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Adiabatic theorem and Gell-Mann–Low formula

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Abstract. Using the adiabatic theorem of quantum mechanics we give a non-perturbative proof of the Gell-Mann–Low formulae. Also a very general form of the adiabatic theorem is proven.

I. Introduction

The present note has a pronounced pedagogical character and has two aims.

The first one is to give a non-perturbative proof of the Gell-Mann–Low (G–L) formulae [1] relating the adiabatic limit of the evolution in the interaction picture to quantities appearing in the time independent theory like the energy shift and the eigenvector of the total hamiltonian. Although the content of the G–L formulae is clearly non-perturbative, the existing proofs are (to our best knowledge) in the framework of perturbation theory (to obtain a real proof one has e.g. to combine the arguments in [2] with the rigourous perturbative results in [3]) and only for the G–L switching function $e^{-\varepsilon|t|}$. On the other hand it is almost evident that the G–L formulae should be related to the adiabatic theorem of quantum mechanics. The difficulty is that while the adiabatic theorem holds for the Schrödinger picture evolution operator U, the G–L formulae deal with the interaction picture evolution operator U_I. The latter contains an explicit time dependence in the hamiltonian even in the adiabatic limit. We will show that this difficulty is not a serious one and that one can obtain G–L-type formulae for general switching functions by just applying an appropriate formulation of the adiabatic theorem. Moreover we will express also Berry's phase [4,5] in terms of U_I.

The second aim of this note is to give a general formulation of the adiabatic theorem. In the last decade this has proven to be the main technical tool in a lot of interesting phenomena, e.g. the Quantum Hall Effect [6] and spontaneous pair production in strong fields [7] and general and rigorous proofs of the adiabatic

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theorem appeared. For its history and physical content we refer to [8,6,9] and references therein. Here we will reformulate the results of [10] in the spirit of [6]. This formulation provides the factorisation of the secular divergences [11,12] for the general degenerate case and to higher orders in the slowness parameter ε. For the adiabatic switching formalism ε has no physical meaning and the higher order corrections are mainly of academic interest. In quantum optics and quantum electronics on the other hand ε has a physical meaning coming from the interaction of matter with classical electromagnetic fields in the low frequency limit. We hope to come back to these problems in future work.

In the remainder of this introductory section we shall fix some notations and technical assumptions. In section II we will formulate the adiabatic theorem and perform the factorisation of the adiabatic evolution. For those interested only in the G–L formulae we collect at the end of Section II the result in the particular form needed in Section III. This particular case of the adiabatic theorem (with a semirigorous proof and almost identical notations) is contained already in [8]. Section III contains the proof of the G–L formulae. The appendix contains the proof of the adiabatic theorem.

We will consider hamiltonians of the form

$$H(\varepsilon t) = \mathring{H} + f(\varepsilon t)V \tag{1.1}$$

where $\mathring{H}$ is self-adjoint and bounded from below; V is self-adjoint and bounded with respect to $\mathring{H}$:

$$\|V\psi\| \leq a\,\|\mathring{H}\psi\| + b\,\|\psi\|, \quad a<1, \quad \psi \in \mathscr{D}(\mathring{H}) \tag{1.2}$$

and $f(s)$ is a switching function.

We will very often abbreviate $s = \varepsilon t$.

We assume that $f(s)$ is defined on $(-\infty, +\infty)$ and has the following properties:

$$\left.\begin{array}{l} 0 \leq f(s) \leq 1 \\ f(0) = 1 \\ \partial_s^m f(s) \equiv f^{(m)}(s) \in L^1(\mathbb{R}), \quad m = 0, 1, \ldots, m_0 \leq \infty. \end{array}\right\} \tag{1.3}$$

Actually only the last property of (1.3) will be needed to prove the adiabatic theorem. For the G–L switching function $e^{-|s|}$ we have $m_0 = 1$ since the first derivative is discontinuous at $s = 0$. However, $e^{-|s|}$ can be obtained as a limit of switching functions with $m_0 = 2$ and with $\int_{-\infty}^{+\infty} |f^{(2)}(s)|\, ds$ uniformly bounded, so actually all the results for the switching functions with $m_0 = 2$ apply to $e^{-|s|}$ as well.

As a consequence of (1.2) $H(s) = \mathring{H} + f(s)V$ is self-adjoint on the domain of $\mathring{H}$ [13,14]. Moreover $(H(s) + i)^{-1}$ is m_0 times norm differentiable. For example

$$\partial_s(H(s) + i)^{-1} = f^{(1)}(s)(H(s) + i)^{-1} V (H(s) + i)^{-1} \tag{1.4}$$

Many Schrödinger-type equations will appear in the following, especially in the proof of the adiabatic theorem, and since the corresponding hamiltonians depend on time some care is needed to assure the existence and uniqueness of the solutions. Due to the norm differentiability of $(H(s)+i)^{-1}$ and to some technical results in [15, see Chapter IV Theorem 2.4] the existence and uniqueness of the unitary evolutions is assured by the standard results [16,14,17].

The basic objects we consider are the evolution operator $U(t, t_0; \varepsilon)$ in the Schrödinger picture given by

$$i\partial_t U(t, t_0; \varepsilon) = H(\varepsilon t)U(t, t_0; \varepsilon), \qquad U(t_0, t_0; \varepsilon) = \mathbb{1} \tag{1.5}$$

and the evolution operator $U_I(t, t_0; \varepsilon)$ in the interaction picture given by

$$i\partial_t U_I(t, t_0; \varepsilon) = f(\varepsilon t)e^{i\mathring{H}t}Ve^{-i\mathring{H}t}U_I(t, t_0; \varepsilon); \qquad U(t_0, t_0; \varepsilon) = \mathbb{1} \tag{1.6}$$

They are related by

$$U_I(t, t_0; \varepsilon) = e^{i\mathring{H}t}U(t, t_0; \varepsilon)e^{-i\mathring{H}t_0} \tag{1.7}$$

It is well known that due to (1.3)

$$\lim_{\substack{t\to\infty \\ t_0\to-\infty}} U_I(t, t_0; \varepsilon)\psi = U_I(\infty, -\infty; \varepsilon)\psi \tag{1.8}$$

exists for all $\psi \in \mathscr{D}(\mathring{H})$. If V is bounded, the limit in (1.8) is a norm limit.

