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ON THE BOUND STATES OF RELATIVISTIC KRÖNIG-PENNEY HAMILTONIANS WITH SHORT RANGE IMPURITIES

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Abstract

In this work we investigate the spectrum of the relativistic counterpart of the Krönig-Penney Hamiltonian perturbed by a non-positive potential W whose $1+\delta$-moment is finite for some $\delta > 0$.

It will be shown that, differently from the non-relativistic case, in the relativistic case one may find more than one bound state in each remote gap under the same assumptions on W. Nevertheless, it will be shown that such additional bound states cannot appear in the range of energies of solid state physics.

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1) Introduction.

In this paper we investigate the bound states of the relativistic counterpart of the Schrödinger operator considered in (1).

The relativistic analogue of the Krönig-Penney model has been investigated in (2) and the main results of that analysis have been summarised in (3) Appendix J.

Here we consider the relativistic Hamiltonian of (2) perturbed by a non-positive potential W satisfying the same assumption as in (1), i.e.

$$\int_{\mathbb{R}} (1+|x|)^{1+\delta}|W(x)|\,dx < \infty \tag{1.1}$$

The most important difference between the relativistic case and the non-relativistic one is that in the former there may be cases in which the Hamiltonian has more than only one bound state in each remote gap of both the positive and negative part of its essential spectrum. Nevertheless, we shall be able to provide a condition because of which it will follow that if the periodic potential and the impurity are in the range of energies of solid-state physics there cannot be more than one bound state in each sufficiently remote gap of the essential spectrum.

Let us now give a brief description of the contents of this paper.

In section 2 we compute the Green's function for the Hamiltonian $\tilde{H}_\alpha$, the relativistic analogue of the Krönig-Penney Hamiltonian considered in (1).

In section 3 the method used in (1) will be applied in order to estimate the number of bound states of $\tilde{H}_\alpha - \lambda W\,(\lambda > 0, W \geq 0)$ in the remote gaps of its essential spectrum. As anticipated, it will be seen that one may find more than one bound state in each remote gap. The nature of the possible additional bound states is different from that of the bound state arising from the one-dimensional divergence of the Birman-Schwinger kernel at both endpoints of each spectral gap since the additional bound states disappear as bound states in the non-relativistic limit ($c \to \infty$) while that particular eigenvalue converges to the non-relativistic bound state as $c \to 0$. Therefore such additional bound states are to be regarded as Dirac localised states according to the terminology introduced in papers such as (4) and (5) while the other eigenvalue is said to be a relativistic bound state.

In relation to the papers just mentioned, it will be shown that in the particular case of an interstitial impurity given by a δ-function of strength λ situated at the midpoint between two adjacent lattice sites there is a Dirac localised state in each gap where the discriminant is greater than two when both the periodic potential and the impurity are attractive.

2) The Green's function of $\tilde{H}_\alpha = -ic\frac{d}{dx} \otimes \sigma_1 + \frac{1}{2}c^2 \otimes \sigma_3 + \alpha \sum_{m\in\mathbb{Z}} \delta(\cdot - (2m+1)\pi)$ inside the gaps of its essential spectrum.

In this section we will determine the relativistic equivalent of the Green's function of (1).

First of all, let us recall that the free Dirac Hamiltonian in one dimension is given by

$$\tilde{H}_0 = -ic\frac{d}{dx} \otimes \sigma_1 + \frac{1}{2}c^2 \otimes \sigma_3 =$$

$$\begin{bmatrix} \frac{c^2}{2} & -ic\frac{d}{dx} \\ -ic\frac{d}{dx} & -\frac{c^2}{2} \end{bmatrix} \tag{2.1}$$

with $\mathcal{D}(\tilde{H}_0) = H^{2,1}(\mathbb{R}) \otimes \mathbb{C}^2 \subset L^2(\mathbb{R}) \otimes \mathbb{C}^2$. σ_1 and σ_3 appearing in (2.1) are the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.2}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3}$$

Following (3), the operator $\tilde{H}_\alpha = \tilde{H}_0 + \alpha \sum_{m\in\mathbb{Z}} \delta(\cdot - (2m+1)\pi)$ is determined as the self-adjoint operator given by

$$\tilde{H}_\alpha = \tilde{H}_0$$

$$\begin{cases} \mathcal{D}(\tilde{H}_\alpha) = \{f \in H^{2,1}(\mathbb{R} - (2\mathbb{Z}+1)\pi \otimes \mathbb{C}^2 | f_1 \in AC_{loc}(\mathbb{R}), f_2 \in AC_{loc}(\mathbb{R} - (2\mathbb{Z}+1)\pi); \\ f_2((2m+1)\pi_+) - f_2((2m+1)\pi_-) = -\frac{i\alpha}{c} f_1((2m+1)\pi)\} \end{cases} \tag{2.4}$$

A complete description of the spectrum of $\tilde{H}_\alpha$ can be found in (2) and (3).

As it has been done in the second section of Ref.(1), we shall carry out all the calculations leading to the Green's function for E belonging to a gap where the discriminant is greater than 2 since it will be clear how to proceed for the gaps of the other type.

In this case the equation corresponding to (2.2) in (1) is given by

$$\begin{cases} \left[-ic\frac{d}{dx} \otimes \sigma_1 + \frac{1}{2}c^2 \otimes \sigma_3\right] \phi = E\phi \\ \phi(2\pi) = e^{\pm\theta}\phi(0) \\ \phi_2(\pi_+) - \phi_2(\pi_-) = -\frac{i\alpha}{c}\phi_1(\pi) \end{cases} \tag{2.5}$$

with

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

in all the gaps where $D(E) > 2$ (see (2), (3) and also (6) for the general theory of periodic Dirac equations), while in the gaps with $D(E) < -2$ the only modification is a minus sign in front of the exponential in the boundary condition in (2.5).

The first equation in (2.5) leads to the following system of first-order differential equations

$$\begin{cases} -ic\frac{d}{dx}\phi_2 = \left(E - \frac{c^2}{2}\right)\phi_1 \\ -ic\frac{d}{dx}\phi_1 = \left(E + \frac{c^2}{2}\right)\phi_2 \end{cases} \tag{2.6}$$

where $|E| > \frac{c^2}{2}$.

