ is uniformly m-sectorial with sector $\mathscr{S}$.

Proof. We assume that $z \notin \Sigma(H_\kappa^1) + \Sigma(H_\kappa^2)$ and we establish the convolution formula

$$R_\kappa(z) = (2\pi i)^{-1} \oint_\Gamma ds\, R_\kappa^1(z-s) R_\kappa^2(s) \tag{6.13}$$

for the resolvent of $\bar{H}_\kappa$ to prove that $z \notin \Sigma(\bar{H}_\kappa)$. The path Γ in (6.13) is chosen as follows:

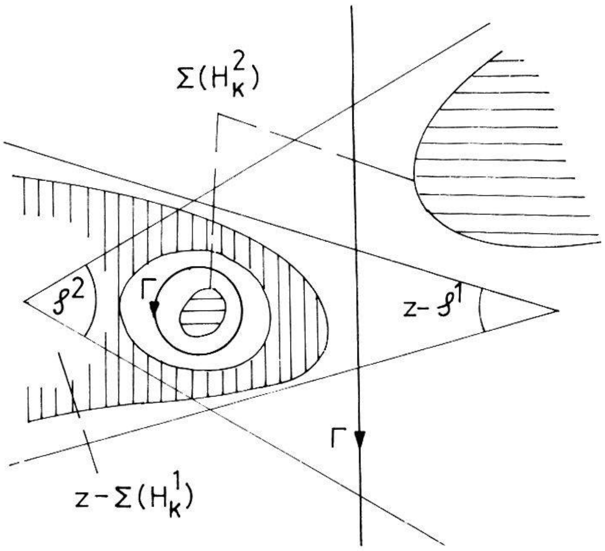


Figure 4

By hypothesis, the closed sets $z - \Sigma(H_\kappa^1) \subset z - \mathscr{S}^1$ and $\Sigma(H_\kappa^2) \subset \mathscr{S}^2$ are disjoint, and $(z - \mathscr{S}^1) \cap \mathscr{S}^2$ is compact. Γ is a generally multiple path with the following properties:

(i) Outside some finite region it is a straight line diverging from $(z - \mathscr{S}^1) \cup \mathscr{S}^2$. There the integrand of (6.13) is bounded uniformly in κ by

$$\{\text{dist}(s, z - \mathscr{S}^1) \cdot \text{dist}(s, \mathscr{S}^2)\}^{-1}, \tag{6.14}$$

which is of order $|s|^{-2}$ as $s \to \infty$.

(ii) Within $z - P(H_\kappa^1)$, Γ can be deformed into a path Γ^1 outside $z - \mathscr{S}^1$, and within $P(H_\kappa^2)$ into a path Γ^2 outside $\mathscr{S}^2$:

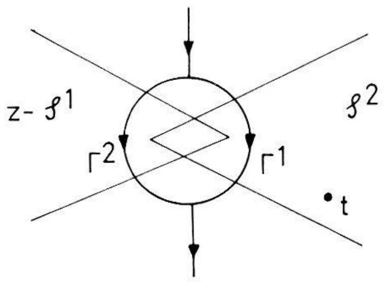


Figure 5

(We give no formal existence proof for Γ since Γ is easily constructed in the applications we have in mind.) The part of Γ where the bound (6.14) exceeds 1 is compact. By (4.3) the integral (6.13) is therefore well defined for small κ and defines a bounded operator $R_\kappa(z)$ with $\|R_\kappa(z)\| <$ const. for small κ. It remains to prove that $R_\kappa(z) = (z - \bar{H}_\kappa)^{-1}$.

H_κ is densely defined since its domain includes the range of

$$R^1_\kappa(\lambda)R^2_\kappa(\lambda) = R^2_\kappa(\lambda)R^1_\kappa(\lambda)$$

for $\lambda \notin \mathscr{S}^1 \cup \mathscr{S}^2$. Since H^1_κ and H^2_κ are closed and since $R_\kappa(z)$ is a norm-limit of finite Riemann sums we have

$$(z - H_\kappa)R_\kappa(z)u = R_\kappa(z)(z - H_\kappa)u$$

for any $u \in D(H_\kappa)$. To evaluate this expression we use the identity

$$R_\kappa(z)(z - H_\kappa)u = (2\pi i)^{-1} \oint_\Gamma ds\,(s-t)^{-1}(z-s-H^1_\kappa)^{-1}(z-t-H^1_k)u$$
$$- (2\pi i)^{-1} \oint_\Gamma ds\,(s-t)^{-1}(s-H^2_\kappa)^{-1}(t-H^2_\kappa)u,$$

which holds for any $t \notin \Gamma$ and all $u \in D(H_\kappa)$. Both integrals exist since the integrands vanish like $|s|^{-2}$ as $s \to \infty$. To be specific we choose the auxiliary point t as indicated in Fig. 5. In the first integral Γ can now be replaced by Γ^1 and in the second integral by Γ^2. The integral over Γ^2 vanishes and the integral over Γ^1 is the residue at the pole $s = t$. Thus we obtain

$$R_\kappa(z)(z - H_\kappa)u = (z - H_\kappa)R_\kappa(z)u = u \tag{6.15}$$

for all $u \in D(H_\kappa)$. To show that H_κ is closable, suppose that $u_n \in D(H_\kappa)$, $u_n \to 0$ and $H_\kappa u_n \to v$. Then (6.15) implies $R_\kappa(z)v = 0$ and therefore

$$R_\kappa(z)w = 0 \quad \text{for} \quad w = R^1_\kappa(\lambda)R^2_\kappa(\lambda)v$$

with $\lambda \notin \mathscr{S}^1 \cup \mathscr{S}^2$. Since $w \in D(H_\kappa)$ it follows from (6.15) that $w = 0$ and therefore $v = 0$. Hence H_κ has a closure $\bar{H}_\kappa$ and (6.15) then implies that $R_\kappa(z) = (z - \bar{H}_\kappa)^{-1}$. This proves (6.12).

If H^1_κ and H^2_κ are independent of κ, (6.12) says that $\sigma(\bar{H}) \subset \sigma(H^1) + \sigma(H^2)$. In the general case we thus have

$$\sigma(\bar{H}_\kappa) \subset \sigma(H^1_\kappa) + \sigma(H^2_\kappa)$$

which shows that $\bar{H}_\kappa$ is uniformly m-sectorial with sector $\mathscr{S}$. □

7. Reduction to finite dimension

Let λ be a stable discrete eigenvalue of H_0. The λ-group of perturbed eigenvalues is given by

$$\lambda(\kappa) = \lambda + \Delta\lambda(\kappa)$$

where $\Delta\lambda(\kappa)$ is the multivalued function representing the eigenvalues of

$$P_\kappa(H_\kappa - \lambda)P_\kappa | M_\kappa = \text{range of } P_\kappa$$

for small κ. We define

$$D(\kappa) = P_0 P_\kappa P_0|_{M_0} \tag{7.1}$$

Since $D(\kappa) \to 1$ as $\kappa \to 0$, the operator

$$S(\kappa) = D(\kappa)^{-1/2} P_0 |_{M_\kappa}$$

is well defined for small κ. It maps M_κ onto M_0 and has the inverse

$$S(\kappa)^{-1} = P_\kappa D(\kappa)^{-1/2}|_{M_0}$$

as can be seen from the diagram:

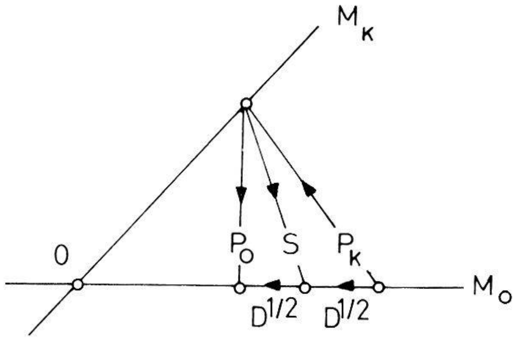


Figure 6

For small κ the perturbations $\Delta\lambda(\kappa)$ can therefore be represented as the eigenvalues of

$$\begin{aligned} E(\kappa) &= S(\kappa) P_\kappa (H_\kappa - \lambda) P_\kappa S(\kappa)^{-1} \\ &= D(\kappa)^{-1/2} N(\kappa) D(\kappa)^{-1/2} \end{aligned} \tag{7.2}$$

with

$$N(\kappa) = P_0 P_\kappa (H_\kappa - \lambda) P_\kappa P_0. \tag{7.3}$$

Here all the operators $D(\kappa)$, $N(\kappa)$ and $E(\kappa)$ are operators acting on the finite dimensional unperturbed spectral subspace M_0.

Our choice of an isomorphism $S(\kappa)$: $M_\kappa \to M_0$ is of course largely arbitrary and in fact not optimal: P_κ will in general be defined in a larger neighbourhood of $\kappa = 0$ than $S(\kappa)$. A more careful but less explicit construction is given in [1]. Compared with the equivalent forms $D(\kappa)^{-1}N(\kappa)$ and $N(\kappa)D(\kappa)^{-1}$ the operator $E(\kappa)$ given by (7.2) has the advantage to be symmetric in selfadjoint problems: if the sector S contains a real half-line where $H_\kappa^* = H_\kappa$, then $E(\kappa)^* = E(\kappa)$ for real κ.

8. The Rayleigh–Schrödinger expansion

We first recall the formal derivation of the RS-expansion which applies to families of the form

$$H_\kappa = H_0 + \kappa V. \tag{8.1}$$

It starts from the iterated resolvent equation

$$R_\kappa = R_0 + \kappa R_\kappa V R_0$$
$$= \sum_{k=0}^{N-1} \kappa^k R_0 (VR_0)^k + \kappa^N R_\kappa (VR_0)^N. \tag{8.2}$$

Inserting this into (5.2) and (7.1) we obtain

$$D(\kappa) = \sum_{k=0}^{N-1} D_k \kappa^k + D_N(\kappa)\kappa^N, \tag{8.3}$$

where $D_k = D_k(0)$ and

$$D_k(\kappa) = (2\pi i)^{-1} \oint_\Gamma dz P_0 R_\kappa(z)[VR_0(z)]^k P_0. \tag{8.4}$$

A similar expansion holds for $N(\kappa)$: the only difference is an extra factor $(z - \lambda)$ in the integral (8.4). Representing now $R_0(z)$ by its Laurent series at the pole $z = \lambda$ one obtains the expansion coefficients D_k and N_k as the first- and second-order residues of $R_0(z)[VR_0(z)]^k$. We sketch the procedure for the most important case where the unperturbed eigenvalue λ is semi-simple, i.e. when

$$(\lambda - H_0)P_0 = 0.$$

Then $R_0(z)$ has the Laurent expansion

$$R_0(z) = (z - \lambda)^{-1} P_0 + S_0 - (z - \lambda) S_0^2 + (z - \lambda)^2 S_0^3 + \cdots,$$

where $S_0 = (\lambda - H_0)^{-1}(1 - P_0)$ is the regular part of the unperturbed resolvent at the pole $z = \lambda$. This gives:

$$\begin{array}{ll}
D_0 = P_0 & N_0 = 0 \\
D_1 = 0 & N_1 = P_0 V P_0 \\
D_2 = -P_0 V S_0^2 V P_0 & N_2 = P_0 V S_0 V P_0 \\
\cdots\cdots & N_3 = P_0 V S_0 V S_0 V P_0 \\
 & \qquad -P_0 V S_0^2 V P_0 V P_0 \\
 & \qquad -P_0 V P_0 V S_0^2 V P_0 \\
 & \cdots\cdots
\end{array}$$

From (7.2) we then find the expansion

$$E(\kappa) = \sum_{k=0}^{N-1} E_k \kappa^k + E_N(\kappa)\kappa^N, \tag{8.5}$$

which for a semi-simple eigenvalue λ begins with

$$E_0 = N_0 = 0$$
$$E_1 = N_1 = P_0 V P_0$$

$$E_2 = N_2 = P_0 V S_0 V P_0$$
$$E_3 = N_3 - \tfrac{1}{2}(N_1 D_2 + D_2 N_1)$$
$$= P_0 V S_0 V S_0 V P_0$$
$$- \tfrac{1}{2} P_0 V P_0 V S_0^2 V P_0 - \tfrac{1}{2} P_0 V S_0^2 V P_0 V P_0$$

. . . . .