Besides (1.2) and (1.3) the main assumption concerning $H(s)$ is that its spectrum consists for all s of n disconnected pieces $\sigma_j^0(s)$, $(j = 1, 2, \ldots, n < \infty)$ so that

$$d = \min_{j\neq k} \inf_s \operatorname{dist}(\sigma_j^0(s), \sigma_k^0(s)) > 0 \tag{1.9}$$

We will choose the numbering such that $\sigma_j^0(s)$ is bounded for $j \le n-1$. $P_{0,j}(s)$ are the spectral projections of $H(s)$ corresponding to $\sigma_j^0(s)$.

Due to the norm differentiability of $(H(s)+i)^{-1}$ and to the Riesz formula [13,14]

$$P_{0,j}(s) = \frac{1}{2\pi i}\oint_{C_j} (H(s)-z)^{-1}\, dz \tag{1.10}$$

where C_j is a bounded contour surrounding $\sigma_j^0(s)$, the $P_{0,j}(s)$ are m_0 times norm differentiable. Due to

$$P_{0,n}(s) = \mathbb{1} - \sum_{j=1}^{n-1} P_{0,j}(s) \tag{1.11}$$

also $P_{0,n}(s)$ is m_0 times norm differentiable.

In Section III we will deal with the particular case $n = 2$, $\sigma_1^0(s) = \{E_0(s)\}$, where $E_0(s)$ is a nondegenerate eigenvalue of $H(s)$ with eigenvector $\psi_0(s)$. In this case $P_{0,1}(s)$ is a one dimensional projector.

II The adiabatic theorem

In this section we state the adiabatic theorem in the form as we will prove it in the appendix and perform the factorisation of U. To this end we first define recursively for $k = 0, 1, \ldots, m_0 - 1$ and sufficiently small ε the self-adjoint operators

$$H_{k+1}(s;\varepsilon) = H_k(s;\varepsilon) + \sum_{j=1}^{n} P_{k,j}(s;\varepsilon)\{i\varepsilon\partial_s P_{k,j}(s;\varepsilon) + [P_{k,j}(s;\varepsilon), H(s)]\} \tag{2.1}$$

with $H_0(s;\varepsilon) = H(s)$. The $P_{k,j}(s;\varepsilon)$ are the spectral projections corresponding to the j'th piece of the spectrum of H_k. Actually the $P_{0,j}$ do not depend on ε. In the appendix we will show for all k that the sum on the r.h.s. of (2.1) is bounded and of order ε and then by perturbation theory for small enough ε the spectrum of $H_k(s;\varepsilon)$ also consists of n disconnected pieces, going to the spectrum of $H(s)$ as $\varepsilon \to 0$.

We have now all the ingredients to formulate the **Adiabatic Theorem:**
The operator $U(t, t_0;\varepsilon)$ has for $k = 0, 1, \ldots, m_0 - 1$ the factorisation

$$U(t, t_0;\varepsilon) = U_k^A(t, t_0;\varepsilon)\Omega_k(t, t_0;\varepsilon) \tag{2.2}$$

where $U_k^A(t, t_0;\varepsilon)$ is the (unique) solution of

$$i\partial_t U_k^A(t, t_0;\varepsilon) = \{H_0(\varepsilon t) - H_{k+1}(\varepsilon t;\varepsilon) + H_k(\varepsilon t;\varepsilon)\}U_k^A(t, t_0;\varepsilon)$$
$$U_k^A(t_0, t_0;\varepsilon) = \mathbb{1} \tag{2.3}$$

and (remember the abbreviation $s = \varepsilon t$, $s_0 = \varepsilon t_0$)

$$P_{k,j}(s;\varepsilon) = U_k^A(t, t_0;\varepsilon)P_{k,j}(s_0;\varepsilon)U_k^{A*}(t, t_0;\varepsilon) \tag{2.4}$$

Furthermore, for $k = 0, 1, \ldots, m_0 - 2$

$$\sup_{t,t_0} \|\Omega_k(t, t_0;\varepsilon) - \mathbb{1}\| \le \text{const } \varepsilon^{k+1}. \tag{2.5}$$

$U_0^A(t, t_0;\varepsilon)$ coincides with the adiabatic evolution of [6]. We therefore name U_k^A the "adiabatic evolution of order k". From (2.5) it follows that for increasing k the U_k^A approximate the Schrödinger evolution operator U better and better.

From (2.2), (2.4) and (2.5) one can write an alternative form of the adiabatic theorem:

$$\|P_{k,j}(s;\varepsilon)U(t, t_0;\varepsilon) - U(t, t_0;\varepsilon)P_{k,j}(s_0;\varepsilon)\| \le \text{const } \varepsilon^{k+1}$$

Families of orthogonal projections satisfying this inequality have been constructed for the first time by Garrido [18] at the formal level and for $H(s)$ with discrete nondegenerate spectrum. At the rigorous level and for the general case the same construction (used also for the proof of the adiabatic theorem given in the appendix) has been rediscovered in [9]. The construction uses solutions of certain differential equations and then $H(s)$ enters nonlocally in s. The crucial point of the construction given here is that the $P_{k,j}(s;\varepsilon)$ are constructed from $H(s)$ without solving any differential equation and depend only on $H(s)$ and its derivatives of order $\le k$ at the point s. In physical terms this means that the

$P_{k,j}(s;\varepsilon)$ depend only on the "dynamics" at time s and then they can be regarded as generalised adiabatic invariants.

We will now exploit (2.4) to further factorise U_k^A. To this end we make use of the theory of transformation functions which goes back to Daletsky and Krein [15] and Kato [19]. The basic construction which will be used also in the proof of the adiabatic theorem goes as follows:

Consider a continuously differentiable family of orthogonal projections $Q_j(s)$, $j=1,2,\ldots,n$ with

$$\sum_{j=1}^{n} Q_j(s) = \mathbb{1}; \qquad Q_l(s)Q_m(s) = \delta_{l,m}Q_m(s).$$

A transformation function [13,14] for the $Q_j(s)$ is a family $T(s,s_0)$ of unitary operators satisfying

$$Q_j(s) = T(s,s_0)Q_j(s_0)T^*(s,s_0), \qquad T(s_0,s_0) = \mathbb{1} \tag{2.6}$$

Lemma 1 [15,13,8]. *If $K(s)$ is defined by*

$$K(s) = -i\sum_{j=1}^{n} Q_j(s)\partial_s Q_j(s) \tag{2.7}$$

then

(i) *$K(s)$ is self-adjoint*
(ii) *$Q_j(s)K(s)Q_j(s) = 0$ for $j=1,\ldots,n$* (2.8)
(iii) *The (unique) solution of*

$$i\partial_s A(s,s_0) = K(s)A(s,s_0); \qquad A(s_0,s_0) = \mathbb{1} \tag{2.9}$$

is a transformation function, i.e.