Although some long calculations are required, it is straightforward to obtain the solution of (2.4), namely

$$\phi^{\pm} = \begin{pmatrix} \phi_1^{\pm} \\ \phi_2^{\pm} \end{pmatrix}$$

with

$$\begin{cases} \phi_1^{\pm}(x) = e^{\pm m\theta}\left[\cos k(E)(x-2m\pi) + \frac{e^{\pm\theta}-1}{e^{\pm\theta}+1}\cot k(E)\pi \,\sin k(E)(x-2m\pi)\right] \\ \phi_2^{\pm}(x) = i(\gamma(E))^{-1}\, e^{\pm\theta}\left[\sin k(E)(x-2m\pi) - \frac{e^{\pm\theta}-1}{e^{\pm\theta}+1}\cot k(E)\pi \,\cos k(E)(x-2m\pi)\right] \end{cases} \tag{2.7}$$

$\forall x \in [(2m-1)\pi,(2m+1)\pi), \forall m \in \mathbb{Z}$ with $k(E) = \frac{1}{c}\sqrt{E^2 - \frac{c^4}{4}}$ and $\gamma(E) = \frac{E+\frac{c^2}{2}}{ck(E)}$ and θ defined by means of the relativistic counterpart of the Krönig-Penney relation, i.e.

$$\cosh\theta = \cos 2\pi k(E) + \frac{\alpha}{2c}\gamma(E)\sin 2\pi k(E) \tag{2.8}$$

since in the gaps we are considering the right-hand side of (2.8) is a half of the discriminant.

By using the formula

$$\mathcal{W}(\phi^+,\phi^-) = \phi_1^+(x)\phi_2^-(x) - \phi_2^+(x)\phi_1^-(x) \tag{2.9}$$

we obtain the Wronskian for our system, namely

$$\mathcal{W}(E) = 2i(\gamma(E))^{-1}\beta(\theta)\cot k(E)\pi \tag{2.10}$$

with $\beta(\theta)$ defined as in (1).

The Green's function can be obtained by means of the analogue of (2.14) in (1), i.e.

$$\tilde{G}_\alpha(x, y; E) = \frac{1}{-icW(E)} \begin{cases} \phi^+(x)[\phi^-]^*(y), & x < y \\ \phi^-(x)[\phi^+]^*(y) & x > y \end{cases} \tag{2.11}$$

where

$$[\phi^\pm]^* = \begin{pmatrix} \phi_1^\pm & \overline{\phi_2^\pm} \end{pmatrix}$$

for any E in the gaps where $D(E) > 2$.

Accordingly, the entries of the 2×2−matrix expressing the Green's function are given by

$$\begin{gathered}\left[(\tilde{H}_\alpha - E)^{-1}\right]_{11}(x, y) = \frac{\gamma(E)}{2c} e^{-|m-n|\theta} \times \\ \times \Big[(\beta(\theta))^{-1} \tan k(E)\pi \, \cos k(E)(x - 2m\pi) \cos k(E)(y - 2n\pi) + \\ -\beta(\theta) \cot k(E)\pi \, \sin k(E)(x - 2m\pi) \sin k(E)(y - 2n\pi) + \\ - \sin k(E)(|x - y| - 2|m - n|\pi)\Big]\end{gathered} \tag{2.12}$$

$$\begin{gathered}\left[(\tilde{H}_\alpha - E)^{-1}\right]_{12}(x, y) = -\frac{i}{2c} e^{-|m-n|\theta} \times \\ \times \Big[(\beta(\theta))^{-1} \tan k(E)\pi \, \cos k(E)(x - 2m\pi) \sin k(E)(y - 2n\pi) + \\ +\beta(\theta) \cot k(E)\pi \sin k(E)(x - 2m\pi) \cos k(E)(y - 2n\pi) + \\ -\operatorname{sgn}(x - y) \cos k(E)(|x - y| - 2|m - n|\pi)\Big]\end{gathered} \tag{2.13}$$

$$\left[(\tilde{H}_\alpha - E)^{-1}\right]_{21}(x,y) = \frac{i}{2c}e^{-|m-n|\theta}\times$$

$$\times\Big[(\beta(\theta))^{-1}\tan k(E)\pi\ \sin k(E)(x-2m\pi)\cos k(E)(y-2n\pi)+$$

$$+\beta(\theta)\cot k(E)\pi\ \cos k(E)(x-2m\pi)\sin k(E)(y-2n\pi)+$$

$$+\operatorname{sgn}(x-y)\cos k(E)(|x-y|-2|m-n|\pi)\Big] \tag{2.14}$$

$$\left[(\tilde{H}_\alpha - E)^{-1}\right]_{22}(x,y) = \frac{1}{2c\gamma(E)}e^{-|m-n|\theta}\times$$

$$\times\Big[(\beta(\theta))^{-1}\tan k(E)\pi\ \sin k(E)(x-2m\pi)\sin k(E)(y-2n\pi)+$$

$$-\beta(\theta)\cot k(E)\pi\cos k(E)(x-2m\pi)\cos k(E)(y-2n\pi)+$$

$$-\sin k(E)(|x-y|-2|m-n|\pi)\Big] \tag{2.15}$$

for any $x \in I_m, y \in I_n, m, n \in \mathbb{Z}$, where $I_m, m \in \mathbb{Z}$ are the same intervals appearing in (1).

Finally, it is not difficult to check that

$$\|\ \| - \lim_{c\to\infty}(\tilde{H}_\alpha - \frac{c^2}{2} - E)^{-1} = (H_\alpha - E)^{-1}\otimes\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \tag{2.16}$$

H_α being the Schrödinger Hamiltonian of section 1, in agreement with (7).

5) Bound states of $\tilde{H}_\alpha + \lambda W$ in the gaps of its essential spectrum.

In this section we will investigate the bound states of the relativistic analogue of the Schrödinger Hamiltonian considered in Ref.(1).

We shall carry out all the proofs assuming $\alpha < 0$ since everything can easily be repeated for the case with $\alpha > 0$. We should like to remind that $\tilde{H}_\alpha, \alpha < 0$ is the Dirac Hamiltonian for a negatively charged one-dimensional particle of spin $\frac{1}{2}$ and mass $m = \frac{1}{2}$ in a Krönig-Penney crystal of positive ions.

In the following proofs we shall only consider the gaps characterised by having $\tilde{F}_\alpha(E)$, the right-hand side of (2.8), greater than 1 but it is clear from what has been seen in the previous section that the same results can be obtained for the bound states in the gaps of the other type by using the similar expression for the Green's function in those gaps.