What then remains is a problem of finite dimensional asymptotic perturbation theory: to find the expansions of the eigenvalues $\Delta\lambda(\kappa)$ of $E(\kappa)$ from the expansion (8.5). This is discussed in Section 11.

We now investigate the validity of these expansions, assuming that

$$H_\kappa = p^2 + V_0(x) + \kappa V(x) \qquad (\kappa \in S) \tag{8.6}$$

is a family of Schrödinger operators with a potential $V_\kappa = V_0 + \kappa V$ satisfying (2.4)–(2.6). With $D(V)$ we denote the natural domain of the multiplication operator $V(x)$.

Lemma 8.1. *Let H_κ be the family* (8.6). *If $z \in \rho(H_\kappa) \cap \rho(H_0)$, then* (8.2) *holds on $D([VR_0(z)]^N)$, i.e. on all vectors u with*

$$R_0(z)[VR_0(z)]^k u \in D(V) \text{ for } k = 0 \cdots N-1.$$

Proof. It suffices to prove the case $N = 1$: the general case then follows by iteration. Since C_0^∞ is a core of H_κ^* we need only show that

$$(v, R_\kappa(z)u) = (v, R_0(z)u) + \kappa(v, R_\kappa(z)VR_0(z)u) \tag{8.7}$$

for all $v = (\bar{z} - H_\kappa^*)w$ with $w \in C_0^\infty$. Then

$$v = (\bar{z} - H_0^*)w - \bar{\kappa}\bar{V}w$$

which reduces (8.7) to an identity. □

From the definition of stable eigenvalues and (4.3) we now obtain directly:

Lemma 8.2. *Let H_κ be the family* (8.6). *Suppose that λ is a stable eigenvalue of H_0 and Γ the circle in* (5.2). *If*

$$B_k = \sup_{z \in \Gamma} \|[VR_0(z)]^k P_0\| < \infty \tag{8.8}$$

for $k = 1, \ldots, N$, then $D(\kappa)$ has the RS-*expansion* (8.3) *with a remainder estimate*

$$\|D_N(\kappa)\| \leqq A B_N \tag{8.9}$$

for small κ, where A is independent of N. With suitable A the same estimate holds for the expansion of $N(\kappa)$.

Remarks

Using (2.16) we can in fact expand $D(\kappa)$ and $N(\kappa)$ up to order $2N$ under the hypothesis of Lemma 8.2. For this we write

$$P_0R_\kappa(z) = (R_\kappa(z)^*P_0^*)^* = (JR_\kappa(z)P_0J)^*$$

in the integrand of (8.4) and expand $R_\kappa(z)P_0$ as before. The result is

$$D(\kappa) = \sum_{k=0}^{2N-1} D_k\kappa^k + D_{2N}(\kappa)\kappa^{2N} \tag{8.10}$$

with the remainder

$$D_{2N}(\kappa) = (2\pi i)^{-1}\oint_\Gamma dz (J[VR_0(z)]^N P_0J)^* R_\kappa(z)[VR_0(z)]^N P_0$$

satisfying

$$\|D_{2N}(\kappa)\| \leqq AB_N^2. \tag{8.11}$$

We also remark that by Lemma 2.3 (vi) $H_\kappa = H_0 + \kappa V$ is an analytic family on the interior of S. Therefore $D(\kappa)$, $N(\kappa)$ and $E(\kappa)$ are analytic for small κ in the interior of S.

9. Asymptotic estimates

Analytic perturbation theory deals with the case where $VR_0(z)$ is bounded for $z \in \rho(H_0)$. Then any discrete eigenvalue of H_0 is stable and the RS-series are convergent for small κ.

If $VR_0(z)$ is unbounded we have to exploit special properties of P_0 to arrive at the estimate (8.8), in particular the characteristic exponential fall-off of the unperturbed eigenfunctions which has been studied independently in great detail (see e.g. [3], Section XIII.11., or [5, 19, 21]). The method described below to obtain asymptotic estimates from exponential bounds is due to Herbst [30].

First we discuss exponential bounds: this concerns only the unperturbed operator H_0. Let f be a positive C^∞-function on R^ν and

$$\begin{aligned} H_f &= e^f H_0 e^{-f} \\ &= (p + i\nabla f)^2 + V_0(x), \end{aligned} \tag{9.1}$$

defined on C_0^∞. This is not quite a Schrödinger operator in our terminology, but under natural conditions on f it will have very similar properties. Let us assume for the moment that H_f has a closure with non-empty resolvent set. We denote with H_f this closure and set $(z - H_f)^{-1} = R_f(z)$. It follows from (9.1) that

$$e^{-f}R_f(z) = R_0(z)e^{-f} \tag{9.2}$$

whenever both resolvents exist. Since $\exp(-f)$ is bounded and maps C_0^∞ onto itself we see that $R_f(z)$ and $R_0(z)$ have the same poles in the complement of $\sigma_{ess}(H_f) \cup \sigma_{ess}(H_0)$. Thus H_f and H_0 have the same discrete eigenvalues in this region, with corresponding spectral projections related by

$$e^{-f}P_f = P_0 e^{-f}. \tag{9.3}$$

Now, since $\exp(-f)$ has dense range and since $\dim P_0 < \infty$,

$$R(P_0) = R(P_0 e^{-f}) \subset R(e^{-f})$$

which proves

$$\|e^f P_0\| < \infty. \tag{9.4}$$

We remark that for a given discrete eigenvalue λ of H_0 the exponential bound f is essentially determined by the condition that $\lambda \notin \sigma_{ess}(H_f)$.

To estimate the numbers B_N defined in (8.8) we now consider the family H_{sf} on $0 \leqq s \leqq 1$, assuming that

$$\operatorname{dist}(\lambda, \sigma_{ess}(H_{sf})) \leqq \delta > 0 \tag{9.5}$$

for all s. Then H_{sf} and H_0 have the same discrete spectrum in the disc $|z - \lambda| < \delta$. Let Γ be a circle of radius $r < \delta$ around λ which separates λ from the rest of the spectrum. Then $\Gamma \in \rho(H_{sf})$ for $0 \leqq s \leqq 1$ and we can use (9.2) to rewrite B_N in the form

$$\begin{aligned} B_N &= \sup_{z \in \Gamma} \|Ve^{-f/N}R_{f/N}(z)Ve^{-f/N}R_{2f/N} \ldots \\ &\qquad \ldots Ve^{-f/N}R_f(z)e^f P_0\| \\ &\leqq \|e^f P_0\| \Big(\sup_{\substack{z \in \Gamma \\ 0 \leqq s \leqq 1}} \|Ve^{-f/N}R_{sf}(z)\| \Big)^N \qquad (9.6) \\ &\leqq \|e^f P_0\| \, \|V^N e^{-f}\| \Big(\sup_{\substack{z \in \Gamma \\ 0 \leqq s \leqq 1}} \|R_{sf}(z)\| \Big)^N, \qquad (9.7) \end{aligned}$$

provided that these bounds exist. The great advantage of these estimates is that in Nth order perturbation theory each factor V in the RS series is reduced by a factor $\exp(-f/N)$. We note in particular that the N-dependence of these bounds is explicit.

Now we discuss the details. For convenience we work with complex f. The following lemma gives the basic properties of H_f in close analogy to Lemma 2.2.

Lemma 9.1. *Suppose that H_0 is a Schrödinger operator with the phase γ_0 defined in* (2.8). *Let g be a real C^∞-function such that the potential*

$$V_0 - e^{2i\gamma_0}(\nabla g)^2 \tag{9.8}$$

still satisfies (2.4)–(2.6), *and define*

$$f(x) = \alpha e^{i\gamma_0} g(x) \tag{9.9}$$

for some positive $\alpha < 1$. *Then the operator* H_f *defined by* (9.1) *on* C_0^∞ *has the following properties*:

(i) $$\langle p^2 \rangle + (1-\alpha^2)(\cos\gamma_0)\langle(\nabla g)^2\rangle \leqq a \operatorname{Re} e^{-i\gamma_0}\langle H_f\rangle + b \tag{9.10}$$

with a, b *independent of the choice of* α. *Thus* H_f *has numerical range in a halfplane* A_0 *given by* a, b, γ_0 (*Fig.* 2).

(ii) $(z - H_f)C_0^\infty$ *is dense in* L^2 *for* $z \notin A_0$. *Therefore* H_f *has a closure with spectrum contained in* A_0.

Proof. (i) By hypothesis the Schrödinger operator with the potential (9.8) satisfies (2.8). Using the fact that $\langle \nabla g \cdot p + p \cdot \nabla g\rangle$ is real (9.10) follows. (ii) is proven as in the case of Lemma 2.2 with the following minor changes. The operator H_F results from H_f by the truncation

$$V_0 \to V_0 F \quad \text{and} \quad \nabla g \to F\nabla g.$$

It has the form $H_F = p^2 + B$ with B small relative to p^2, and the estimate (9.10) continues to hold for H_F with the same constants a, b independent of F. In addition to (2.12) this gives the bound

$$\| |\nabla g| \, Fh \| < c$$

with c independent of the choice of F, h. This allows to estimate the contribution of the last term in

$$[G, H_f] = [G, p^2] - 2\alpha e^{i\gamma_0}(\nabla g \cdot \nabla G)F. \qquad \square$$

Let H_f now be the closed operator given by Lemma 9.1. We remark that H_f^* is an operator of the same type, resulting from H_f by the substitutions

$$V_0 \to \bar{V}_0, \qquad f \to -\bar{f}$$

which is allowed since so far g was not assumed to be bounded below. The family H_{sf} $(0 \leqq s \leqq 1)$ is obtained by changing α into αs. By (9.10) it satisfies the estimate

$$\langle p^2 \rangle + (1-\alpha^2)(\cos\gamma_0)\langle(\nabla g)^2\rangle \leqq a \operatorname{Re} e^{-i\gamma_0}\langle H_{sf}\rangle + b \tag{9.11}$$

uniformly in $0 \leqq s \leqq 1$. As a consequence all the results of Sections 3 and 4 also apply to the family

$$H_\kappa = H_{(s+\kappa)f} \tag{9.12}$$

where s is fixed in $0 \leqq s \leqq 1$ and κ variable in $0 \leqq s + \kappa \leqq 1$.