$$Q_j(s) = A(s,s_0)Q_j(s_0)A^*(s,s_0) \tag{2.10}$$

Suppose now that T and S are transformation functions for Q_j. Then from (2.6) it follows that $[T^*(s,s_0)S(s,s_0), Q_j(s_0)] = 0$, or put in other words

$$S(s,s_0) = T(s,s_0)W(s,s_0) \quad \text{with} \quad [W(s,s_0), Q_j(s_0)] = 0 \tag{2.11}$$

Now (2.4) implies that $U_k^A(s,s_0;\varepsilon)$ is a transformation function for the $P_{k,j}(s;\varepsilon)$. Let $A_k(s,s_0;\varepsilon)$ be the transformation function for the $P_{k,j}(s;\varepsilon)$ constructed according to Lemma 1 and let $\Phi_k(s,s_0;\varepsilon)$ be given by

$$\Phi_k(s,s_0;\varepsilon) = A_k^*(s,s_0;\varepsilon)U_k^A(s,s_0;\varepsilon) \tag{2.12}$$

which is equivalent to

$$U_k^A(s,s_0;\varepsilon) = A_k(s,s_0;\varepsilon)\Phi_k(s,s_0;\varepsilon). \tag{2.13}$$

From (2.11) and (2.12) it follows that

$$[\Phi_k(s,s_0;\varepsilon), P_{k,j}(s_0;\varepsilon)] = 0. \tag{2.14}$$

Until now we have always carefully written out all the arguments of the operators in detail. From now on for typographical convenience we will suppress these arguments if no confusion can arise. In particular we will drop the argument ε in expressions like $U_k^A(s, s_0; \varepsilon)$ nearly everywhere.

The differential equation for $\Phi_k(s, s_0)$ can be found from (2.12), (2.3), (2.9) and the definition of A_k in terms of the $P_{k,j}$:

$$i\partial_t\Phi_k = A_k^*\Big(H_0 - H_{k+1} + H_k + i\sum_{j=1}^{n} P_{k,j}\partial_s P_{k,j}\Big)A_k\Phi_k$$

$$\Phi_k(s_0, s_0) = \mathbb{1} \tag{2.15}$$

This can be simplified. Due to (2.14)

$$\Phi_k(s, s_0) = \sum_{j=1}^{n} P_{k,j}(s_0)\Phi_k(s, s_0)P_{k,j}(s_0) \tag{2.16}$$

Furthermore, due to (2.8) with (2.7) and (A.19)

$$i\partial_t P_{k,j}(s_0)\Phi_k(s, s_0)P_{k,j}(s_0)$$
$$= P_{k,j}(s_0)A_k^*(s, s_0)P_{k,j}(s)H(s)P_{k,j}(s)A_k(s, s_0)P_{k,j}(s_0)\Phi_k(s, s_0)P_{k,j}(s_0) \tag{2.17}$$

The main result of this section is contained in (2.13), (2.16) and (2.17): In order to have U_k^A one constructs A_k from the $P_{k,j}$ according to Lemma 1 and solves the differential equation for Φ_k, in each subspace of $P_{k,j}(s_0)$. The second step is trivial in a one dimensional subspace. Actually, if $k = 0$ and $P_{0,1}$ is one dimensional, then

$$\Phi_0 P_{0,1} = \varphi_0 P_{0,1}$$

where $\varphi_0(s, s_0)$ is a phasefactor given by the equation

$$i\partial_t\varphi_0(\varepsilon t, \varepsilon t_0) = \text{trace}\{A_0^* P_{0,1} H_0 P_{0,1} A_0^*\}\varphi_0$$
$$= E_0(\varepsilon t)\varphi_0(\varepsilon t, \varepsilon t_0); \qquad \varphi_0(\varepsilon t_0, \varepsilon t_0) = 1$$

which has the solution

$$\varphi_0(\varepsilon t, \varepsilon t_0) = \exp\left[-i\int_{t_0}^{t} E_0(\varepsilon t')\,dt'\right] = \exp\left[-i/\varepsilon\int_{s_0}^{s} E_0(s')\,ds'\right] \tag{2.18}$$

The point of the factorisation (2.13) is that by construction the $P_{k,j}$ are continuous for $\varepsilon \to 0$; thus also A_k is continuous for $\varepsilon \to 0$. This means that all the singular ε-behaviour of U_k^A is contained in Φ_k which in the nondegenerate case is just a phase-factor.

We end this section by summing up the results for the simplest case we treat in the next section: $n = 2$ and $k = 0$, and $P_{0,1}$ corresponding to a nondegenerate eigenvalue E_0 (see also [8]) (we write all arguments out!):

$$U(s, s_0; \varepsilon) = U_0^A(s, s_0; \varepsilon)\Omega_0(s, s_0; \varepsilon) = A_0(s, s_0)\Phi_0(s, s_0; \varepsilon)\Omega_0(s, s_0; \varepsilon) \tag{2.19}$$

$$\Phi_0(s, s_0; \varepsilon)P_{0,1}(s_0) = \varphi_0(s, s_0; \varepsilon)P_{0,1}(s_0) \tag{2.20}$$

with

$$\varphi_0(s, s_0; \varepsilon) = \exp\left[-i/\varepsilon \int_{s_0}^{s} E_0(s')\, ds' \right] \tag{2.21}$$

Using

$$-i \sum_{j=1}^{2} P_{0,j}(s) \partial_s P_{0,j}(s) = i(\mathbb{1} - 2P_{0,1}(s)) \partial_s P_{0,1}(s) \tag{2.22}$$

we have

$$\left.\begin{aligned} i\partial_s A_0(s, s_0) &= i(\mathbb{1} - 2P_{0,1}(s))(\partial_s P_{0,1}(s)) A_0(s, s_0) \\ A_0(s_0, s_0) &= \mathbb{1} \end{aligned}\right\} \tag{2.23}$$

$$A_0^*(s, s_0) P_{0,1}(s) A_0(s, s_0) = P_{0,1}(s_0) \tag{2.24}$$

and

$$\sup_{s, s_0} \|\Omega_0(s, s_0; \varepsilon) - \mathbb{1}\| \le \text{const}\ \varepsilon \tag{2.25}$$

III. The Gell-Mann–Low Formulae

In this section we will apply our results to a piece of the spectrum, which consists only of one (isolated) nondegenerate eigenvalue of $H(s)$. For convenience we suppress the index j corresponding to the one-dimensional subspace in question. We will only use the quantities of the lowest adiabatic approximation; therefore we also can suppress the index $k = 0$.