In the following we shall consider the case of a perturbing potential W that is non-positive corresponding to the case of an attractive impurity, i.e. $W \leq 0$, so that we shall look for those E's in the gaps for which there exists a nonzero $\phi \in L^2(\mathbb{R}) \otimes \mathbb{C}^2$ satisfying

$$\lambda |W|^{\frac{1}{2}} (\tilde{H}_\alpha - E)^{-1} |W|^{\frac{1}{2}} \phi = \phi \tag{3.1}$$

Following (3), the gaps we are going to consider are given by

$$\begin{cases} \left(E_{2N}^{\alpha}(0), \frac{c}{2}\left[(2N)^2 + c^2\right]^{\frac{1}{2}} \right) & E > \frac{c^2}{2} \\ \left(E_{-(2N+1)}^{\alpha}(0), -\frac{c}{2}\left[(2N)^2 + c^2\right]^{\frac{1}{2}} \right) & E < \frac{c^2}{2} \end{cases} \tag{3.2}$$

It follows from the results of the previous section that for any E belonging to a gap among those in (3.2) the Birman-Schwinger kernel can be written as follows

$$|W|^{\frac{1}{2}} \left(\tilde{H}_\alpha - E \right)^{-1} |W|^{\frac{1}{2}} = \sum_{l=1}^{5} \tilde{T}_\alpha^{(l)}(E) \tag{3.3}$$

where $\tilde{T}_\alpha^{(l)}(E)$, $l = 1, 2$ are the rank-one operators given by

$$\tilde{T}_\alpha^{(1)}(E) = \frac{(\beta(\theta))^{-1} \tan k(E)\pi}{2c\gamma(E)} |W|^{\frac{1}{2}} \tilde{\phi}_\alpha^{(1)}(E) \left(|W|^{\frac{1}{2}} \tilde{\phi}_\alpha^{(1)}(E), \cdot \right) \tag{3.4}$$

$$\tilde{T}_\alpha^{(2)}(E) = -\frac{\beta(\theta) \cot k(E)\pi}{2c\gamma(E)} |W|^{\frac{1}{2}} \tilde{\phi}_\alpha^{(2)}(E) \left(|W|^{\frac{1}{2}} \tilde{\phi}_\alpha^{(2)}(E), \cdot \right) \tag{3.5}$$

with

$$\tilde{\phi}_{\alpha}^{(1)}(x;E) = \begin{pmatrix} \gamma(E)\, e^{-|m|\theta} \cos k(E)(x-2m\pi) \\ i\, e^{-|m|\theta} \sin k(E)(x-2m\pi) \end{pmatrix} \tag{3.6}$$

and

$$\tilde{\phi}_{\alpha}^{(2)}(x;E) = \begin{pmatrix} \gamma(E)\, e^{-|m|\theta} \sin k(E)(x-2m\pi) \\ -i\, e^{-|m|\theta} \cos k(E)(x-2m\pi) \end{pmatrix} \tag{3.7}$$

for any $x \in I_m$, $\forall m \in \mathbb{Z}$, $\tilde{\phi}_{\alpha}^{(l)}(E) \in \left(L^2(\mathbb{R}) \cap L^{\infty}(\mathbb{R})\right) \otimes \mathbb{C}^2$, $l = 1,2$ and $\tilde{T}_{\alpha}^{(l)}(E)$, $l = 3,4,5$ are the operators on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ whose integral kernels are given by

$$\tilde{T}_{\alpha}^{(3)}(x,y;E) = \begin{pmatrix} \left[\tilde{T}_{\alpha}^{(3)}\right]_{11}(x,y;E) & \left[\tilde{T}_{\alpha}^{(3)}\right]_{12}(x,y;E) \\ \left[\tilde{T}_{\alpha}^{(3)}\right]_{21}(x,y;E) & \left[\tilde{T}_{\alpha}^{(3)}\right]_{22}(x,y;E) \end{pmatrix} \tag{3.8}$$

with

$$\left[\tilde{T}_{\alpha}^{(3)}\right]_{11}(x,y;E) = \frac{\gamma(E)\,(\beta(\theta))^{-1} \tan k(E)\pi}{2c} \cdot$$

$$\cdot |W(x)|^{\frac{1}{2}} \cos k(E)(x-2m\pi) \left[e^{-|m-n|\theta} - e^{-(|m|+|n|)\theta}\right] |W(y)|^{\frac{1}{2}} \cos k(E)(y-2n\pi) \tag{3.9}$$

$$\left[\tilde{T}_{\alpha}^{(3)}\right]_{12}(x,y;E) = -i\frac{(\beta(\theta))^{-1} \tan k(E)\pi}{2c} \cdot$$

$$\cdot |W(x)|^{\frac{1}{2}} \cos k(E)(x-2m\pi) \left[e^{-|m-n|\theta} - e^{-(|m|+|n|)\theta}\right] |W(y)|^{\frac{1}{2}} \sin k(E)(y-2n\pi) \tag{3.10}$$

$$\left[\tilde{T}_{\alpha}^{(3)}\right]_{21}(x,y;E) = i\frac{(\beta(\theta))^{-1} \tan k(E)\pi}{2c} \cdot$$

$$\cdot |W(x)|^{\frac{1}{2}} \sin k(E)(x-2m\pi) \left[e^{-|m-n|\theta} - e^{-(|m|+|n|)\theta}\right] |W(y)|^{\frac{1}{2}} \cos k(E)(y-2n\pi) \tag{3.11}$$

$$\left[\tilde{T}_{\alpha}^{(3)}\right]_{22}(x,y;E) = \frac{(\beta(\theta))^{-1} \tan k(E)\pi}{2c\gamma(E)} \cdot$$

$$\cdot |W(x)|^{\frac{1}{2}} \sin k(E)(x-2m\pi) \left[e^{-|m-n|\theta} - e^{-(|m|+|n|)\theta}\right] |W(y)|^{\frac{1}{2}} \sin k(E)(y-2n\pi) \tag{3.12}$$