Theorem 9.2. *Let* H_f *be the closed operator given by Lemma* 9.1 *and suppose in addition that* $\operatorname{Re} f$ *is bounded from below. Let* λ *be a discrete eigenvalue of* H_0. *If*

$$\|(\lambda - H_{sf})u\| \varepsilon > 0 \tag{9.13}$$

uniformly in $0 \leqq s \leqq 1$ *and for all states* $u \in C_0^\infty(|x| > n)$, *n fixed but arbitrary, then* (9.2)–(9.7) *hold with*

$$\sup_{z \in \Gamma, 0 \leqq s \leqq 1} \|R_{sf}(z)\| < \infty \tag{9.14}$$

Remarks

This theorem reduces the proof of asymptotic bounds to the stability estimate (9.13) for which the methods of Section 6 are available.

Proof. By hypothesis there is a disc

$$\{z \mid 0 < |z - \lambda| < \delta < \varepsilon\} \subset \rho(H_0) \tag{9.15}$$

with ε given by (9.13). We will show that this disc belongs to $\rho(H_{sf})$ for $0 \leqq s \leqq 1$, thus proving (9.2)–(9.5). We fix z in this disc. Then

$$\|(z - H_{sf})u\| \geqq \varepsilon - \delta > 0 \tag{9.16}$$

for $0 \leqq s \leqq 1$ and all states $u \in C_0^\infty(|x| > n)$. Let B be the subset of $0 \leqq s \leqq 1$ where $\rho(H_{sf}) \ni z$.B is non-empty since $0 \in B$. Also, B is open as a consequence of (9.16) and Lemma 4.2(iii), applied to the family (9.12).

Let t be a boundary point of B and $\{s\}$ a sequence in B with $s \to t$. Then $\|R_{sf}(z)\| \to \infty$ by the remark following Lemma 4.1, so that

$$\|(z - H_{sf})u_s\| \to 0 \tag{9.17}$$

for a corresponding sequence of states $u_s \in C_0^\infty$. By passing to a subsequence we have $u_s \xrightarrow[w]{} u$. It then follows as in the proof of Lemma 4.2 that

$$(z - H_{tf})u = 0.$$

Taking scalar products with C_0^∞-functions and using the properties of the adjoint H_{tf}^* we find

$$(z - H_0)e^{-tf}u = 0.$$

Since $z \in \rho(H_0)$ we see that $u = 0$, i.e. $u_s \xrightarrow[w]{} 0$. This together with (9.16) and (9.17) leads to a contradiction as in the proof of Lemma 4.2. Therefore B has no boundary points, i.e. $B = [0, 1]$.

Suppose now that (9.14) is false. Then

$$\|(z_s - H_{sf})u_s\| \to 0 \tag{9.18}$$

for a sequence $\{s\}$ in $0 \leqq s \leqq 1$ and corresponding $z_s \in \Gamma$ and states $u_s \in C_0^\infty$. By passing to a subsequence we have $s \to t$, $z_s \to z$, and $u_s \xrightarrow[w]{} u$. Since $z \in \rho(H_{tf})$ we deduce that $u = 0$ as in the proof of Lemma 4.2. By the now familiar argument this is in contradiction to (9.16) and (9.18). □

10. Borel summability

For an introduction to Borel summability in perturbation theory we refer to [3], where the case of simple eigenvalues is treated.

In the degenerate case the operator $E(\kappa)$ given by (7.2) takes the place of the perturbed eigenvalue. We therefore consider functions $f(\kappa)$ taking values in the algebra $L(\mathscr{H})$ of bounded operators on some Hilbert space $\mathscr{H}$ – in fact $\mathscr{H}$ will be the finite dimensional subspace M_0 introduced in Section 7. $f(\kappa)$ is said to be Borel-summable in a sector S if it has the following properties:

(i) $f(\kappa)$ is holomorphic for small κ in the interior of a complex sector S with origin 0 and opening angle $> \pi$

(ii) $f(\kappa)$ has an asymptotic expansion

$$f(\kappa) = \sum_{k=0}^{N-1} f_k \kappa^k + f_N(\kappa)\kappa^N; \qquad f_k \in L(\mathscr{H})$$

to all orders N, satisfying a "strong asymptotic estimate"

$$\|f_N(\kappa)\| \leqq C\sigma^N N! \tag{10.1}$$

for small $\kappa \in S$ and all N.

The principal fact about Borel summable functions is that they can be constructed uniquely from their asymptotic expansion, i.e. from the set of all expansion coefficients [3].

In perturbation theory this means that the perturbed eigenvalues are uniquely determined by the RS-series if $E(\kappa)$ is Borel summable. In this case we will also obtain more precise spectral information for $E(\kappa)$ than in the general asymptotic case (Theorem 11.1). In order to derive a strong asymptotic estimate for $E(\kappa)$ we need the elementary.

Lemma 10.1. *Suppose that the functions $f(\kappa)$, $g(\kappa)$ satisfy strong asymptotic estimates in a common sector S (of arbitrary opening angle). Then the same is true for the functions $f(\kappa)+g(\kappa)$, $f(\kappa)g(\kappa)$ and – if $f(0)^{-1} \in L(\mathscr{H})$ – for $f(\kappa)^{-1}$.*

Proof. We prove only the last statement. By hypothesis $f(\kappa)^{-1} = g(\kappa)$ exists for small κ. We will prove

$$\|g_N(\kappa)\| \leqq C(1+C^2)^N \sigma^N N! \tag{10.2}$$

provided that

$$\|g(\kappa)\| \leqq C \quad \text{and} \quad \|f_N(\kappa)\| \leqq C\sigma^N N!. \tag{10.3}$$

For $N \geqq 1$ all term of order $\geqq N$ in the identity

$$1 = \sum_{k=0}^{N-1} f(\kappa) g_k \kappa^k + f(\kappa) g_N(\kappa) \kappa^N$$

must cancel. This gives

$$g_N(\kappa) = -g(\kappa) \sum_{k=0}^{N-1} f_{N-k}(\kappa) g_k \tag{10.4}$$

and in the limit $\kappa \to 0$ the recursion for the g_k. By (10.3) we find for $N \geqq 1$

$$\|g_N(\kappa)\| \leqq C^2 \sum_{k=0}^{N-1} \|g_k\| \, \sigma^{N-k}(N-k)!$$

while $\|g_0(\kappa)\| = \|g(\kappa) \leqq C$. Using that $(N-k)!\,k! \leqq (N-1)!$ for $1 \leqq k \leqq N-1$ one now confirms by induction that for $N \geqq 1$

$$\|g_N(\kappa)\| \leqq C(C^2 + C^4 + \cdots + C^{2N})\sigma^N N! \qquad \square$$

Lemma 10.2. *Suppose that the operators $D(\kappa)$ and $N(\kappa)$ given in* (7.1) *and* (7.3) *satisfy strong asymptotic estimates in the sector S. Then the same is true for $E(\kappa) = D(\kappa)^{-1/2} N(\kappa) D(\kappa)^{-1/2}$.*

Proof. By Lemma 10.1 we need only show that $D(\kappa)^{-1/2}$ has a strong asymptotic estimate. Since $D(0) = 1$ the spectrum of $D(\kappa)$ is contained in the disc of radius $1/2$ around $z = 1$ for small κ. Then we have

$$D(\kappa)^{-1/2} = (2\pi i)^{-1} \oint_\Gamma dz\, z^{-1/2}(z - D(\kappa))^{-1}$$

where Γ is the circle of radius $3/4$ around 1 and $z^{-1/2}$ the branch taking the value 1 at $z = 1$. The resolvent in this integral is bounded in norm for $z \in \Gamma$ and has a strong asymptotic estimate given explicitely by (10.2). This leads directly to the estimate for $D(\kappa)^{-1/2}$. $\square$

Remarks

If H_κ has the form (8.1) we see from (8.9) that $E(\kappa)$ will satisfy a strong asymptotic estimate in S if

$$B_N \leqq C\sigma^N N!. \tag{10.5}$$

Suppose that f is an exponential bound in the sense of Theorem 9.2. Then (10.5) follows from (9.7) if

$$\|V^N e^{-f}\| \leqq C\sigma^N N!. \tag{10.6}$$

Since $E(\kappa)$ is analytic in the interior of S this will prove Borel summability if S has opening angle $> \pi$: the standard example is the quartic anharmonic oscillator (Example (d)). In other cases where H_κ is defined only on a sector S of opening angle $< \pi$, it is sometimes possible to prove Borel summability by constructing an analytic continuation of $E(\kappa)$ to a larger sector (Examples (e)–(h)).

11. Finite dimensional asymptotic perturbation theory

Here we return to the question raised at the end of Section 8: given the expansion (8.5) of $E(\kappa)$, what can we say about the asymptotics of the eigenvalues of $E(\kappa)$? This is a finite dimensional problem, unrelated to our discussion of Schrödinger operators. We therefore confine ourselves to an exposition of the results proven in [34].

We assume that $E(\kappa)$ is defined for small κ in a complex sector S as an operator on a Hilbert space of dimension $m<\infty$. $E(\kappa)$ is further supposed to have an asymptotic expansion (8.5) with

$$\|E_N(\kappa)\| = o(1) \qquad (\kappa \to 0) \tag{11.1}$$

for values of N specified below. The basic problem can be stated as follows. The expanded part

$$E^N(\kappa) = \sum_{k=0}^{N} E_k \kappa^k \tag{11.2}$$

is analytic in κ. Therefore its eigenvalues and eigenprojections are given for small κ by convergent perturbation series ([1], Chapter II). The task is to show that these expansions are asymptotic to the corresponding quantities for $E(\kappa)$ up to some order in κ which can be estimated in terms of N and m.

Theorem 11.1. [34]

(i) *Eigenvalues to order p*

To find the eigenvalues of $E(\kappa)$ to some prescribed error $o(|\kappa|^p)$, $p \geqq 0$, we expand $E(\kappa)$ to some order $N \geqq mp$ and the eigenvalues of $E^N(\kappa)$ to the highest order $\leqq p$. The result is a number of finite Puiseux series

$$e(\kappa) = \sum_{\substack{k=r/b \\ r=1,2\dots[bp]}} e_k \kappa^k \tag{11.3}$$

with integer b = number of branches of $e(\kappa)$ and $[x]$ = largest integer $\leqq x$. Each branch of $e(\kappa)$ represents a group of eigenvalues of $E^N(\kappa)$ and a corresponding group of eigenvalues of $E(\kappa)$ up to an error $o(|\kappa|^p)$. Both groups have the same total algebraic multiplicity, which is the same for each branch of $e(\kappa)$. In short: for $N \geqq mp$ the eigenvalues of $E(\kappa)$ and $E^N(\kappa)$ coincide up to errors $o(|\kappa|^p)$. Fig. 7 gives a picture of the b groups of eigenvalues described by (11.3):

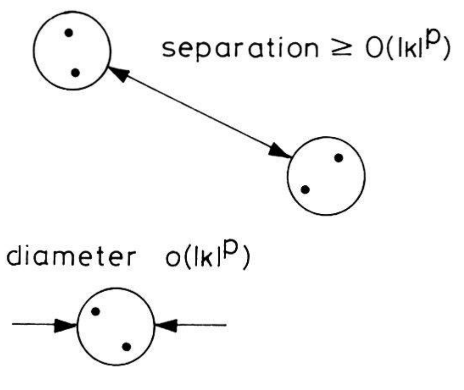


Figure 7

(ii) *Spectral projections to order q*
The spectral projections of $E(\kappa)$ corresponding to the b groups of eigenvalues given by (11.3) form a b-valued function $P(\kappa)$ defined for small $\kappa \in S$. This function has a Puiseux-Laurent expansion

$$P(\kappa) = \sum_{\substack{k=r/b \\ r=s,s+1,\ldots[bq]}} P_k \kappa^k + o(|\kappa|^q) \tag{11.4}$$

which is obtained by expanding the corresponding spectral projections of $E^N(\kappa)$ for some $N \geqq q + p(2m-1)$ up to the highest order $\leqq q$. Here s is a possibly negative integer; in fact s is strictly negative if $b > 1$.