We thus have the eigenvalue $E(s)$ with $E(\infty) = E(-\infty) = \mathring{E}$ and the one-dimensional projector $P(s)$ with

$$P(\infty) = P(-\infty) = \mathring{P} \tag{3.1}$$

where $\mathring{E}$ and $\mathring{P}$ are the eigenvalue and the projector of $\mathring{H}$.

The evolution operator in the interaction picture is given by (1.7). Since the limit $t_0 \to -\infty$ does not exist for $U(t, t_0; \varepsilon)$, in order to make use of the adiabatic theorem we first compute $U_I(t, t_0; \varepsilon)\mathring{P}$ and then take the limit $t_0 \to -\infty$. *In the following we conveniently suppress the dependence on ε again. But remember that $s \equiv \varepsilon t$ and $s_0 \equiv \varepsilon t_0$.*

Using the factorisation (2.19), the definition

$$A_I(s, s_0) = e^{it\mathring{H}} A(s, s_0) e^{-it\mathring{H}}$$

and the fact that

$$\mathring{H}\mathring{P} = \mathring{P}\mathring{H} = \mathring{E}\mathring{P}$$

we have

$$U_I(t, t_0)\mathring{P} = e^{-it_0\mathring{E}}A_I(s, s_0)e^{it\mathring{H}}\Phi(s, s_0)\mathring{P} + R(t, t_0) \tag{3.2}$$

Here

$$R(t, t_0) = e^{-it_0\mathring{E}}e^{i\mathring{H}t}A(s, s_0)\Phi(s, s_0)(\Omega(s, s_0) - \mathbb{1})\mathring{P} \tag{3.3}$$

with (due to (2.25))

$$\sup_{t, t_0} \|R(t, t_0)\| \le \text{const } \varepsilon$$

Using (2.20) we have

$$\begin{aligned} e^{it\mathring{H}}\Phi(s, s_0)\mathring{P} &= e^{it\mathring{H}}\Phi(s, s_0)P(s_0) + e^{it\mathring{H}}\Phi(s, s_0)(\mathring{P} - P(s_0)) \\ &= e^{it\mathring{E}}\varphi(s, s_0)\mathring{P} + e^{it\mathring{H}}(P(s_0) - \mathring{P})\varphi(s, s_0) + e^{it\mathring{H}}\Phi(s, s_0)(\mathring{P} - P(s_0)) \end{aligned} \tag{3.4}$$

Inserting (3.4) into (3.2) and using (3.1) as well as (2.21) one gets:

$$U_I(t, -\infty)\mathring{P} = A_I(s, -\infty)\mathring{P} \exp\left[-i \int_{-\infty}^{t} (E(\varepsilon t') - \mathring{E})\, dt'\right] + R(t, -\infty) \tag{3.5}$$

This is the adiabatic theorem in the interaction picture for a one dimensional subspace.

For $t = 0$ we get

$$U_I(0, -\infty)\mathring{P} = A(0, -\infty)\mathring{P} \exp\left[-i/\varepsilon \int_{-\infty}^{0} (E(s') - \mathring{E})\, ds'\right] + R(0, -\infty) \tag{3.6}$$

Note that $A(0, -\infty)$ does **not** depend on ε so that (up to terms of order ε coming from $R(0, -\infty)$) all the dependence of U_I on ε is contained in an explicit form in the phase.

Now let $\mathring{\psi}$ be a unit vector corresponding to $\mathring{P}$, i.e. $\mathring{P}\mathring{\psi} = \mathring{\psi}$. Remembering that $A(0, -\infty)\mathring{P} = P(0)A(0, -\infty)$ it follows that

$$P(0)A(0, -\infty)\mathring{\psi} = A(0, -\infty)\mathring{P}\mathring{\psi} = A(0, -\infty)\mathring{\psi}$$

This means (because of the unitarity of A) that $A(0, -\infty)\mathring{\psi}$ is a unit vector corresponding to $P(0)$ and thus it is a normalised eigenvector of $H(0)$.

Since

$$\begin{aligned} P(0)\mathring{\psi} = A(0, -\infty)\mathring{P}A^*(0, -\infty)\mathring{\psi} &= (\mathring{\psi}, A^*(0, -\infty)\mathring{\psi})A(0, -\infty)\mathring{\psi} \\ &= \overline{(\mathring{\psi}, A(0, -\infty)\mathring{\psi})}A(0, -\infty)\mathring{\psi} \end{aligned} \tag{3.7}$$

one concludes that

$$(\mathring{\psi}, P(0)\mathring{\psi}) = |(\mathring{\psi}, A(0, -\infty)\mathring{\psi})|^2$$

We assume that $(\mathring{\psi}, P(0)\mathring{\psi}) \neq 0$, which is equivalent to the condition $\|\mathring{P} - P(0)\| < 1$.

From (3.6) and (3.7) we then get

$$\lim_{\varepsilon\to 0}\frac{U_I(0,-\infty;\varepsilon)\mathring{\psi}}{(\mathring{\psi},U_I(0,-\infty;\varepsilon)\mathring{\psi})}=\frac{A(0,-\infty)\mathring{\psi}}{(\mathring{\psi},A(0,-\infty)\mathring{\psi})}=\frac{P(0)\mathring{\psi}}{|(\mathring{\psi},P(0)\mathring{\psi})|^2}$$
$$=\frac{P(0)\mathring{\psi}}{\|P(0)\mathring{\psi}\|^2} \tag{3.8}$$

This is the G–L result for the eigenvector of $H(0)$; it has been derived here for the general form (1.3) of the switching factor.

From (3.5) we have

$$(\mathring{\psi},U_I(t,-\infty)\mathring{\psi})=(\mathring{\psi},A(s,-\infty)\mathring{\psi})\exp\left[-i/\varepsilon\int_{-\infty}^{\varepsilon t}(E(s')-\mathring{E})\,ds'\right]$$
$$+(\mathring{\psi},R(t,-\infty)\mathring{\psi})$$

Here one can take the limit $t\to\infty$ and we obtain:

$$(\mathring{\psi},U_I(\infty,-\infty)\mathring{\psi})=(\mathring{\psi},A(\infty,-\infty)\mathring{\psi})\exp\left[-i/\varepsilon\int_{-\infty}^{+\infty}(E(s')-\mathring{E})\,ds'\right]$$
$$+(\mathring{\psi},R(\infty,-\infty)\mathring{\psi}) \tag{3.9}$$

From $P(\infty)A(\infty,-\infty)=A(\infty,-\infty)P(-\infty)$ and (4.1) it follows that

$$\mathring{P}A(\infty,-\infty)=A(\infty,-\infty)\mathring{P}$$

so that $A(\infty,-\infty)\mathring{\psi}$ is another unit vector corresponding to $\mathring{P}$ and can differ from $\mathring{\psi}$ only by a phase factor:

$$A(\infty,-\infty)\mathring{\psi}=e^{-i\chi}\mathring{\psi} \tag{3.10}$$

χ is associated with switching on and off the interaction and is independent of ε. It is known as "Berry's phase" [4,5].