$\forall x \in I_m, y \in I_n$, $\forall m,n \in \mathbb{Z}$,

$$\tilde{T}_\alpha^{(4)}(x,y;E)=\begin{pmatrix}\left[\tilde{T}_\alpha^{(4)}\right]_{11}(x,y;E) & \left[\tilde{T}_\alpha^{(4)}\right]_{12}(x,y;E)\\ \left[\tilde{T}_\alpha^{(4)}\right]_{21}(x,y;E) & \left[\tilde{T}_\alpha^{(4)}\right]_{22}(x,y;E)\end{pmatrix} \tag{3.13}$$

with

$$\left[\tilde{T}_\alpha^{(4)}\right]_{11}(x,y;E)=-\frac{\gamma(E)\beta(\theta)\cot k(E)\pi}{2c}\cdot$$
$$\cdot|W(x)|^{\frac{1}{2}}\sin k(E)(x-2m\pi)\left[e^{-|m-n|\theta}-e^{-(|m|+|n|)\theta}\right]|W(y)|^{\frac{1}{2}}\sin k(E)(y-2n\pi) \tag{3.14}$$

$$\left[\tilde{T}_\alpha^{(4)}\right]_{12}(x,y;E)=-i\frac{\beta(\theta)\cot k(E)\pi}{2c}\cdot$$
$$\cdot|W(x)|^{\frac{1}{2}}\sin k(E)(x-2m\pi)\left[e^{-|m-n|\theta}-e^{-(|m|+|n|)\theta}\right]|W(y)|^{\frac{1}{2}}\cos k(E)(y-2n\pi) \tag{3.15}$$

$$\left[\tilde{T}_\alpha^{(4)}\right]_{21}(x,y;E)=i\frac{\beta(\theta)\cot k(E)\pi}{2c}\cdot$$
$$\cdot|W(x)|^{\frac{1}{2}}\cos k(E)(x-2m\pi)\left[e^{-|m-n|\theta}-e^{-(|m|+|n|)\theta}\right]|W(y)|^{\frac{1}{2}}\sin k(E)(y-2n\pi) \tag{3.16}$$

$$\left[\tilde{T}_\alpha^{(4)}\right]_{22}(x,y;E)=-\frac{\beta(\theta)\cot k(E)\pi}{2c\gamma(E)}\cdot$$
$$\cdot|W(x)|^{\frac{1}{2}}\cos k(E)(x-2m\pi)\left[e^{-|m-n|\theta}-e^{-(|m|+|n|)\theta}\right]|W(y)|^{\frac{1}{2}}\cos k(E)(y-2n\pi) \tag{3.17}$$

$\forall x\in I_m, y\in I_n,\ m,n\in\mathbb{Z}$,

$$\tilde{T}_\alpha^{(5)}(x,y;E)=\begin{pmatrix}\left[\tilde{T}_\alpha^{(5)}\right]_{11}(x,y;E) & \left[\tilde{T}_\alpha^{(5)}\right]_{12}(x,y;E)\\ \left[\tilde{T}_\alpha^{(5)}\right]_{21}(x,y;E) & \left[\tilde{T}_\alpha^{(5)}\right]_{22}(x,y;E)\end{pmatrix} \tag{3.18}$$

with

$$\left[\tilde{T}_\alpha^{(5)}\right]_{11}(x,y;E)=$$

$$= -\frac{\gamma(E)}{2c}|W(x)|^{\frac{1}{2}}\, e^{-|m-n|\theta} \sin k(E)(|x-y| - 2|m-n|\pi)|W(y)|^{\frac{1}{2}} \tag{3.19}$$

$$\left[\tilde{T}_\alpha^{(5)}\right]_{12}(x,y;E) =$$

$$= \frac{i}{2c}|W(x)|^{\frac{1}{2}}\, e^{-|m-n|\theta}\mathrm{sgn}(x-y) \cos k(E)(|x-y| - 2|m-n|\pi)|W(y)|^{\frac{1}{2}} \tag{3.20}$$

$$\left[\tilde{T}_\alpha^{(5)}\right]_{21}(x,y;E) = -\left[\tilde{T}_\alpha^{(5)}\right]_{12}(x,y;E) \tag{3.21}$$

$$\left[\tilde{T}_\alpha^{(5)}\right]_{22}(x,y;E) = \frac{1}{\gamma^2(E)}\left[\tilde{T}_\alpha^{(5)}(E)\right]_{11}(x,y;E) \tag{3.22}$$

$\forall x \in I_m, y \in I_n,\ m,n \in \mathbb{Z}$.

First of all, we notice that the operators $\tilde{T}_\alpha^{(l)}(E)$, $l = 3,4,5$ are self-adjoint trace-class operators on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ since each of them is of the type

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with $A_{11} = A_{11}^*$, $A_{22} = A_{22}^*$, $A_{21} = A_{12}^*$ as trace-class operators on $L^2(\mathbb{R})$ for any E belonging to one of the gaps in (3.2) as follows from the fact that the integral kernels involved are of the type of those appearing in (1).

By using the same technique used in the proof of Theorem 2.1. in (1) we can get a completely analogous result under the assumption (1.1) on W. In order to simplify the notation we shall use the symbol $\mathcal{J}_1$ to denote both $\mathcal{J}_1\left(L^2(\mathbb{R}) \otimes \mathbb{C}^2\right)$, the space to which $\tilde{T}_\alpha^{(l)}(E)$, $l = 3,4,5$ belong, and $\mathcal{J}_1\left(L^2(\mathbb{R})\right)$, the space of $\left[\tilde{T}_\alpha^{(l)}\right]_{ij}(E)$.

LEMMA 3.1. *Let $W \leq 0$ be a function satisfying (1.1).*

Then $\tilde{T}_{\alpha}^{(l)}(E)$, $l = 3,4$, can be made $\| \|_{\mathcal{J}_1}$−continuous functions of E on each closed interval

$$\left[E_{2N}^{\alpha}(0), \frac{c}{2}\left((2N)^2+c^2\right)^{\frac{1}{2}}\right],$$

$$\left[E_{-(2N+1)}^{\alpha}(0), -\frac{c}{2}\left((2N)^2+c^2\right)^{\frac{1}{2}}\right], \forall N \in \mathbb{N}$$

by defining

$$\tilde{T}_{\alpha}^{(3)}(E_{2N}^{\alpha}(0)) = \begin{pmatrix} \left[\tilde{T}_{\alpha}^{(3)}\right]_{11}(E_{2N}^{\alpha}(0)) & \left[\tilde{T}_{\alpha}^{(3)}\right]_{12}(E_{2N}^{\alpha}(0)) \\ \left[\tilde{T}_{\alpha}^{(3)}\right]_{21}(E_{2N}^{\alpha}(0)) & \left[\tilde{T}_{\alpha}^{(3)}\right]_{22}(E_{2N}^{\alpha}(0)) \end{pmatrix} \tag{3.23}$$

with entries defined by means of their integral kernels as follows

$$\left[\tilde{T}_{\alpha}^{(3)}\right]_{11}(x,y;E_{2N}^{\alpha}(0)) = \frac{\gamma(E_{2N}^{\alpha}(0))\tan k(E_{2N}^{\alpha}(0))\pi}{c}\cdot$$