(iii) *The symmetric case*
Suppose that $E_k^* = E_k$ for all (relevant) k. Then (11.3) and (11.4) reduce to Taylor expansions: $b = 1$, $s = 0$. In this case it suffices to take $N \geqq p$ in (i) and $N \geqq p + q$ in (ii).

(iv) *Asymptotic series*
Suppose that $E(\kappa)$ has an asymptotic series, i.e. an expansion (8.5) to all orders N. Then, for small $\kappa \in S$, the eigenvalues of $E(\kappa)$ fall into groups which are separated by a distance $\geqq$ const $|\kappa|^\nu$ for some $\nu \geqq 0$ and which have diameters vanishing faster than any power of $|\kappa|$ as $\kappa \to 0$. These groups and the corresponding spectral projections have asymptotic expansions (11.3) and (11.4) to all orders which can be computed from the coefficients E_k by analytic perturbation theory as described above.

(v) *Borel summable $E(\kappa)$*
If $E(\kappa)$ is Borel summable in S, then the eigenvalues and eigenprojections of $E(\kappa)$ are holomorphic for small κ in the interior of S. Different eigenvalues cannot be degenerate to all orders in κ. The asymptotic series given in (iv) then describe individual eigenvalues and eigenprojections.

(vi) *Borel-summability of eigenvalues and eigenprojections*
Suppose that $E(\kappa)$ is Borel summable in S and that $E_k^* = E_k$ for all k. According to (iii) and (v) the eigenvalues and eigenprojections then have asymptotic Taylor series. These Taylor series are Borel summable to the eigenvalues and eigenprojections of $E(\kappa)$. Moreover, the perturbed eigenvalues remain semi-simple: the eigennilpotents vanish identically for small κ.

Remarks

(i) and (ii): We point out that the lower bounds required for N refer to the least favourable case, where the separation of the groups of eigenvalues in Fig. 7 is only of order $|\kappa|^p$. For larger separation the conditions on N are less restrictive.

(iii): We emphasize the fact that $E_k^* = E_k$ (all k) does not imply that $E(\kappa)^* = E(\kappa)$ for real κ. This situation arises naturally in perturbation problems where bound states turn into resonances as in Examples (c), (f), (g). For the expanded part it is of course true that $E^N(\kappa)^* = E^N(\kappa)$ if κ is real: this is why (11.3) and (11.4) reduce to Taylor expansions in this case. Moreover, the resolvent $(z - E^N(\kappa))^{-1}$ is then bounded in norm by some constant times the inverse distance of z from the spectrum of $E^N(\kappa)$ for small κ: this is the reason why the conditions on N are less severe in the symmetric case (In the general case the inverse m-th power of this distance can appear).

12. Examples

(a) *The set* $\Sigma\,(H_\kappa)$

The Hamiltonian describing the free fall in a constant field of force provides a good example which shows in particular that the set $\Sigma\,(H_k)$ defined by (4.1) is not determined by the spectrum of H_κ. This example was first discussed by Herbst [30] in connection with the Stark effect. On $L^2(R^1)$ we consider the family

$$H_\kappa = p^2 - \kappa x \quad \text{on} \quad S: \varepsilon \leqq \arg \kappa \leqq \pi - \varepsilon \tag{12.1}$$

Lemma 12.1.
(i) $\Sigma(H_\kappa)$ *is the range of the Hamiltonian function*

$$(\kappa, p, x) \to p^2 - \kappa x \tag{12.2}$$

on $S \times R^1 \times R^1$ *(given by the shaded region in Fig.* 8)
(ii) $\sigma(H_\kappa)$ *is empty for* $\kappa \neq 0$.

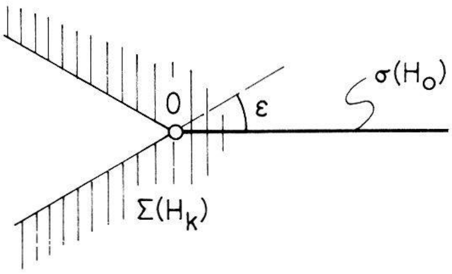


Figure 8

Proof. (i) The numerical range of H_κ and therefore $\Sigma\,(H_\kappa)$ are contained in the range of (12.2). Conversely, let

$$z = a^2 - \mu b \qquad \text{for some } (\mu, a, b) \in S \times R^1 \times R^1.$$

The idea is to take a sequence $\kappa_n \to 0$ and to construct corresponding states u_n such that

$$\|(a^2 - p^2)u_n\| \to 0 \quad \text{and} \quad \|(\mu b - \kappa_n x)u_n\| \to 0, \tag{12.3}$$

i.e. quantum states u_n for which both the kinetic and the potential energy take prescribed classical values with vanishing mean square deviation in the limit $\kappa \to 0$. *A fortiori* we then have

$$\|(z - H_{\kappa_n})u_n\| \to 0,$$

which shows that $z \in \Sigma(H_\kappa)$. A possible choice is

$$\begin{aligned} \kappa_n &= \mu n^{-4}, \qquad n = 1, 2, \ldots \\ u_n(x) &= e^{iax} n^{-1} f(n^{-2}(x - n^4 b)) \end{aligned} \tag{12.4}$$

with arbitrary $f \in C_0^\infty$, $\|f\| = 1$.

(ii) First we remark that $\sigma_{\mathrm{disc}}(H_\kappa)$ is empty since H_κ is unitary equivalent to $H_\kappa + \kappa a$ for any real a by a translation. To show that $\sigma_{\mathrm{ess}}(H_\kappa)$ is empty we use Lemma 3.3. Any state $u \in C_0^\infty(|x| > n)$ decomposes into $u = u^+ + u^-$ with $u^{+/-} \in C_0^\infty(\pm x > n)$. From the numerical range estimate we thus obtain

$$\begin{aligned} \|(z - H_\kappa)u^2\| &= \|(z - H_\kappa)u^+\|^2 + \|(z - H_\kappa)u\|^2 \\ &\geqq d^2(\|u^+\|^2 + \|u^-\|^2) = d^2, \end{aligned}$$

where d is the distance of z from the set given in Fig. 9

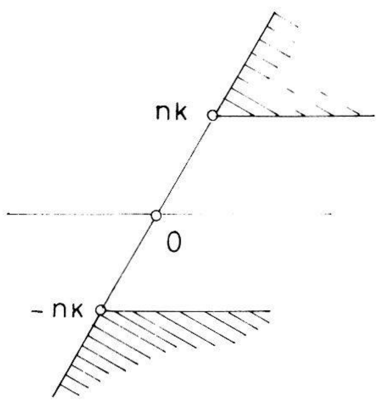


Figure 9

Since d is positive for any z and sufficiently large n, it follows from Lemma 3.3 that $\sigma_{\mathrm{ess}}(H_\kappa)$ has no boundary points and is therefore empty. □

(b) *Strong convergence of* P_κ

This is a rather trivial example to show that eigenvalues of H_κ can simply disappear in the limit $\kappa \to 0$ if P_κ and P_κ^* converge strongly but not in norm. Let

$$H_\kappa = \begin{cases} p^2 + V(x - \kappa^{-1}) & \text{if } \kappa > 0 \\ p^2 & \text{if } \kappa = 0 \end{cases} \tag{12.5}$$

on $L^2(R^1)$, where V is some negative C_0 function so that $H = p^2 + V$ has an eigenvalue $\lambda < 0$ with eigenfunction v, $\|v\| = 1$. For $\kappa > 0$ H_κ is unitary equivalent to H: λ is an eigenvalue of H_κ with eigenfunction $v_\kappa(x) = v(x - \kappa^{-1})$. Moreover $\|R_\kappa(z)\|$ is independent of κ for any $z \in \rho(H_\kappa) = \rho(H) \subset \rho(H_0)$. Since $H_\kappa u \to H_0 u$ for all $u \in C_0^\infty$ as $\kappa \to 0$ it follows as in Section 5 that

$$P_\kappa = P_\kappa^* \underset{s}{\to} P_0 = 0.$$

This is of course also seen from the explicit form $P_\kappa u = (v_\kappa, u)v_\kappa$ since $v_\kappa \underset{w}{\rightarrow} 0$.

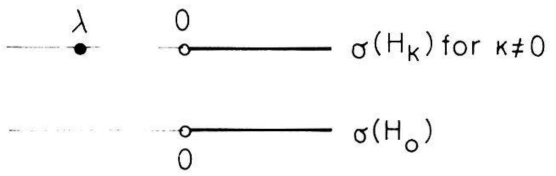


Figure 10

(c) *Shape resonances*

As a first illustration of Theorem 5.1 we discuss a system which exhibits typical shape resonances:

$$H_\kappa = p^2 - g\,|x|^{-1} - \kappa\,|x|\,(1+\kappa\,|x|)^{-1} \tag{12.6}$$

with $g > 0$ and $\kappa \geqq 0$. The potential $V_\kappa(x)$ has the form:

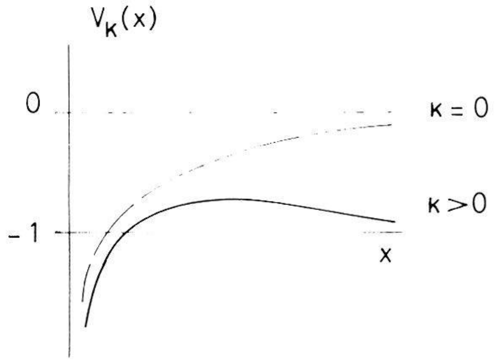


Figure 11

As soon as $\kappa > 0$ we expect that the bound states of H_0 in the energy range $-1 < \lambda < 0$ become unstable: they can tunnel through the potential barrier which has thickness of order κ^{-1}.

According to the Balslev–Combes theory ([3], Section XIII.10, [4], Section 8.1) these eigenvalues should turn into resonances, given by eigenvalues of the dilated Hamiltonian

$$H_\kappa^\mu = \mu^{-2}p^2 - g\mu^{-1}\,|x|^{-1} - \mu\kappa\,|x|\,(1+\mu\kappa\,|x|)^{-1} \tag{12.7}$$

for suitable complex $\mu \neq 0$. We assume $|\arg \mu| < \pi/2$. Then H_0^μ has the spectrum

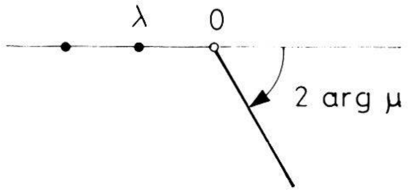


Figure 12

with negative real discrete eigenvalues independent of μ: the discrete eigenvalues of H_0.