Taking the logarithm of (3.9) one obtains with (3.10) and (3.3)

$$-i\varepsilon\log(\mathring{\psi},U_I(\infty,-\infty)\mathring{\psi})=\int_{-\infty}^{+\infty}(E(s')-\mathring{E})\,ds'+\varepsilon\chi+O(\varepsilon^2) \tag{3.11}$$

this is the G–L result for the phase; it has been derived here for the general form (1.3) of the switching factor.

For the particular switching factor $f(s)=e^{-|s|}$ one can proceed further. We replace V by λV where λ is a coupling constant. The hamiltonian $H(s\mid\lambda)$ and the eigenvalue $E(s\mid\lambda)$ depend on s and λ through the combination $\xi=\lambda e^{-|s|}$. Using

$$\partial_\lambda E=e^{-|s|}\partial_\xi E$$

and

$$\partial_s E=-\frac{s}{|s|}\lambda e^{-|s|}\partial_\xi E$$

we have

$$\lambda\partial_\lambda E = -\frac{s}{|s|}\partial_s E$$

so that

$$\lambda\partial_\lambda \int_{-\infty}^{+\infty} (E(s \mid \lambda) - \mathring{E})\, ds = -\int_{-\infty}^{+\infty} \frac{s}{|s|}\partial_s E(s \mid \lambda)\, ds = -2(E(0 \mid \lambda) - \mathring{E})$$

Together with (3.11) we have (writing also the dependence on ε)

$$i\lambda\partial_\lambda \lim_{\varepsilon\to 0} \varepsilon \log(\mathring{\psi}, U_I(\infty, -\infty; \varepsilon \mid \lambda)\mathring{\psi}) = 2(E(0 \mid \lambda) - \mathring{E})$$

This is the usual G–L result for the difference of the eigenvalues of $H(0 \mid \lambda)$ and $\mathring{H}$.

Appendix

In this appendix we prove the adiabatic theorem. We first define H^0 by

$$H^0(s, s_0; \varepsilon) = H(s)$$

The spectral projection of H^0 corresponding to σ_j^0 we call $P_j^0(s, s_0; \varepsilon)$. Actually H^0 and P_j^0 do not depend on s_0 and ε. We now define following (2.7)

$$K^0(s, s_0; \varepsilon) = -i\sum_{j=1}^{n} P_j^0(s, s_0; \varepsilon)\partial_s P_j^0(s, s_0; \varepsilon) \tag{A.1}$$

K^0 is $m_0 - 1$ times differentiable and also actually does not depend on s_0 and ε. We then define (following (2.9)) $M^0(s, s_0; \varepsilon)$ by

$$i\partial_s M^0(s, s_0; \varepsilon) = K^0(s, s_0; \varepsilon)M^0(s, s_0; \varepsilon); \qquad M^0(s_0, s_0; \varepsilon) = \mathbb{1} \tag{A.2}$$

M^0 actually does not depend on ε. As a transformation function it has the following crucial property (see (2.10)):

$$P_j^0(s, s_0; \varepsilon) = M^0(s, s_0; \varepsilon)P_j^0(s_0, s_0; \varepsilon)M^{0*}(s, s_0; \varepsilon) \tag{A.3}$$

We now define the family of self-adjoint operators

$$H^1(s, s_0; \varepsilon) = M^{0*}(s, s_0; \varepsilon)(H^0(s, s_0; \varepsilon) - \varepsilon K^0(s, s_0; \varepsilon))M^0(s, s_0; \varepsilon)$$

If ε is sufficiently small, more exactly if

$$\varepsilon \sup_s \|K^0(s)\| < d/4,$$

by perturbation theory the spectrum of H^1 still consists for all s of n disconnected pieces $\sigma_j^1(s, s_0; \varepsilon)$, separated by intervals of length $> d/2$. The corresponding spectral projections we call $P_j^1(s, s_0; \varepsilon)$.

We can now repeat the whole construction starting from H^1 instead of H^0. One can continue this process defining $H^2, H^3, \ldots, H^{m_0}$. The reason is that if H^k is constructed for $k < m_0$, then for

$$0 < \varepsilon < \varepsilon_k = d/4\left(\sum_{l=0}^{k-1} \sup_s \|K^l(s, s_0; \varepsilon)\|\right)^{-1} \tag{A.4}$$

its spectrum is still well separated in n pieces $\sigma_j^k(s, s_0; \varepsilon)$ with $\lim_{\varepsilon\to 0} \sigma_j^k(s, s_0; \varepsilon) = \sigma_j^0(s)$ and moreover $(H^k(s, s_0; \varepsilon) + i)^{-1}$ is differentiable for $k \le m_0 - 1$.

Starting with $k = 0$ and ending with $k = m_0 - 1$ we thus have:

$$K^k(s, s_0; \varepsilon) = -i \sum_{j=1}^{n} P_j^k(s, s_0; \varepsilon)\partial_s P_j^k(s, s_0; \varepsilon) \tag{A.5}$$

$$i\partial_s M^k(s, s_0; \varepsilon) = K^k(s, s_0; \varepsilon)M^k(s, s_0; \varepsilon); \qquad M^k(s_0, s_0; \varepsilon) = \mathbb{1} \tag{A.6}$$

$$H^{k+1}(s, s_0; \varepsilon) = M^{k*}(s, s_0; \varepsilon)(H^k(s, s_0; \varepsilon) - \varepsilon K^k(s, s_0; \varepsilon))M^k(s, s_0; \varepsilon) \tag{A.7}$$

From these quantities we define the unitary operators Z_k, Γ_k, U_k^A and Ω_k for $k = 0, 1, \ldots, m_0 - 1$ via:

$$Z_k(s, s_0; \varepsilon) = M^0(s, s_0; \varepsilon)M^1(s, s_0; \varepsilon) \cdots \quad \cdots M^k(s, s_0; \varepsilon) \tag{A.8}$$

$$\left.\begin{aligned} &i\partial_t \Gamma_k(s, s_0; \varepsilon) = M^{k*}(s, s_0; \varepsilon)H^k(s, s_0; \varepsilon)M^k(s, s_0; \varepsilon)\Gamma_k(s, s_0; \varepsilon) \\ &\Gamma_k(s_0, s_0; \varepsilon) = \mathbb{1} \end{aligned}\right. \tag{A.9}$$

$$U_k^A(s, s_0; \varepsilon) = Z_k(s, s_0; \varepsilon)\Gamma_k(s, s_0; \varepsilon) \tag{A.10}$$

$$\Omega_k(s, s_0; \varepsilon) = U_k^{A*}(s, s_0; \varepsilon)U(s, s_0; \varepsilon) \tag{A.11}$$

U_k^A is the k'th adiabatic approximation to the Schrödinger evolution operator U.