$$|W(x)|^{\frac{1}{2}}\cos k(E_{2N}^{\alpha}(0))(x-2m\pi)\left[|m|+|n|-|m-m|\right]|W(y)|^{\frac{1}{2}}\cos k(E_{2N}^{\alpha}(0))(y-2n\pi) \tag{3.24}$$

$$\left[\tilde{T}_{\alpha}^{(3)}\right]_{12}(x,y;E_{2N}^{\alpha}(0)) = -i\frac{\tan k(E_{2N}^{\alpha}(0))\pi}{c}\cdot$$

$$\cdot|W(x)|^{\frac{1}{2}}\cos k(E_{2N}^{\alpha}(0))(x-2m\pi)\left[|m|+|n|-|m-n|\right]|W(y)|^{\frac{1}{2}}\sin k(E_{2N}^{\alpha}(0))(y-2n\pi) \tag{3.25}$$

$$\left[\tilde{T}_{\alpha}^{(3)}\right]_{21}(x,y;E_{2N}^{\alpha}(0)) = i\frac{\tan k(E_{2N}^{\alpha}(0))\pi}{c}\cdot$$

$$\cdot|W(x)|^{\frac{1}{2}}\sin k(E_{2n}^{\alpha}(0))(x-2m\pi)\left[|m|+|n|-|m-n|\right]|W(y)|^{\frac{1}{2}}\cos k(E_{2N}^{\alpha}(0))(y-2n\pi) \tag{3.26}$$

$$\left[\tilde{T}_{\alpha}^{(3)}\right]_{22}(x,y;E_{2N}^{\alpha}(0)) = \frac{\tan k(E_{2N}^{\alpha}(0))\pi}{c\gamma(E_{2N}^{\alpha}(0))}\cdot$$
$$\cdot|W(x)|^{\frac{1}{2}}\sin k(E_{2N}^{\alpha}(0))(x-2m\pi)\,[|m|+|n|-|m-n|]\,|W(x)|^{\frac{1}{2}}\sin k(E_{2N}^{\alpha}(0))(y-2n\pi) \tag{3.27}$$

$\forall x \in I_m, y \in I_n, \forall m,n \in \mathbb{Z}$,

$$\tilde{T}_{\alpha}^{(3)}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right) = 0 \tag{3.28}$$

and similarly for the intervals on the negative semiaxis, while for $\tilde{T}_{\alpha}^{(4)}(E)$ *we have*

$$\tilde{T}_{\alpha}^{(4)}(E_{2N}^{\alpha}(0)) = 0 \tag{3.29}$$

$$\tilde{T}_{\alpha}^{(4)}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right) =$$
$$\begin{pmatrix} \left[\tilde{T}_{\alpha}^{(4)}\right]_{11}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right) & \left[\tilde{T}_{\alpha}^{(4)}\right]_{12}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right) \\ \left[\tilde{T}_{\alpha}^{(4)}\right]_{21}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right) & \left[\tilde{T}_{\alpha}^{(4)}\right]_{22}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right) \end{pmatrix} \tag{3.30}$$

with entries defined by means of their integral kernels as follows

$$\left[\tilde{T}_{\alpha}^{(4)}\right]_{11}\left(x,y;\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right) = \frac{|\alpha|}{2c^2}\gamma^2\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\cdot$$
$$\cdot|W(x)|^{\frac{1}{2}}\sin Nx\,[|m|+|n|-|m-n|]\,|W(y)|^{\frac{1}{2}}\sin Ny \tag{3.31}$$

$$\left[\tilde{T}_{\alpha}^{(4)}\right]_{12}\left(x,y;\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right) = i\frac{|\alpha|}{2c^2}\gamma\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\cdot$$
$$\cdot|W(x)|^{\frac{1}{2}}\sin Nx\,[|m|+|n|-|m-n|]\,|W(y)|^{\frac{1}{2}}\cos Ny \tag{3.32}$$

$$\left[\tilde{T}_{\alpha}^{(4)}\right]_{21}\left(x,y;\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)=-i\frac{|\alpha|}{2c^2}\gamma\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\cdot$$
$$\cdot|W(x)|^{\frac{1}{2}}\cos Nx\left[|m|+|n|-|m-n|\right]|W(y)|^{\frac{1}{2}}\sin Ny \tag{3.33}$$

$$\left[\tilde{T}_{\alpha}^{(4)}\right]_{22}\left(x,y;\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)=\frac{|\alpha|}{2c^2}\cdot$$
$$\cdot|W(x)|^{\frac{1}{2}}\cos Nx\left[|m|+|n|-|m-n|\right]|W(y)|^{\frac{1}{2}}\cos Ny \tag{3.34}$$

$\forall x\in I_m, y\in I_n$, $\forall m,n\in\mathbb{Z}$ and similarly for the intervals on the negative semiaxis.

$\tilde{T}_{\alpha}^{(5)}(E)$ is $\|\ \|_{\mathcal{J}_1}$-continuous on each closed interval written above.

From (3.4) we see that the nonzero eigenvalue of the negative rank-one operator $\tilde{T}_{\alpha}^{(1)}(E)$ diverges as

$$E\to\begin{cases}E_{2N}^{\alpha}(0)+ & E>\frac{c^2}{2}\\ E_{-(2N+1)}^{\alpha}(0)+ & E<-\frac{c^2}{2}\end{cases} \tag{3.35}$$

since

$$\begin{cases}(\beta(\theta))^{-1}\tan k(E)\pi\to-\infty & \text{as } E\to E_{2N}^{\alpha}(0)+\\ (\beta(\theta))^{-1}\tan k(E)\pi\to 0 & \text{as } E\to E_{2N+1}^{\alpha}(0)-\end{cases} \tag{3.36}$$

$$\begin{cases}(\beta(\theta))^{-1}\tan k(E)\pi\to 0 & \text{as } E\to E_{-2N}^{\alpha}(0)-\\ (\beta(\theta))^{-1}\tan k(E)\pi\to-\infty & \text{as } E\to E_{-(2N+1)}^{\alpha}(0)+\end{cases} \tag{3.37}$$

while the nonzero eigenvalue of the positive rank-one operator $T_{\alpha}^{(2)}(E)$ diverges as

$$E\to\begin{cases}\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}_{-} & E>\frac{c^2}{2}\\ -\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}_{-} & E<-\frac{c^2}{2}\end{cases} \tag{3.38}$$

since

$$\begin{cases}-\beta(\theta)\cot k(E)\pi\to 0 & \text{as} E\to E_{2N}^{\alpha}(0)+\\ -\beta(\theta)\cot k(E)\pi\to+\infty & \text{as} E\to\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\end{cases} \tag{3.39}$$

$$\begin{cases}-\beta(\theta)\cot k(E)\pi\to 0 & \text{as} E\to E_{-(2N+1)}^{\alpha}(0)+\\ -\beta(\theta)\cot k(E)\pi\to+\infty & \text{as} E\to-\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}_{-}\end{cases} \tag{3.40}$$

Therefore, $|W|^{\frac{1}{2}}(\tilde{H}_\alpha - E)^{-1}|W|^{\frac{1}{2}}$ has a large positive eigenvalue if E is in a left neighbourhood of the right endpoint of each gap and a large negative eigenvalue if E is in a right neighbourhood of the left endpoint.