To the family (12.7) we assign the sector

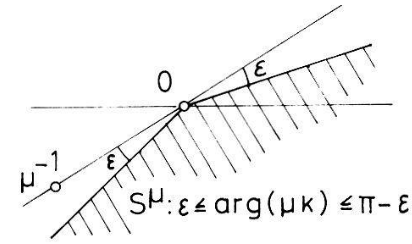


Figure 13

Since $|x|^{-1}$ is small relative to p^2 it is easily seen that (apart from an overall factor μ^{-2}) H_κ^μ is a family of Schrödinger operators.

Definition. Σ_∞^μ is the set of all complex λ which violate the stability condition (5.6) with respect to H_κ^μ.

Lemma 12.2. *Σ_∞^μ is the range of the function*

$$(\kappa, p, x) \to \mu^{-2}p^2 - \mu\kappa\,|x|\,(1 + \mu\kappa\,|x|)^{-1} \tag{12.8}$$

on $S^\mu \times R^3 \times R^3$: the shaded region in Fig. 14.

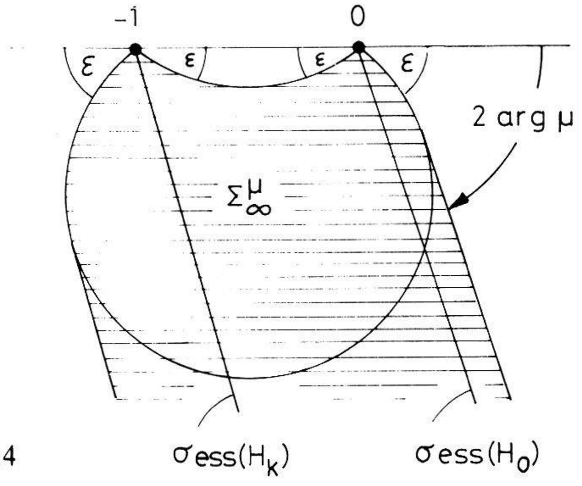


Figure 14

Remarks

The two circles passing through -1 and 0 are determined by the angle ε given in Fig. 13. According to Lemma 3.3 the essential spectrum of H_κ^μ depends only on the behavior of the potential $V_\kappa(x)$ as $|x| \to \infty$:

$$\sigma_{\text{ess}}(H_\kappa^\mu) = \begin{cases} \sigma(\mu^{-2}p^2) & \text{if } \kappa = 0 \\ \sigma(\mu^{-2}p^2 - 1) & \text{if } \kappa \neq 0 \end{cases}$$

Proof. If λ is in the range (12.8) it is easy to construct a sequence $\kappa_n \to 0$ and states $u_n \in C_0(|x| > n)$ such that $\|(\lambda - H_{\kappa_n}^\mu)u_n\| \to 0$ (see Example (a)). Therefore $\lambda \in \Sigma_\infty^\mu$.

To prove the converse we use localization (Lemma 6.1). Let f be the function

$$f(\phi, s) = -e^{i\phi}s(1 + e^{i\phi}s)^{-1}$$

on $0 \leqq s \leqq \infty$, $\varepsilon \leqq \phi \leqq \pi - \varepsilon$. In this region f and $\partial f / \partial s$ are uniformly bounded. Given any $\delta > 0$ we can thus choose a finite set of nonnegative C^∞ functions $J^\alpha(s)$, $\alpha = 1 \ldots \beta$, and corresponding points s^α in $0 \leqq s \leqq \infty$ such that

$$\sum_\alpha J^\alpha(s) = 1 \quad \text{and} \quad |f(\phi, s) - f(\phi, s^\alpha)| < \delta$$

for all ϕ if $s \in \operatorname{supp} J^\alpha$. The scaled functions

$$J_\kappa^\alpha(x) = J^\alpha(|\kappa\mu|\,|x|)$$

form a partition of unity on $|x| > n$ in the sense of (6.2). For states $u \in C_0^\infty(|x| > n)$ we have

$$\begin{aligned} &\|(H_\kappa^\mu - H_\kappa^{\mu\alpha}) J_\kappa^\alpha u\| \leqq (|g\mu n|^{-1} + \delta)\, \|J_\kappa^\alpha u\|, \quad \text{where} \\ &H_\kappa^{\mu\alpha} = \mu^{-2} p^2 - e^{i\phi} s^\alpha (1 + e^{i\phi} s^\alpha)^{-1}, \ \phi = \arg(\mu\kappa). \end{aligned} \tag{12.9}$$

Since the numbers s^α are constants the numerical range of $H_\kappa^{\mu\alpha}$ is contained in the range of (12.8). If λ is not in this range it follows from (12.9) that the stability conditions (6.3) are satisfied for suitably chosen n, δ. □

By Theorem 5.1 and Fig. 14 all eigenvalues λ of H_0^μ in the open intervals $(-\infty, -1)$ and $(-1, 0)$ are stable with respect to H_κ^μ, but with a different qualitative behavior for real $\kappa > 0$:

$-\infty < \lambda < -1$: Then the perturbed eigenvalues are also eigenvalues of H_κ and therefore real. They represent bound states.

$-1 < \lambda < 0$: In this case the perturbed eigenvalues have strictly negative imaginary parts for $\arg \mu > 0$ (the situation depicted in Fig. 14); these are the shape resonances.

These statements are based on the following facts:

(i) A discrete eigenvalue of H_κ^μ is independent of μ as long as it is separated from the essential spectrum of H_κ^μ (shown in Fig. 14)
(ii) For real $\kappa \geqq 0$ the real eigenvalues $\neq -1$ of H_κ^μ and H_κ coincide
(iii) For real $\kappa > 0$, H_κ has no eigenvalues > -1

(i) and (ii) are general aspects of the Balslev-Combes theory ([3][4]). (iii) is elementary for spherical symmetric potentials (with limit -1 as $|x| \to \infty$.) and in fact a special case of "nonexistence of positive eigenvalues" ([4], Section 4.4).

Remarks

The details of Fig. 14 depend of course on the special form chosen for the barrier potential. By using "exterior complex scaling" [43] or other variants of the Balslev-Combes method [20, 33] only the behavior of the potential in the energetically allowed outer region will come into play.

From our proof it is also clear that the global numerical range argument (6.1) must fail in this case. In the limit of large n the numerical range of H^{μ}_{κ} for states $u \in C_0^{\infty}(|x| > n)$ is the convex hull of the set Σ^{μ}_{∞} which covers the interval $(-1, 0)$.

We mention that a similar perturbation problem with respect to an infinitely high barrier is treated in [9]. The most interesting aspect of shape resonances are estimates or asymptotic formulas for the widths [27, 28, 38]. This problem has been studied in particular for shape resonances appearing in the semi-classical limit $\hbar \to 0$ (see e.g. [29, 18, 32]).

(d) *Stable anharmonic oscillators* [3]

This is our first example for asymptotic RS-expansions. Let

$$\begin{aligned} &H_{\kappa} = p^2 + x^2 + \kappa V(x) \text{ on} \\ &S: -\pi + \varepsilon \leqq \arg \kappa \leqq \pi - \varepsilon \end{aligned} \tag{12.10}$$

with $V \geqq 0$ in $L^{\infty}_{\text{loc}}(R^{\nu})$. The conditions (2.4)–(2.7) are obviously satisfied. For states $u \in C_0(|x| > n)$ H_{κ} has expectation values in the region

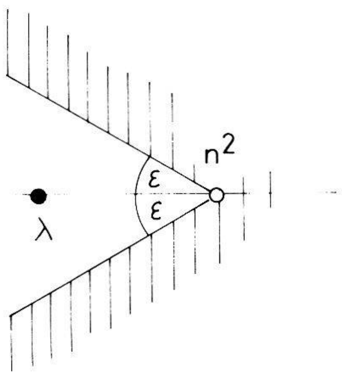


Figure 15

Since n is arbitrary large, each eigenvalue of H_0 is stable by the numerical range argument (6.1). To establish the RS-expansion we construct an exponential bound.

Lemma 12.3. *For any positive* $\alpha < 1$, $f(x) = \alpha x^2/2$ *is an exponential bound satisfying the hypothesis of Theorem* 9.2.

Proof. The function $g(x) = x^2/2$ is chosen such that the potential (9.8) vanishes. H_{sf} is given by

$$H_{sf} = p^2 + i\beta(x \cdot p + p \cdot x) + (1 - \beta^2)x^2 \tag{12.12}$$

where $\beta = s\alpha : 0 \leqq \beta \leqq \alpha < 1$. For states $u \in C_0^{\infty}(|x| > n)$ we have

$$\text{Re}\,\langle H_{sf} \rangle \geqq (1 - \alpha^2)n^2$$

uniformly in $0 \leqq s \leqq 1$. This proves (9.13) for any λ if n is taken sufficiently large. □

It follows from (8.11), (9.7) and Theorem 9.2 that $D(\kappa)$, $N(\kappa)$ and $E(\kappa)$

have RS-expansions up to order $2N$ if

$$\sup_x |V(x) \exp(-\alpha x^2/2N)| < \infty$$

for some $\alpha < 1$.

We now turn to the quartic anharmonic oscillator

$$H_\kappa = p^2 + x^2 + \kappa x^4$$

which has become the model example for Borel summability [24, 3]. To show this we need the sharper estimate (9.6). First we prove that

$$\|x^2 u\| \leqq c(\|H_{sf} u\| + \|u\|) \tag{12.13}$$

uniformly in $0 \leqq s \leqq 1$. Proof. Let $A = p^2 + i\beta(x \cdot p + p \cdot x)$. By (12.12) we have on C_0^∞

$$\begin{aligned} H_{sf}^* H_{sf} &= A^*A + (1-\beta^2)^2 x^4 \\ &\quad + (1-\beta^2)\Big(2\sum_k p_k x^2 p_k - 2\nu\Big) - 4\beta(1-\beta^2)x^2 \\ &\geqq (1-\beta^2)x^4 - 4\beta(1-\beta^2)x^2 - 2\nu(1-\beta^2) \\ &\geqq (1-\alpha^2)^2 x^4 - 4\alpha x^2 - 2\nu. \end{aligned}$$

This proves (12.13) for $u \in C_0^\infty$, which is sufficient since C_0^∞ is a core of H_{sf}. □ (12.13) implies that

$$\|x^2 R_{sf}(z)\| < c\{1 + (1+|z|)\,\|R_{sf}(z)\|\}. \tag{12.14}$$

With (9.6) and (9.14) this leads to the estimate

$$\begin{aligned} B_N &< Cb^N \sup |x^{2N} \exp(-\alpha x^2/2)| \\ &< Cd^N N^N < C\sigma^N N! \end{aligned}$$

since $N^N \leqq e^N N!$. By Lemma 8.2 and Lemma 10.2, $E(\kappa)$ is thus Borel summable in the sector S. We also remark that $E(\kappa)^* = E(\kappa)$ for real κ so that Theorem 11.1 (vi) applies.

(e) *The one-body Stark effect* [25, 30]

This is a shape resonance problem with a linear barrier potential. Corresponding to (12.6) and (12.7) we have

$$\begin{aligned} H_\kappa &= p^2 - |x|^{-1} - \kappa(e, x) \\ H_\kappa^\mu &= \mu^{-2}p^2 - \mu^{-1}|x|^{-1} - \kappa\mu(e, x) \end{aligned} \tag{12.15}$$

where $e \in R^3$ defines the direction of the electric field. $\mu^2 H_\kappa^\mu$ is a Schrödinger operator as long as $\kappa\mu^3$ is not real.