Having written out in the appendix all arguments up to now, we will suppress them from now on in the spirit of the remark in section II.

We first show by induction, that

$$i\partial_t(Z_k^* U) = H^{k+1}(Z_k^* U) \tag{A.12}$$

(A.12) is certainly true for $k = 0$, because

$$\begin{aligned} &i\partial_t(Z_0^* U) \\ &\quad = i\partial_t(M^{0*}U) = M^{0*}(H^0 - \varepsilon K^0)U = M^{0*}(H^0 - \varepsilon K^0)M^0 M^{0*} U = H^1 Z_0^* U \end{aligned}$$

We now assume that (A.12) is true for $k = l - 1$ and prove that it is true for $k = l$:

$$\begin{aligned} i\partial_t(Z_l^* U) &= i\partial_t(M^{l*} Z_{l-1}^* U) = i(\partial_t M^{l*})(Z_{l-1}^* U) + M^{l*} H^l Z_{l-1}^* U \\ &= M^{l*}(H^l - \varepsilon K^l)Z_{l-1}^* U = M^{l*}(H^l - \varepsilon K^l)M^l M^{l*} Z_{l-1}^* U = H^{l+1}(Z_l^* U) \end{aligned}$$

From (A.12) it follows that

$$i\partial_t Z_k = (H - Z_k H^{k+1} Z_k^*)Z_k \tag{A.13}$$

We can now prove by induction

Lemma 2

$$\left.\begin{aligned} H_k &= Z_{k-1}H^k Z^*_{k-1} \\ P_{k,j} &= Z_{k-1}P^k_j Z^*_{k-1} \end{aligned}\right\} \tag{A.14}$$

where H_k and $P_{k,j}$ have been defined in (2.1).

Proof. (A.14) is true for $k = 1$, because

$$Z_0 H^1 Z_0^* = M^0 H^1 M^{0*} = M^0 M^{0*}(H^0 - \varepsilon K^0) M^0 M^{0*} = H^0 + i \sum_{j=1}^{n} P_j^0 \partial_t P_j^0 = H_1$$

as defined in (2.1). Note that $H^0 = H = H_0$.

We now assume that (A.14) is true for $k = l$ and prove it for $k = l + 1$:

$$\begin{aligned} Z_l H^{l+1} Z_l^* &= Z_l M^{l*}(H^l - \varepsilon K^l) M^l Z_l^* = Z_{l-1}(H^l - \varepsilon K^l) Z^*_{l-1} \\ &= H_l - \varepsilon Z_{l-1} K^l Z^*_{l-1} = H_l + i\varepsilon Z_{l-1} \sum_{j=1}^{n} P_j^l(\partial_s P_j^l) Z^*_{l-1} \\ &= H_l + i\varepsilon \sum_{j=1}^{n} P_{l,j} Z_{l-1}(\partial_s P_j^l) Z^*_{l-1} \\ &= H_l + i\varepsilon \sum_{j=1}^{n} P_{l,j}\{\partial_s P_{l,j} - (\partial_s Z_{l-1}) P_j^l Z^*_{l-1} - Z_{l-1} P_j^l \partial_s Z^*_{l-1}\} \\ &= H_l + \sum_{j=1}^{n} P_{l,j}\{i\varepsilon \partial_s P_{l,j} - (H - Z_{l-1} H^l Z^*_{l-1}) Z_{l-1} P_j^l Z^*_{l-1} \\ &\quad + Z_{l-1} P_j^l Z^*_{l-1}(H - Z_{l-1} H^l Z^*_{l-1})\} \\ &= H_l + \sum_{j=1}^{n} P_{l,j}\{i\varepsilon \partial_s P_{l,j} - (H - H_l) P_{l,j} + P_{l,j}(H - H_l)\} \\ &= H_l + \sum_{j=1}^{n} P_{l,j}\{i\varepsilon \partial_s P_{l,j} + [P_{l,j}, H]\} = H_{l+1} \end{aligned}$$

In the calculation we have used (A.7), (A.8), (A.5) and (A.13).

With (A.14) we can write (A.13) in the form

$$i\partial_t Z_k = (H - H_{k+1}) Z_k \tag{A.15}$$

We now prove (2.3)

$$i\partial_t U_k^A = (H_0 - H_{k+1} + H_k) U_k^A \tag{A.16}$$

To this end we calculate

$$\begin{aligned} i\partial_t U_k^A &= i\partial_t Z_k \Gamma_k = i(\partial_t Z_k)\Gamma_k + Z_k i\partial_t \Gamma_k \\ &= (H - H_{k+1}) Z_k \Gamma_k + Z_k M^{k*} H^k M^k \Gamma_k \\ &= (H - H_{k+1}) Z_k \Gamma_k + Z_{k-1} H^k M^k Z_k^* Z_k \Gamma_k \\ &= (H - H_{k+1} + H_k) Z_k \Gamma_k = (H - H_{k+1} + H_k) U_k^A \end{aligned}$$

where we have used (A.13), (A.9) and (A.14).