At this point we are going to prove some lemmas which will be useful in the following.

LEMMA 3.2. *If $W \leq 0$ satisfies (1.1) the operators*

$$\tilde{T}_\alpha^{(4)}\left(\pm\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right) \in \mathcal{J}_1\left(L^2(\mathbb{R})\otimes\mathbb{C}^2\right)$$

defined in Lemma 3.1. are positive.

PROOF: It is an easy concequence of the positivity of the kernel $|m|+|n|-|m-n|$, which has been shown in (1), and of the form of the 2×2 operator-valued matrix.■

Similarly, we have that $\tilde{T}_\alpha^{(1)}(E)+\tilde{T}_\alpha^{(3)}(E)$ is a negative trace-class operator in any gap of the type which is being considered.

Differently from its non-relativistic analogue the trace-class norm of the operators $\tilde{T}_\alpha^{(4)}\left(\pm\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)$ does not go to zero as N goes to infinity as the following lemma shows.

LEMMA 3.3. *If $W \leq 0$ satisfies (1.1), then*

$$\lim_{N\to\infty}\left\|\tilde{T}_\alpha^{(4)}\left(\pm\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\right\|_{\mathcal{J}_1} = \frac{|\alpha|}{c^2}\sum_{m\in\mathbb{Z}}\int_{I_m}|m||W(x)|\,dx \tag{3.41}$$

PROOF: First of all, we notice that the trace-class norm of the 2×2 $\mathcal{J}_1$-operator-valued matrix (3.30) is given by the sum of the trace-class norms of its diagonal entries as follows by using

$$\begin{pmatrix}\psi_l\\0\end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ \psi_l \end{pmatrix}$$

as an orthonormal basis for $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, $\{\psi_l\}_{l=1}^{\infty}$ being any orthonormal basis for $L^2(\mathbb{R})$.

From the results of (1) we can compute the trace-class norm of both entries explicitly and we obtain

$$\left\| \tilde{T}_{\alpha}^{(4)} \left(\pm \frac{c}{2} \left[(2N)^2 + c^2 \right]^{\frac{1}{2}} \right) \right\|_{\mathcal{J}_1} =$$

$$= \frac{|\alpha|}{2c^2} \left[\gamma^2 \left(\pm \frac{c}{2} \left[(2N)^2 + c^2 \right]^{\frac{1}{2}} \right) \sum_{m \in \mathbb{Z}} \int_{I_m} 2|m||W(x)| \sin^2 Nx \, dx + \right.$$

$$\left. + \sum_{m \in \mathbb{Z}} \int_{I_m} 2|m||W(x)| \cos^2 Nx \, dx \right] \underset{N \to \infty}{\longrightarrow} \frac{|\alpha|}{c^2} \sum_{m \in \mathbb{Z}} \int_{I_m} |m||W(x)| \, dx \tag{3.42}$$

where we have exploited the identities $\sin^2 t = \frac{1 - \cos 2t}{2}$, $cos^2 t = \frac{1 + \cos 2t}{2}$, the Riemann-Lebesgue lemma and the fact that $\lim_{E \to \pm\infty} \gamma(E) = 1$. ■

Furthermore, since the diagonal entries of $\tilde{T}_{\alpha}^{(5)}(E_N)$, E_N being either endpoint of the N-th closed interval of the type being considered, contain the factor

$$\gamma(E_N) \underset{N \to \infty}{\longrightarrow} 1$$

it is clear that the Hilbert-Schmidt norm of the operator does not go to zero as $N \to \pm\infty$. However, the $\mathcal{J}_4$-norm of the operator, i.e. the trace of its fourth power, does due to the oscillations of the convolution kernel.

LEMMA 3.4. *If $W \in L^1(\mathbb{R})$ then*

$$\left\| \tilde{T}_{\alpha}^{(5)}(E_N) \right\|_{\mathcal{J}_4} \underset{N \to \infty}{\longrightarrow} 0 \tag{3.43}$$

where E_N is either endpoint of

$$\begin{cases} \left[E_{2N}^{\alpha}(0), \frac{c}{2} \left((2N)^2 + c^2 \right)^{\frac{1}{2}} \right] & E > \frac{c^2}{2} \\ \left[E_{-(2N+1)}^{\alpha}(0), -\frac{c}{2} \left((2N)^2 + c^2 \right)^{\frac{1}{2}} \right] & E < -\frac{c^2}{2} \end{cases}$$

PROOF: Since

$$\left\|\tilde{T}_\alpha^{(5)}(E_N)\right\|_{\mathcal{J}_4} = \left(\left\|\left[\tilde{T}_\alpha^{(5)}(E_N)\right]^2\right\|_{\mathcal{J}_2}^2\right)^{\frac{1}{4}} \tag{3.44}$$

it easily follows that

$$\left\|\tilde{T}_\alpha^{(5)}(E_N)\right\|_{\mathcal{J}_4} \leq 2 \max_{i,j=1,2}\left\{\left\|\left[\tilde{T}_\alpha^{(5)}\right]_{ij}(E_N)\right\|_{\mathcal{J}_4}\right\} \tag{3.45}$$

Therefore we must simply show that the $\mathcal{J}_4$-norm of each entry goes to zero as N goes to infinity. Since all the entries have kernels of the type

$$|W|^{\frac{1}{2}}(x)f(x-y)|W|^{\frac{1}{2}}(y)$$

where $f(x-y)$ is either

$$\sin k(E_N)(|x-y|-2|m-n|\pi),\ \forall x \in I_m, \forall y \in I_n,\ \forall m,n \in \mathbb{Z}$$

or

$$\operatorname{sgn}(x-y)\cos k(E_N)(|x-y|-2|m-n|\pi),\ \forall x \in I_m, \forall y \in I_n\ \forall m,n \in \mathbb{Z}$$

the method of the proof is the same for all the entries and therefore we shall only show that

$$\left\|\left[\tilde{T}_\alpha^{(5)}\right]_{11}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\right\|_{\mathcal{J}_4} \underset{N\to\infty}{\to} 0 \tag{3.46}$$