For $|\arg \mu| < \pi/2$ the spectrum of H_0^μ is shown in Fig. 12. Here we also need the following result of the Balslev–Combes analysis. Let P_0 and P_0^μ be the spectral projections of H_0 and H_0^μ corresponding to a given (common!) eigenvalue $\lambda < 0$. Then

$$P_0 v = U(\mu)^{-1} P_0^\mu U(\mu) v \tag{12.16}$$

for all $v \in D(U(\mu))$. Here $U(\mu)$ is the dilation group, given for real $\mu > 0$ by

$$(U(\mu)v)(x) = \mu^{-3/2} v(\mu^{-1}x)$$

and extended to complex μ via its spectral representation. (12.16) shows that the (finite dimensional!) spectral subspaces $M_0 = R(P_0)$ and $M_0^\mu = R(P_0^\mu)$ are in the domains of $U(\mu)$ and $U(\mu)^{-1}$, respectively, so that

$$U(\mu)|_{M_0} = \text{bijection } M_0 \to M_0^\mu. \tag{12.17}$$

As a first step we discuss the perturbation of a discrete eigenvalue λ of H_0^μ under (12.15) for fixed μ. It is clear from Example (a) that the stability condition (5.6) cannot hold if λ is in the range of the function

$$(\kappa, p, x) \to \mu^{-2} p^2 - \kappa \mu (e, x).$$

To exclude this we define the sector S^μ for the family (12.15) by

$$S^\mu : \begin{array}{l} \varepsilon \leqq \arg \kappa + 3 \arg \mu \leqq \pi - \varepsilon \\ \varepsilon \leqq \arg \kappa + \arg \mu \leqq \pi - \varepsilon \end{array} \tag{12.18}$$

with small $\varepsilon > 0$. The region allowed by these inequalities is shown in Fig. 16 for $\varepsilon = 0$:

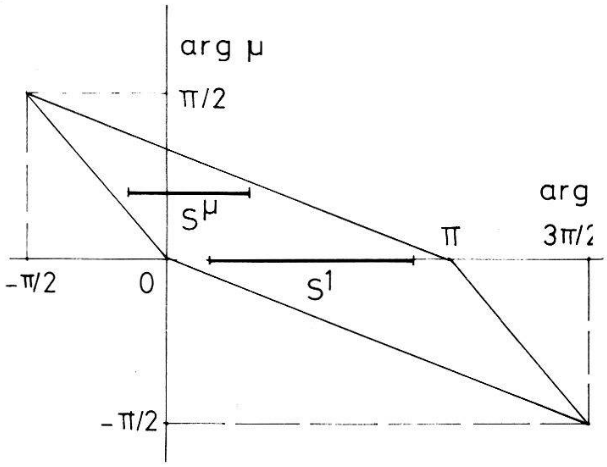


Figure 16

For a fixed μ with $|\arg \mu| < \pi/2$ the sector S^μ is given by a horizontal interval inside the allowed region. In particular,

$$S^1 : \varepsilon \leqq \arg \kappa \leqq \pi - \varepsilon \tag{12.19}$$

corresponds to the undilated Hamiltonian H_κ in which, however, κ is not allowed to be real $\neq 0$. For states $u \in C_0^\infty\ (|x| > n)$, and up to an error of order n^{-1}, H_κ^μ has expectation values in the shaded region of Fig. 8. Therefore each (negative) eigenvalue λ of H_0 is stable by the numerical range argument (6.1).

To establish the RS-expansion we show that for a given eigenvalue λ

$$f(x) = \alpha\,|x| \tag{12.20}$$

is an exponential bound satisfying (9.14), provided that $\alpha > 0$ is chosen sufficiently small. To be precise we should regularize $f(x)$ near $x = 0$, but this is of no consequence. The optimal α can be determined from (9.13) and is known as O'Connor's bound ([3], section XIII.11). However, since any $\alpha > 0$ will serve here, we use a simpler argument:

$$\alpha \to H_f = H_0^\mu + i\alpha\mu^{-2}(px\,|x|^{-1} + |x|^{-1}xp) - \alpha^2\mu^{-2}$$

is an analytic family of type A for small complex α, since the operator in brackets is bounded relative to H_0^μ as a consequence of (2.8). Given any compact set $\Gamma \subset \rho(H_0)$ it follows directly that

$$\sup_{\substack{z\in\Gamma \\ 0\leqq s\leqq 1}} \|R_{sf}(z)\| < \infty \tag{12.21}$$

for sufficiently small $\alpha > 0$. Thus (9.7) gives the estimate

$$\begin{aligned} B_N &< Cb^N \sup_x |x|^N \exp(-\alpha\,|x|) \\ &< C\sigma^N N!, \end{aligned}$$

which leads to a strong asymptotic estimate for $E^\mu(\kappa)$ in the sector S^μ by Lemma 10.2. This estimate holds in particular for

$$E(\kappa) \equiv E^1(\kappa) \text{ on } S^1,$$

which has the asymptotic RS-series computed from the undilated Hamiltonian H_κ.

To prove Borel summability we construct an analytic continuation of $E(\kappa)$ to a sector of opening angle $> \pi$. We pick μ with $|\arg\mu| < \pi/3$ so that the sectors S^1 and S^μ overlap (Fig. 16). For small κ in $S^1 \cap S^\mu$ it follows from (12.16) that

$$E(\kappa) = U(\mu)^{-1}E^\mu(\kappa)U(\mu) \tag{12.22}$$

where $U(\mu)$ is the bijection (12.17). Thus (12.22) defines an analytic continuation of $E(\kappa)$ to small κ in S^μ. In the sector S^μ, $E(\kappa)$ has a strong asymptotic estimate obtained via (12.22) from the corresponding estimate for $E^\mu(\kappa)$, since $U(\mu)$ and $U(\mu)^{-1}$ are bounded. The expansion coefficients of $E(\kappa)$ in the sectors S^1 and S^μ are the same, since both expansions hold in $S^1 \cap S^\mu$. By joining a third sector to $S^1 \cup S^\mu$ and so forth (see Fig. 16) we arrive at the following result:

Lemma 12.4 [30]. *$E(\kappa)$ is analytic for small κ in $-\pi/2 + \varepsilon \leqq \arg\kappa \leqq 3\pi/2 - \varepsilon$ and given by the Borel sum of the* RS *series for the Stark Hamiltonian H_κ.*

The Stark resonances are the perturbed eigenvalues for real $\kappa > 0$. As in Example (c) they have strictly negative imaginary parts since it is known that $H_\kappa = H_\kappa^*$ has spectrum $(-\infty, +\infty)$ with no embedded eigenvalues [10]. This also shows that

$$E(\kappa)^* \neq E(\kappa)$$

for real $\kappa \neq 0$, while of course

$$E_\kappa^* = E_k \qquad \text{for all } k.$$

Therefore Theorem 11.1 (vi) applies: the perturbed eigenvalues are semi-simple and Borel summable in a sector $-\pi/2+\varepsilon \leqq \arg\kappa \leqq 3\pi/2-\varepsilon$ with real RS-expansion coefficients. For small real $\kappa > 0$ they have strictly negative imaginary parts which vanish faster than any power of κ as $\kappa \to 0$. This also shows that the RS-series are actually divergent. In fact there is a quantitative relation between the growth of the RS-coefficients and the asymptotic behavior of the imaginary parts (the widths of the resonances) as $\kappa \to 0$. For a discussion of this aspect we refer to [8].

(f) *Unstable anharmonic oscillators* [17]

As a second illustration of the method used for the Stark problem we treat

$$H_\kappa = p^2 + x^2 + \kappa V(x), \tag{12.23}$$

where $V(x)$ is a real homogeneous polynomial of order $m > 2$ on R^ν. We do not assume V to be positive. As in the Stark case we bypass the problem of assigning a definite self-adjoint extension to the operator given by (12.23) e.g. on C_0^∞ for real $\kappa \neq 0$. Instead we start again from the dilated operator

$$H_\kappa^\mu = \mu^{-2}p^2 + \mu^2 x^2 + \kappa\mu^m V(x) \tag{12.24}$$

on the sector

$$S^\mu : \begin{matrix} \varepsilon \leqq \arg\kappa + (m-2)\arg\mu \leqq \pi - \varepsilon \\ \varepsilon \leqq \arg\kappa + (m-2)\arg\mu \leqq \pi - \varepsilon \end{matrix}. \tag{12.25}$$

The allowed region for $(\arg\kappa, \arg\mu)$ is shown in Fig. 17:

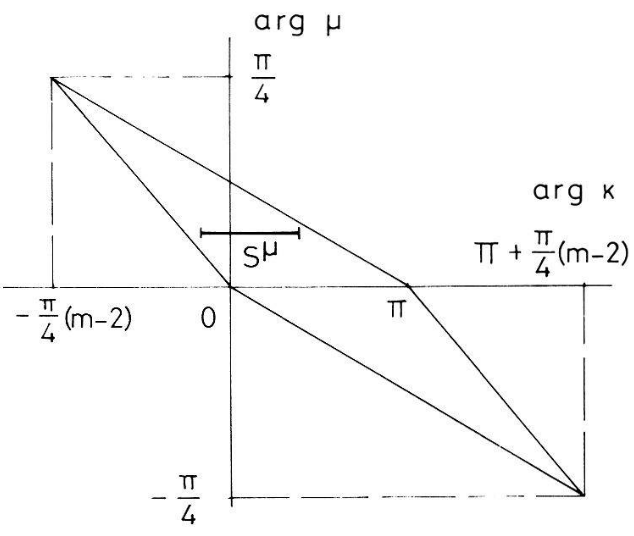


Figure 17

We note that $|\arg \mu| < \pi/4$ and that any sector

$$-\frac{\pi}{4}(m-2) + \varepsilon \leqq \arg \kappa \leqq \pi + \frac{\pi}{4}(m-2) - \varepsilon \tag{12.26}$$

with $\varepsilon > 0$ can be covered by finitely many S^{μ}. This sector has opening angle arbitrarily close to $m\pi/2$. For $m > 4$ it cannot be drawn in the simple κ-plane but on a Riemann surface with branch point 0.