Lemma 3

$$i\partial_t\Omega_k = -\varepsilon\Gamma_k^* M^{k*} K^k M^k \Gamma_k \Omega_k \tag{A.17}$$

To prove this we calculate

$$i\partial_t\Omega_k = i\partial_t\Gamma_k^* Z_k^* U = -\Gamma_k^* M^{k*} H^k M^k Z_k^* U + \Gamma_k^* H^{k+1} Z_k^* U$$

where use has been made by (A.9) *and* (A.12). Using (A.10), (A.11) *and* (A.7) *we get*

$$\begin{aligned} i\partial_t\Omega_k &= (-\Gamma_k^* M^{k*} H^k M^k \Gamma_k + \Gamma_k^* H^{k+1}\Gamma_k)\Omega_k \\ &= \Gamma_k^*(-M^{k*}H^k M^k + H^{k+1})\Gamma_k\Omega_k = -\varepsilon\Gamma_k^* M^{k*} K^k M^k \Gamma_k \Omega_k. \end{aligned}$$

We can now calculate, using (A.14), (A.7) and (A.8)

$$\begin{aligned} H_{k+1} - H_k &= Z_k H^{k+1} Z_k^* - Z_{k-1} H^k Z_{k-1}^* \\ &= Z_k M^{k*}(H^k - \varepsilon K^k) M^k Z_k^* - Z_{k-1} H^k Z_{k-1}^* \\ &= Z_{k-1}(H^k - \varepsilon K^k) Z_{k-1}^* - Z_{k-1} H^k Z_{k-1}^* = -\varepsilon Z_{k-1} K^k Z_{k-1}^* \end{aligned} \tag{A.18}$$

and, using (A.18) together with (A.14), (A.5) and $P(\partial_s P)P \equiv 0$

$$\begin{aligned} P_{k-1,j}(H_k - H_{k-1})P_{k-1,j} &= -\varepsilon P_{k-1,j} Z_{k-2} K^{k-1} Z_{k-2}^* P_{k-1,j} \\ &= -\varepsilon Z_{k-2} P_j^{k-1} K^{k-1} P_j^{k-1} Z_{k-2}^* = i\varepsilon Z_{k-2} P_j^{k-1}(\partial_s P_j^{k-1}) P_j^{k-1} Z_{k-2}^* = 0 \end{aligned} \tag{A.19}$$

In the following we have to write the arguments s_0 and s explicitly, but we *still suppress* ε, *which is kept fixed.*

We can now calculate, using (2.10) for $A = M^k$

$$\begin{aligned} M^{k*}(s, s_0)H^k(s, s_0)M^k(s, s_0)P_j^k(s_0, s_0) &= M^{k*}(s, s_0)H^k(s, s_0)P_j^k(s, s_0)M^k(s, s_0) \\ &= M^{k*}(s, s_0)P_j^k(s, s_0)H^k(s, s_0)M^k(s, s_0) \\ &= P_j^k(s_0, s_0)M^{k*}(s, s_0)H^k(s, s_0)M^k(s, s_0) \end{aligned}$$

so that

$$[M^{k*}(s, s_0)H^k(s, s_0)M^k(s, s_0),\ P_j^k(s_0, s_0)] = 0 \tag{A.20}$$

Now, according to (A.9), $M^{k*}H^kM^k$ is the generator for $\Gamma_k(s, s_0)$. Since $P_j^k(s_0, s_0)$ commutes with the generator, it also commutes with $\Gamma_k(s, s_0)$:

$$[\Gamma_k(s, s_0),\ P_j^k(s_0, s_0)] = 0 \tag{A.21}$$

We now prove (2.4). To this end we calculate

$$\begin{aligned} U_k^A(t, t_0)P_j^k(s_0, s_0) &= Z_k(s, s_0)\Gamma_k(s, s_0)P_j^k(s_0, s_0) \\ &= Z_k(s, s_0)P_j^k(s_0, s_0)\Gamma_k(s, s_0) = Z_{k-1}(s, s_0)M_k(s, s_0)P_j^k(s_0, s_0)\Gamma_k(s, s_0) \\ &= Z_{k-1}(s, s_0)P_j^k(s, s_0)M^k(s, s_0)\Gamma_k(s, s_0) \\ &= Z_{k-1}(s, s_0)P_j^k(s, s_0)Z_{k-1}^*(s, s_0)Z_{k-1}(s, s_0)M^k(s, s_0)\Gamma_k(s, s_0) \\ &= P_{k,j}(s)Z_k(s, s_0)\Gamma_k(s, s_0) = P_{k,j}(s)U_k^A(t, t_0). \end{aligned}$$

Now $P_j^k(s_0, s_0) = P_{k,j}(s_0)$, so that (2.4) follows at once.

The following is the main (and only) estimate in the proof of the adiabatic theorem.

Lemma 4. *For each* $j = 1, 2, \ldots, n-1$ *let* C_j *be a contour (of finite length) surrounding* σ_j^0 *such that*

$$\mathrm{dist}(C_l, \sigma_j) \geq d/2 \text{ for all } l, j$$

Then there exist

$$0 < b_{m,k}(s) < \infty, \quad k = 0, 1, \ldots, m_0; \qquad m = 1, \ldots, m_0 - k$$

and

$$0 \leq d_{m,k}(s) < \infty, \qquad k = 0, 1, \ldots, m_0; \qquad m = 0, 1, \ldots, m_0 - k$$

such that

$$\sup_s b_{m,k}(s) \leq B_{m,k}; \qquad \int_{-\infty}^{+\infty} b_{m,k}(s)\, ds \leq B_{m,k} < \infty \tag{A.22}$$

$$\sup_s d_{m,k}(s) \leq D_{m,k} < \infty \tag{A.23}$$

and for $0 < \varepsilon < \varepsilon_k$ *as defined in* (A.4)

$$\max_{j=1,\ldots,n} \|\partial_s^m P_j^k(s, s_0; \varepsilon)\| \leq b_{m,k}(s)\varepsilon^k \tag{A.24}$$

$$\sup \|\partial_s^m (H^k(s, s_0; \varepsilon) - z)^{-1}\| \leq d_{m,k}(s) \tag{A.25}$$

where the supremum is taken with respect to all $z \in \bigcup_{j=1}^{n-1} C_j$.

The proof of the lemma is by induction over k. For $k = 0$ (A.25) is almost evident, using repeatedly (1.4) and the Leibniz rule. To verify (A.24) for $j = 1, \ldots, n-1$ one has to use (1.10) and take into account that the differentiation with respect to s and the integral over z can be interchanged. For P_n^0 one has to use (1.11). The integrability of $b_{m,k}(s)$ comes from the fact that all terms appearing contain at least one derivative of $f(s)$ as a factor.

Assume now that (A.22–A.25) are true for $0, 1, \ldots, k-1$. From

$$(A + B - z)^{-1} \equiv (A - z)^{-1}(1 + B(A - z)^{-1})^{-1} \tag{A.26}$$

and (A.7) one gets:

$$(H^k - z)^{-1} = M^{k-1}(H^{k-1} - z)^{-1}(1 - \varepsilon K^{k-1}(H^{k-1} - z)^{-1})^{-1}M^{k-1}$$

From this, taking the derivatives and using (A.5) as well as the induction hypothesis (A.25) follows.