In fact, we have

$$\left\|\left[\tilde{T}_\alpha^{(5)}\right]_{11}\left(\frac{c}{2}\left[(2N)^+c^2\right]^{\frac{1}{2}}\right)\right\|_{\mathcal{J}_4}^4 =$$

$$= \left\|\left[\tilde{T}_\alpha^{(5)}\right]_{11}^2\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\right\|_{\mathcal{J}_2}^2 = \gamma^4\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\cdot$$

$$\cdot\iint_{\mathbb{R}^2}|W(x)||W(y)|\left[\int_{\mathbb{R}}|W(t)|\sin N|x-t|\sin N|t-y|\,dt\right]^2 dx\,dy \underset{N\to\infty}{\to} 0 \tag{3.47}$$

as a consequence of the Riemann-Lebesgue lemma and of the dominated convergence theorem.∎

Before giving the main result of this section we want to single out the special case in which the impurity W has compact support contained in a lattice cell.

PROPOSITION 3.5. *If $W \leq 0$ has compact support entirely contained in a lattice cell and is bounded, then $\tilde{H}_\alpha - \lambda|W|$ has only one bound state in each sufficiently remote gap of its essential spectrum.*

PROOF: Without loss of generality, we can assume that supp W is entirely contained inside the lattice cell centred at 0.

First of all, we notice that in this case

$$|W|^{\frac{1}{2}}(\tilde{H}_\alpha - E)^{-1}|W|^{\frac{1}{2}} = \tilde{T}_\alpha^{(1)}(E) + \tilde{T}_\alpha^{(2)}(E) + \tilde{T}_\alpha^{(5)}(E) \tag{3.48}$$

with

$$\tilde{T}_\alpha^{(1)}(E) = \frac{(\beta(\theta))^{-1} \tan k(E)\pi}{2c\gamma(E)} |W|^{\frac{1}{2}}\phi_\alpha^{(1)}(E) \left(|W|^{\frac{1}{2}}\phi_\alpha^{(1)}(E), \cdot\right) \tag{3.49}$$

where

$$\phi_\alpha^{(1)}(x, E) = \begin{pmatrix} \gamma(E) \cos k(E)x \\ i \sin k(E)x \end{pmatrix}, \tag{3.50}$$

$$\tilde{T}_\alpha^{(2)}(E) = -\frac{\beta(\theta) \cot k(E)\pi}{2c\gamma(E)} |W|^{\frac{1}{2}}\phi_\alpha^{(2)}(E) \left(|W|^{\frac{1}{2}}\phi_\alpha^{(2)}(E), \cdot\right) \tag{3.51}$$

where

$$\phi_\alpha^{(2)}(x, E) = \begin{pmatrix} \gamma(E) \sin k(E)x \\ -i \cos k(E)x \end{pmatrix}, \tag{3.52}$$

and the entries of $\tilde{T}_\alpha^{(5)}(E)$ have the following integral kernels

$$\left[\tilde{T}_\alpha^{(5)}\right]_{11}(x, y; E) = -\frac{\gamma(E)}{2c} |W(x)|^{\frac{1}{2}} \sin k(E)|x - y| \, |W(y)|^{\frac{1}{2}} \tag{3.53}$$

$$\left[\tilde{T}_\alpha^{(5)}\right]_{12}(x, y; E) = \frac{i}{2c} |W(x)|^{\frac{1}{2}} \operatorname{sgn}(x - y) \cos k(E)(x - y) \, |W(y)|^{\frac{1}{2}} \tag{3.54}$$

$$\left[\tilde{T}_\alpha^{(5)}\right]_{21}(x, y; E) = -\left[\tilde{T}_\alpha^{(5)}\right]_{12}(x, y; E) \tag{3.55}$$

$$\left[\tilde{T}_{\alpha}^{(5)}\right]_{22}(x,y;E) = \frac{1}{\gamma^2(E)}\left[\tilde{T}_{\alpha}^{(5)}\right]_{11}(x,y;E) \tag{3.56}$$

We recall that the eigenvalues of $\lambda|W|^{\frac{1}{2}}(\tilde{H}_\alpha - E)^{-1}|W|^{\frac{1}{2}}$ are increasing functions of E inside each gap and that for our purpose we may neglect the negative rank-one operator $\tilde{T}_{\alpha}^{(1)}(E)$. Furthermore, we have that $\tilde{T}_{\alpha}^{(2)}(E)$ vanishes at the left endpoint of each gap so that the positive part of $\lambda|W|^{\frac{1}{2}}(\tilde{H}_\alpha - E)^{-1}|W|^{\frac{1}{2}}$ at the left endpoint is simply given by that of $\lambda\tilde{T}_{\alpha}^{(5)}(E)$, whose greatest eigenvalue is less than one provided N is sufficiently large as a consequence of Lemma 3.4.. This fact and the divergence of $\lambda\tilde{T}_{\alpha}^{(2)}(E)$ at the right endpoint of each gap imply that for N sufficiently large the greatest positive eigenvalue as an increasing function of E will always cross the horizontal line $\mu(E) = 1$ from which we obtain the existence of at least a bound state in each sufficiently remote gap.

By repeating the argument of (1) at the right endpoint we have that the number of bound states in the gap $\left(E_{2N}^{\alpha}(0), \frac{c}{2}\left[(2N)^2 + c^2\right]^{\frac{1}{2}}\right)$ is bounded by

$$1 + \left\|\lambda\tilde{T}_{\alpha}^{(5)}\left(\frac{c}{2}\left[(2N)^2 + c^2\right]^{\frac{1}{2}}\right)\right\|_{\mathcal{J}_4}^{4} \underset{N\to\infty}{\to} 1 \tag{3.57}$$

as follows from Lemma 3.4., which implies that there will only be one bound state in each sufficiently remote gap .∎

It also follows easily from Proposition 3.5. that if W is supported in only one lattice cell and λ is sufficiently small $\tilde{H}_\alpha - \lambda|W|$ has only one bound state in each gap of its essential spectrum.

At this point we state and prove the main result of this section.