It is easily verified that $\mu^2 H_{\kappa}^{\mu}$ is a family of Schrödinger operators. For states $u \in C_0^{\infty}(|x| > n)$ the expectation values of H_{κ}^{μ} are in the shaded region of Fig. 18, drawn for $\arg \mu > 0$.:

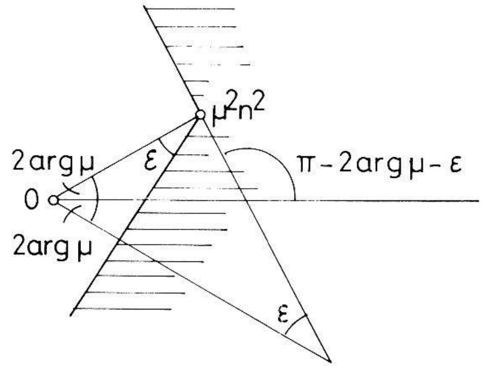


Figure 18

Since n is arbitrary large, each eigenvalue of H_0 is stable by the numerical range argument (6.1). As in the case $V(x) = x^4$ (Example (d)) we find the estimate

$$B_N < Cb^N \sup_x (|x|^{N(m-2)} \exp(-\alpha x^2/2))$$

for some $\alpha > 0$, which gives

$$B_N < Cd^N N^{N(m-2)/2}. \tag{12.27}$$

This is converted into a strong asymptotic estimate with respect to a new expansion parameter ξ defined by

$$\kappa = \begin{cases} \xi^{(m-2)/2} & (m \text{ even}) \\ \xi^{(m-1)/2} & (m \text{ odd}) \end{cases}.$$

Then (12.27) transforms into

$$B_N < C\sigma^n n^n,$$

where n now denotes the order in ξ. Thus $E^{\mu}(\kappa)$ has a strong asymptotic estimate in a sector S^{μ} of the ξ-plane corresponding to (12.25). As in the Stark case we can patch these results for a finite number of sectors S^{μ} to cover any sector of the form

$$-\frac{\pi}{2} + \varepsilon \leqq \arg \xi \leqq \frac{2\pi}{m-2} + \frac{\pi}{2} - \varepsilon \qquad (m \text{ even})$$

$$-\frac{\pi(m-2)}{2(m-1)} + \varepsilon \leqq \arg \xi \leqq \frac{2\pi}{m-1} + \frac{\pi(m-2)}{2(m-1)} - \varepsilon \qquad (m \text{ odd}).$$

For small ε these sectors have opening angle $> \pi$: $E(\kappa) = E^1(\kappa)$ is the Borel sum (in these sectors) of the RS-series computed from the undilated Hamiltonian H_κ, so that again $E_k^* = E_k$ for all k. For real $\xi \neq 0$ the perturbed eigenvalues can be interpreted as resonances of the (ill-defined) Hamiltonian H_κ by using some cut-off for large negative values of $V(x)$ [16].

(g) *The N-body Stark effect* [31, 36]

For N particles with masses $m_1 \cdots m_N$ and charges $q_1 \cdots q_N$ the dilated Stark Hamiltonian corresponding to (12.15) is

$$H_\kappa = -\mu^{-2}\Delta + \mu^{-1} \sum_{i<k} q_i q_k \, |x_i - x_k|^{-1} - \kappa\mu(e, x) \tag{12.28}$$

defined for κ in the sector (12.18). Here and in the following analysis μ is fixed with $|\arg \mu| < \pi/2$ and therefore omitted as a superscript in H_κ and S. H_κ acts on the Hilbert space $L^2(X)$, where

$$X = \left\{ x = (x_1 \cdots x_N) \mid x_k \in R^3, \sum_k m_k x_k = 0 \right\} \tag{12.29}$$

is the space of N-particle configurations in the center-of-mass (CM) frame, equipped with the scalar product

$$(x, y) = 2 \sum_k m_k x_k \cdot y_k, \tag{12.30}$$

$x_k \cdot y_k =$ scalar product in R^3. Δ is the Laplacian for the metric (12.30) and (e, x) an arbitrary real linear form on X. As in the case $N = 1$, $\mu^2 H_\kappa$ is a family of Schrödinger operators since the Coulomb potentials are small relative to Δ and since real $\kappa\mu^3$ are excluded by (12.18).

The following notions are standard in the theory of N-body systems. With

$$D = (C_1 \cdots C_n), \; 2 \leqq n \leqq N$$

we denote the nontrivial partitions of $(1 \cdots N)$ into n subsets (clusters) C_k. For given D we can represent any $x \in X$ by the components

$$x = (x^D, x^{C_1} \cdots x^{C_n}),$$

where $x^D = (\xi_1 \cdots \xi_n)$ is the configuration of the centers-of-mass of $C_1 \cdots C_N$ ($\sum M_k \xi_k = 0$, $M_k =$ total mass of C_k) and x^C the configuration of the cluster C in its own CM-frame. The corresponding configuration spaces X^D, X^C are defined in obvious analogy to (12.29) and (12.30). The advantage of the metric (12.30) is reflected in the identity

$$(x, y) = (x^D, y^D) + \sum_{C \in D} (x^C, y^C)$$

which says that X is the orthogonal sum of the subspaces X^D, $X^{C_1} \cdots X^{C_n}$. Thus each decomposition D induces a factorization

$$L^2(X) = L^2(X^D) \otimes L^2(X^{C_1}) \otimes \cdots \otimes L^2(X^{C_n})$$

of the N-particle Hilbert space. With respect to this factorization the Laplacian takes the form

$$\begin{aligned}\Delta = \Delta^D \otimes 1 \otimes \cdots \otimes 1 \\ + 1 \otimes \Delta^{C_1} \otimes \cdots \otimes 1 \\ + 1 \otimes 1 \otimes \cdots \otimes \Delta^{C_n},\end{aligned}$$

for which we simply write

$$\Delta = \Delta^D + \sum_{C \in D} \Delta^C.$$

Similarly, the system of non-interacting clusters $C_1 \cdots C_n$ is described by the Hamiltonian

$$H_\kappa^D = -\mu^{-2}\Delta^D - \mu\kappa(e^D, x^D) + \sum_{C \in D} H_\kappa^C,$$

where H_κ^C is the Stark Hamiltonian for the subsystem C in its own CM-frame:

$$H_\kappa^C = -\mu^{-2}\Delta^C + \mu^{-1} \sum_{\substack{i,k \in C \\ i<k}} q_i q_k |x_i - x_k|^{-1} - \mu\kappa(e^C, x^C).$$

According to the Balslev–Combes theory the unperturbed Hamiltonian H_0 has the spectrum shown in Fig. 19:

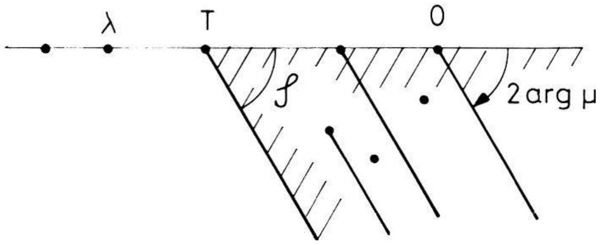


Figure 19

The details of this picture are explained in the Balslev–Combes paper [14] (see also [3], Section XIII.10). Here we only need the following facts:

(i) T and the real eigenvalues of H_0 below T are independent of μ. These eigenvalues form the discrete spectrum of the undilated Hamiltonian $H_0 = H_0^*$ which has the continuous spectrum $[T, +\infty)$

(ii) For any decomposition $D = (C_1 \cdots C_n)$ we have

$$\sigma\left(\sum_{C \in D} H_0^C\right) \subset \mathscr{S} \qquad (12.31)$$

where $\mathscr{S}$ is the complex sector with origin T shown in Fig. 19.

We also mention that H_0 is m-sectorial with an arbitrary narrow sector $\mathscr{S}_0$ shown in Fig. 20. This is evident for the operator $-\mu^{-2}\Delta$ and extends to H_0 since the Coulomb potentials are small relative to Δ.

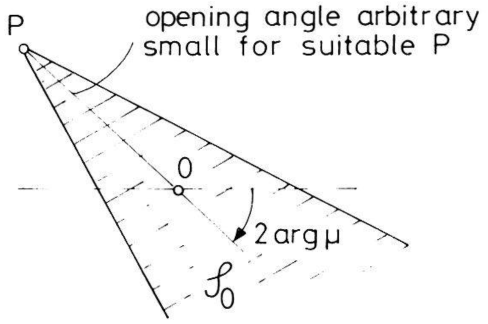


Figure 20

Lemma 12.5 [31]. *For fixed μ with $|\arg \mu| < \pi/2$ any eigenvalue $\lambda < T$ of H_0 is stable with respect to H_κ.*

Proof. In the first step we use a partition of unity on X which is standard in N-body spectral theory (see e.g. [4], Section 3.3):

$$1 = \sum_{D=(C_1, C_2)} J^D(x),$$

where $J^D(x)$ is homogeneous of degree zero, C^∞ on $X \setminus \{0\}$ and

$$\operatorname{dist}(C_1, C_2) \geqq a\,|x| \text{ on } \operatorname{supp}(J^D)$$

for some $a > 0$ depending on the masses. Then (6.2) is evident and (6.6) holds in the form

$$\|(H_\kappa - H_\kappa^D)J^D u\| \leqq \text{const. } n^{-1}\,\|J^D u\|$$

for all $u \in C_0(|x| > n)$. By (6.7) it therefore suffices to prove

$$\lambda \in P(H_\kappa^D) \tag{12.32}$$

for any two-cluster partition $D = (C_1, C_2)$. In fact we will prove this for *all* $D = (C_1 \cdots C_n)$ by induction in n. For $n = N$ we have

$$H_\kappa^D = -\mu^{-2}\Delta - \mu\kappa(e, x),$$

so that (12.32) follows from the numerical range argument (6.1) as in Examples (a), (e). Now we proceed by induction. Let $D = (C_1 \cdots C_n)$ be fixed with $2 \leqq n < N$. We assume that

$$\lambda \in P(H_\kappa^{D'})$$

for all D' obtained from D by splitting some cluster $C \in D$ into two clusters C', C'':

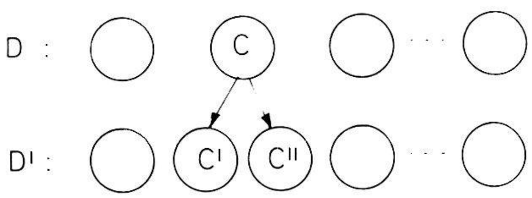


Figure 21

To prove (12.32) from the induction hypothesis we construct a second partition of unity on X as follows. First we define subsets of X:

$$\Omega^{+/-}: \text{diameter of } C < 1 \text{ for all } C \in D$$
$$\text{and } \pm(e, x) > -1$$
$$\Omega^{D'}: \text{dist}(C', C'') > (2N)^{-1}.$$

These sets form an open covering of X. For suppose that $x \notin \Omega^+ \cup \Omega^-$. Then diam $(C) \geqq 1$ for some $C \in D$. Therefore C can be split into two clusters having distance $\geqq (N-1)^{-1}$, which shows that $x \in \Omega^{D'}$ for some D'. Now we choose a partition of unity

$$1 = J^+(x) + J^-(x) + \sum_{D'} J^{D'}(x)$$

whose members J^α are nonnegative C^∞ functions with bounded derivatives, supported in the corresponding sets Ω^α. Then we scale this partition, defining

$$J_\kappa^\alpha(x) = J^\alpha(|\kappa|^{1/2}x).$$

In analogy to (6.3) we need only prove

$$\|(\lambda - H_\kappa^D)J_\kappa^\alpha u\| \geqq \delta \, \|J_\kappa^\alpha u\| \tag{12.33}$$

for some $\delta > 0$, small κ, all $u \in C_0^\infty$ and all members J_κ^α of our partition of unity.

On supp $(J_\kappa^{D'})$ we have

$$\text{dist}(C', C'') \geqq (2N)^{-1} \, |\kappa|^{-1/2}.$$

There H_κ^D reduces to $H_\kappa^{D'}$ up to an error of order $|\kappa|^{1/2}$ in operator norm (tails of Coulomb potentials between C' and C''), and (12.33) follows from the induction hypothesis.