Now, using (A.26) and (1.10) we have (suppressing the dependence on ε in the arguments of the operators again!):

$$P_j^k(s, s_0) - P_j^{k-1}(s_0, s_0)$$

$$= \frac{\varepsilon}{2\pi i} M^{k-1*}(s, s_0) \oint_{C_j} dz (H^{k-1}(s, s_0) - z)^{-1} \{1 - \varepsilon K^{k-1}(s, s_0) \times$$

$$\times (H^{k-1}(s, s_0) - z)^{-1}\}^{-1} K^{k-1}(s, s_0)(H^{k-1}(s, s_0) - z)^{-1} M^{k-1}(s, s_0) \quad \text{(A.27)}$$

Differentiating (A.27) with respect to s one finds that all terms contain either K^{k-1} or its derivatives and therefore by (A.5) the derivatives of P_j^{k-1}. From this one obtains (A.24) by the induction hypothesis.

As a consequence of Lemma 4 and (A.5) we have the following inequalities:

$$\|K^k(s, s_0)\| \leq \varepsilon^k g_k(s) \text{ with } \sup_s g_k(s) \leq G_k \text{ and } \int_{-\infty}^{+\infty} g_k(s)\, ds \leq G_k < \infty \quad \text{(A.28)}$$

To get an estimate for $\|\Omega_k - \mathbb{1}\|$ we can convert (A.17) into the equivalent integral equation and use the unitarity of Γ_k, M^k and Ω_k:

$$\|\Omega_k(s, s_0) - \mathbb{1}\| \leq \int_{s_0}^{s} \|K^k(s', s_0)\|\, ds' \leq \varepsilon^k G_k \quad \text{(A.29)}$$

The last thing we have to do is to improve (A.29) so that (2.5) holds. We assume that Ω^{k+1} exists, i.e. that $k \leq m_0 - 2$. Then

$$\|\Omega_k - \mathbb{1}\| \leq \|\Omega_{k+1} - \mathbb{1}\| + \|\Omega_{k+1} - \Omega_k\| \leq \|\Omega_{k+1} - \mathbb{1}\| + \|\Omega_{k+1}\Omega_k^* - \mathbb{1}\| \quad \text{(A.30)}$$

We will now estimate the last term in (A.30) using (A.11).

$$\|\Omega_{k+1}\Omega_k^* - \mathbb{1}\| \leq \left\| \sum_{j,l=1}^{n} P_{k,l}(s_0)(U_{k+1}^A U_k^{A*} - \mathbb{1}) P_{k,j}(s_0) \right\| \quad \text{(A.31)}$$

With (2.4) we get for the off diagonal terms $j \neq l$:

$$\|P_{k,l}(s_0) U_{k+1}^A(t, t_0) U_k^{A*}(t, t_0) P_{k,j}(s_0)\|$$
$$\leq \|P_{k,l}(s_0) - P_{k+1,l}(s_0)\|$$
$$+ \|U_{k+1}^A(t, t_0) P_{k+1,l}(s) P_{k,j}(s) U_k^{A*}(t, t_0)\|$$
$$\leq \|P_{k,l}(s_0) - P_{k+1,l}(s_0)\| + \|P_{k+1,l}(s) - P_{k,l}(s)\| \quad \text{(A.32)}$$

Now from (A.18) and (A.28)

$$\|H_{k+1}(s) - H_k(s)\| \leq \varepsilon^{k+1} g_k(s) \quad \text{(A.33)}$$

and then by perturbation theory [13]

$$\max_j \sup_s \|P_{k+1,j}(s) - P_{k,j}(s)\| \leq \text{const}\ \varepsilon^{k+1} G_k \quad \text{(A.34)}$$

We now treat the diagonal part. To this end we use (2.3) to derive the equation of motion for $U^A_{k+1}U^{A*}_k$:

$$i\partial_t(U^A_{k+1}U^{A*}_k)) = U^A_{k+1}(H_{k+2} - H_{k+1} + H_k - H_{k+1})U^{A*}_k \tag{A.35}$$

and to write it in its integral form

$$U^A_{k+1}U^{A*}_k - \mathbb{1} = -i\int_{t_0}^{t} U^A_{k+1}(H_{k+2} - H_{k+1} + H_k - H_{k+1})U^{A*}_k\, dt'$$

This gives the estimate

$$\|P_{k,j}(s_0)(U^A_{k+1}U^{A*}_k - \mathbb{1})P_{k,j}(s_0)\|$$
$$\leq \int_{t_0}^{t} \|P_{k,j}(s_0)U^A_{k+1}(H_{k+2} - H_{k+1} + H_k - H_{k+1})U^{A*}_k P_{k,j}(s_0)\|\, dt',$$

Now, using again (A.33), (A.34), (2.4) and (A.19) we get for the integrand

$$\begin{aligned}
&\|P_{k,j}(s_0)U^A_{k+1}(H_{k+2} - H_{k+1} + (H_k - H_{k+1}))U^{A*}_k P_{k,j}(s_0)\| \\
&\quad\leq \|P_{k,j}(s_0) - P_{k+1,j}(s_0)\|(\|H_{k+2} - H_{k+1}\| + \|H_k - H_{k+1}\|) \\
&\qquad + \|U^A_{k+1}P_{k+1,j}(s')(H_{k+2} - H_{k+1} + (H_k - H_{k+1}))P_{k,j}(s')U^{A*}_k\| \\
&\quad\leq \text{const}\, g_k(s')\varepsilon^{2k+2} + \|P_{k,j}(s')(H_{k+2} - H_{k+1})P_{k,j}(s')\| \\
&\quad\leq \text{const}\, \varepsilon^{k+2} \max\{g_k(s'), g_{k+1}(s')\} = \varepsilon^{k+2} b_k(s')
\end{aligned} \tag{A.36}$$

so that

$$\|P_{k,j}(s_0)(U^A_{k+1}U^{A*}_k - \mathbb{1})P_{k,j}(s_0)\| \leq \varepsilon^{k+2}\int_{t_0}^{t} b_k(\varepsilon t')\, dt' \leq \varepsilon^{k+1}\int_{-\infty}^{+\infty} b_k(s')\, ds' \tag{A.37}$$

(A.32), (A.34) and (A.37) show, that the norm of each term in the sum of (A.31) is of order ε^{k+1}, uniformly in t and t_0. This completes the proof.

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