THEOREM 3.6. *If $W \leq 0$ satisfies (1.1) then* *i) $\forall\lambda > 0$, the number of bound states of the Dirac Hamiltonian $\tilde{H}_\alpha - \lambda|W|$ inside any gap of its essential spectrum is finite;*

ii) there exists at least a bound state of $\tilde{H}_\alpha - \lambda|W|$ in each sufficiently remote gap;

iii) the number of bound states in each sufficiently remote gap of the type

$$\left(E^{\alpha}_{-(2N+1)}(0), -\frac{c}{2}\left[(2N)^2 + c^2\right]^{\frac{1}{2}}\right)$$

or

$$\left(E^{\alpha}_{2N}(0), \frac{c}{2}\left[(2N)^2 + c^2\right]^{\frac{1}{2}}\right)$$

is bounded by

$$1 + \left[\frac{\lambda|\alpha|}{c^2}\sum_{m\in\mathbf{Z}}\int_{I_m}|m||W(x)|\,dx\right] \tag{3.58}$$

where $[x]$ denotes the integral part of x and a similar estimate holds for the gaps of the other type.

PROOF: The proof of the existence of eigenvalues of $\tilde{H}_\alpha - \lambda|W|$ in each sufficiently remote gap follows exactly the same lines of that in Proposition 3.5. since for our purpose we may neglect the negative trace-class operator $\tilde{T}^{(1)}_\alpha(E) + \tilde{T}^{(3)}_\alpha(E)$ and since the positive trace-class operator $\tilde{T}^{(2)}_\alpha(E) + \tilde{T}^{(4)}_\alpha(E)$ vanishes at the left endpoint of each gap so that ii) follows from Lemma 3.4..

By repeating the argument of Theorem 4.1. in (1) we get that the number of bound states of $\tilde{H}_\alpha - \lambda|W|$ in the gap $\left(E^{\alpha}_{2N}(0), \frac{c}{2}\left[(2N)^2 + c^2\right]^{\frac{1}{2}}\right)$ is bounded by

$$1 + \dim_{(1,\infty)}\left(\lambda\tilde{T}^{(4)}_\alpha\left(\frac{c}{2}\left[(2N)^2 + c^2\right]^{\frac{1}{2}}\right) + \lambda\tilde{T}^{(5)}_\alpha\left(\frac{c}{2}\left[(2N)^2 + c^2\right]^{\frac{1}{2}}\right)\right) \tag{3.59}$$

and similarly for the gaps of the same type located in the negative part of the spectrum.

By using an argument similar to that of Theorem XIII.80 in (8) we obtain that the quantity in (3.59) is bounded by

$$1+\dim_{(1,\infty)}\left(\lambda\tilde{T}_{\alpha}^{(4)}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\right)+\dim_{(1,\infty)}\left(\lambda T_{\alpha}^{(5)}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\right) \quad (3.60)$$

which in its turn is bounded by

$$1+\lambda\left\|\tilde{T}_{\alpha}^{(4)}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\right\|_{\mathcal{J}_1}+\lambda^4\left\|\tilde{T}_{\alpha}^{(5)}\left(\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)\right\|_{\mathcal{J}_4}^4 \quad (3.61)$$

proving i).

From Lemma 3.3. and Lemma 3.4. we obtain that for N sufficiently large the integral part of the quantity in (3.61) is given by the quantity in (3.58) which implies the upper bound iii).■

Therefeore, we may have more than one bound state for $\tilde{H}_{\alpha}-\lambda|W|$ in each remote gap of its essential spectrum.

The possibility of having more than one bound state in each sufficiently remote gap of $\sigma_{ess}(\tilde{H}_{\alpha}-\lambda|W|)$ is related to the existence of the so-called "Dirac impurity states"or "Dirac localised states", i.e. relativistic bound states which have no non-relativistic analogue since they merge with the band edges in the NR limit ($c\to\infty$), according to the terminology introduced in papers such as (4) and (5).

We would like to recall here that (5) is concerned with the relativistic version of the Krönig -Penney model with a substitutional impurity given by a $\delta-$function of different strength at one of the lattice sites.

In our analysis, in which we have used a more realistic impurity given by a short-range potential in the sense of (1.1), we have seen that the existence of a possible Dirac impurity state inside the gap $\left(E_{2N}^{\alpha}(0),\frac{c}{2}\left[(2N)^2+c^2\right]^{\frac{1}{2}}\right)$ for N large (and similarly for the correspondent gap in the negative part of the spectrum) is related to the existence of E_N inside that gap for which the operator $\lambda\tilde{T}_{\alpha}^{(4)}(E_N)$ has an eigenvalue equal to one since the the bound state arising from the divergence of the rank-one operator $\tilde{T}_{\alpha}^{(2)}(E)$ at the right endpoint of the gap has always a relativistic counterpart.

Remark. It follows from (3.58) that if

$$\lambda|\alpha|\sum_{m\in\mathbb{Z}}\int_{I_m}|m||W(x)|\,dx << c^2$$

such Dirac impurity states cannot appear.

Finally, we want to point out that the case of an interstitial impurity given by

$$-\lambda\delta(\cdot - 2\tilde{m}\pi),\ \lambda > 0$$

for a certain $\tilde{m} \in \mathbb{Z}$ is quite peculiar and perhaps a bit misleading if one were to consider it as a starting point in order to carry out the general analysis about Dirac localised states.

In fact, assuming without loss of generality that $\tilde{m} = 0$ and that

$$\int_{\mathbb{R}}|W(x)|\,dx = 1,$$

by using a method similar to that of (3), ChapterI.3.2 we get that in this particular case the Birman-Schwinger kernel is simply given by

$$-\lambda|W|^{\frac{1}{2}}\tilde{G}_{\alpha}(0,0;E)|W|^{\frac{1}{2}} = -\lambda\frac{\gamma(E)}{2c}(\beta(\theta))^{-1}\tan k(E)\pi\,\{|W|^{\frac{1}{2}}\oplus 0\}\left(\{|W|^{\frac{1}{2}}\oplus 0\},\cdot\right) +$$
$$+\lambda\frac{1}{2c\gamma(E)}\beta(\theta)\cot k(E)\pi\,\{0\oplus|W|^{\frac{1}{2}}\}\left(\{0\oplus|W|^{\frac{1}{2}}\},\cdot\right) \qquad (3.62)$$

which shows that the origin of the relativistic bound state is purely "positronic" and therefore there is no non-relativistic counterpart in agreement with the Saxon-Hutner result (see (3) and (9)) for the non-relativistic case, according to which the gaps of the type

$(E_{2N}^{\alpha}(0), N^2)$ do not contain bound states of the Schrödinger Hamiltonian $H_\alpha - \lambda\delta(\cdot)$ if $\alpha < 0, \lambda > 0$.

Such a situation is completely different from the case of real interactions for which the "electronic component" of $\tilde{T}_\alpha^{(2)}(E)$ is always different from zero.

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