On supp (J_κ^+) we have the two inequalities

$$\text{diam}(C) \geqq |\kappa|^{-1/2} \text{ for all } C \in D \text{ and}$$
$$(e, x) \geqq -|\kappa|^{-1/2},$$

which imply

$$|\kappa(e^C, x^C)| \leqq \text{const}\, |\kappa|^{1/2} \quad \text{and}$$
$$|\kappa(e^D, x^D)_-| \leqq \text{const}\, |\kappa|^{1/2}.$$

(with $(a, x)_{+/-}$ we denote the positive/negative part of the function $x \to (a, x)$). On supp (J_κ^+) the operator H_κ^D thus reduces to

$$\underbrace{-\mu^{-2}\Delta^D - \mu\kappa(e^D, x^D)_+}_{h_\kappa^+} + \underbrace{\sum_{C \in D} H_0^C}_{h^D}$$

up to an error of order $|\kappa|^{1/2}$, and it remains to prove that

$$\lambda \in P(h_\kappa^+ + h^D). \tag{12.34}$$

h_κ^+ is uniformly m-sectorial with a sector $\mathcal{S}^+$:

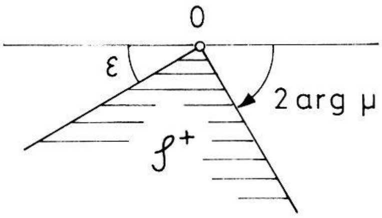


Figure 22

On the other hand, h^D is independent of κ and m-sectorial with a sector $\mathcal{S}_0$ shown in Fig. 20. Therefore we can apply Theorem 6.2, which together with (12.31) gives

$$\sum (h_\kappa^+ + h^D) \subset \mathcal{S}^+ + \sigma(h^D) \subset \mathcal{S}^+ + \mathcal{S}.$$

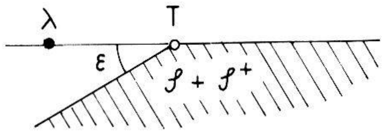


Figure 23

This proves (12.34). The proof for J_κ^- is analogous. □

Borel summability

Given the stability result of Lemma 12.5 the RS-expansion and the asymptotic estimates follow exactly as in the one-body case. As a result, Lemma 12.4 holds in the N-body case for any eigenvalue $\lambda < T$.

Remarks

For the atomic case (defined in the next example) Sigal has shown that the Stark resonances have positive widths [37] with an upper bound depending exponentially on the shortest Agmon distance across the energetically forbidden region [39].

(h) *The Zeeman effect in atoms* [13]

$$H_\kappa = \sum_{k=1} [(p_k - \kappa(e \wedge x_k))^2 - Z\,|x_k|^{-1}] + \sum_{i<k} |x_i - x_k|^{-1}$$

describes an atom with fixed nucleus and N electrons in a homogeneous magnetic field κe. This problem is simpler than the Zeeman effect in an arbitrary N-body system for the following reason.

$$H_\kappa = H_0 + \kappa^2 \sum_k (e \wedge x_k)^2 - 2\kappa L,$$

where

$$L = \left(e, \sum_k x_k \wedge p_k\right)$$

is the total angular momentum in the direction of the field. Since L commutes with H_κ we can drop the term $-2\kappa L$ from the Hamiltonian, treating it like a constant. Again we start from the dilated operator

$$H_\kappa = H_0 + \kappa^2\mu^2 \sum_k (e \wedge x_k)^2 \tag{12.35}$$

$$H_0 = \sum_{k=1} (\mu^{-2}p_k^2 - \mu^{-1}Z\,|x_k|^{-1}) + \mu^{-1} \sum_{i<k} |x_i - x_k|^{-1},$$

where the dilation parameter μ is fixed in $|\arg \mu| \leqq (\pi/2) - \varepsilon$ and with κ restricted to the sector

$$S: -\frac{\pi}{2} + \varepsilon \leqq \arg(\kappa^2\mu^3) \leqq \frac{\pi}{2} - \varepsilon.$$

The region of allowed κ, μ is shown in Fig. 24:

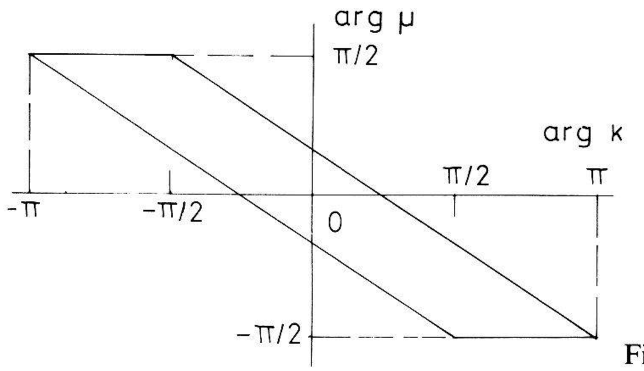


Figure 24

It is easy to check that $\mu^2 H_\kappa$ is a family of Schrödinger operators. The spectrum of H_0 is given by Fig. 19.

Lemma 12.6. *Each real eigenvalue $\lambda < T$ of H_0 is stable with respect to the family H_κ given by* (12.35)

Proof. The proof is essentially the same as in the Stark case. Since now there is a nucleus (called particle 0 with coordinate $x_0 = 0$) we consider two-cluster partitions $D = (C_0, C_1)$ of the set $(0 \cdots N)$ where C_0 is always the cluster containing the nucleus. There is a partition of unity

$$1 = \sum_{D=(C_0,C_1)} J^D(x)$$

on $X = L^2(R^{3N})$ with nonnegative J^D, homogeneous of degree zero and smooth for $x \neq 0$ such that

$$\operatorname{dist}(C_0, C_1) \geqq a\,|x|$$

on supp (J^D) for some $a>0$. In this region H_κ reduces to

$$H_\kappa^D = H_\kappa^{C_0} + h_\kappa^{C_1}$$

in the sense of (6.6), where $H_\kappa^{C_0}$ is the original Hamiltonian restricted to the subsystem C_0 (containing the nucleus) and h_κ^C the Zeeman Hamiltonian of a cluster of electrons without nucleus:

$$h_\kappa^C = \sum_{k\in C} (\mu^{-2}p_k^2 + \kappa^2\mu^2(e\wedge x_k)^2) + \mu^{-1} \sum_{\substack{i<k\\ i,k\in C}} |x_i - x_k|^{-1}.$$

By (6.7) we need only prove

$$\lambda \in P(H_\kappa^D) \tag{12.36}$$

for all two-cluster partitions D. Again we will prove this by induction for all $D=(C_0\cdots C_n)$ with $n\geqq 1$, where always $0\in C_0$ and where H_κ^D is obtained from H_κ by dropping all intercluster potentials. In particular

$$H_\kappa^D = \sum_k (\mu^{-2}p_k^2 + \kappa^2\mu^2(e\wedge x_k)^2)$$

in the case $n=N$: $D=(0)(1)\cdots(N)$. The numerical range of each term in this sum is in the shaded sector $\mathscr{S}_1$ of Fig. 25:

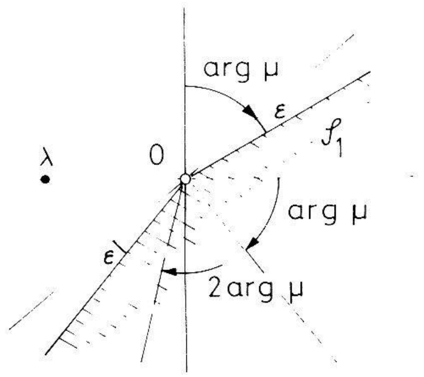


Figure 25

This proves (12.36) for $n=N$. By the same argument, (12.36) also holds for any $D=(C_0\cdots C_n)$ with C_0 containing *only* the nucleus (here we use that the electron-electron potentials are positive. Notice that the ray $\arg z = -\arg\mu$ is within the sector $\mathscr{S}_1$ in Fig. 25). It remains to prove (12.36) in the case when C_0 contains electrons, starting from the induction hypothesis that

$$\lambda \in P(H_\kappa^{D'})$$

for all D' obtained from D by splitting C_0 into two clusters C_0', C_0'' with $0\in C_0'$:

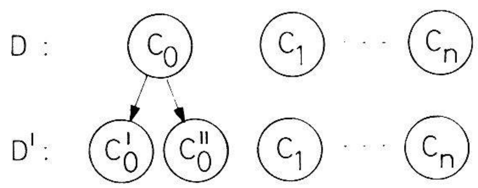


Figure 26

As in the Stark case we introduce a second partition of unity

$$1 = J^0(x) + \sum_{D'} J^{D'}(x)$$

with the properties

$$\operatorname{diam}(C_0) \leqq 1 \text{ on } \operatorname{supp}(J^0)$$

$$\operatorname{dist}(C_0', C_0'') \geqq (2N)^{-1} \text{ on } \operatorname{supp}(J^{D'}).$$

Then we scale this partition defining

$$J_\kappa^\alpha(x) = J^\alpha(|\kappa|^{1/2} x)$$

To prove (12.36) it then suffices to show that

$$\|(\lambda - H_\kappa^D) J_\kappa^\alpha u\| \geqq \delta \, \|J_\kappa^\alpha u\| \tag{12.37}$$

for some $\delta > 0$, small κ, all $u \in C_0^\infty$ and for all members J_κ^α of our partition of unity.

On $\operatorname{supp}(J_\kappa^{D'})$ the operator H_κ^D reduces to $H_\kappa^{D'}$ up to an error of order $|\kappa|^{1/2}$ in operator norm, so that (12.37) is satisfied by the induction hypothesis.

On $\operatorname{supp}(J^0)$ we have $\operatorname{diam}(C_0) \leqq |\kappa|^{-1/2}$, therefore H_κ^D reduces to

$$H_0^{C_0} + \sum_{k=1}^{n} h_\kappa^{C_k}$$

up to an error of order $|\kappa|$. As we have already noted, the operators h_κ^C are uniformly m-sectorial with a sector $\mathscr{S}_1$ (Fig. 25), while the sectorality of the κ-independent operator $H_0^{C_0}$ is given in Fig. 20. By Theorem 6.2 and (12.31) we thus obtain

$$\sum \left(H_0^{C_0} + \sum_{k=1}^{n} h_\kappa^{C_k} \right) \subset \mathscr{S}_1 + \mathscr{S},$$

which proves (12.37) for J^0. □

Borel summability

As in the Stark case we use an exponential bound

$$f(x) = \alpha \, |x| \; (\alpha > 0)$$

for any eigenvalue $\lambda < T$ of H_0. Since the RS-series for (12.35) is a power series in κ^2 we find from (9.7) the estimate

$$B_{N=1} = B_N < Cb^N \operatorname{supp}_x |x|^N \exp(-\alpha \, |x|/2)$$

$$< C\sigma^N N! \; (N \text{ even}).$$

Patching these results for different values of the dilation parameter μ as in Example (e), it is evident from Fig. 24 that we arrive at

Lemma 12.7. *For any eigenvalue $\lambda < T$ of H_0, $E(\kappa)$ is the Borel sum of the* RS-*series, computed from the undilated Hamiltonian, in a sector* $-\pi + \varepsilon \leqq \arg \kappa \leqq \pi - \varepsilon$.

Remark

In this case we have of course $E(\kappa)^* = E(\kappa)$ for real κ: the perturbed eigenvalues correspond to bound states.

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