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On the Effective Action of the WZNW-Model [1]

H. Leutwyler

Institute for Theoretical Physics, University of Bern,
Sidlerstrasse 5, CH-3012 Bern Switzerland

The WZNW-model is analyzed within perturbation theory, allowing for a curved background geometry. The generating functional is expressed in terms of the effective action of a free massless scalar field. The current correlation functions are shown to be given by the corresponding tree graphs while the correlation functions of the energy momentum tensor only receive contributions from loops .

1 Introduction

Field theory models with an exact conformal symmetry resemble horses with an exact spherical symmetry. They may represent an important aspect of the structure underlying the standard model, but they do not show any trace of the scales of space and time which manifest themselves in this world and which, in particular, are at the origin of birthday celebrations. Nevertheless, as witnessed by the number of publications which are devoted to these models per second, they do provide for entertainment and this is why I present Raoul Gatto and Henri Ruegg with a discussion of such a model on the occasion of their 60th birthday.

For a theory to be conformally invariant, it is necessary that the Lagrangian does not contain any dimensionful constants, but this condition is not sufficient. A well known example is QCD, where, if the quark masses are set equal to zero, the Lagrangian only contains a dimensionless coupling constant. Nevertheless, this theory has a scale which shows up, e.g. in the mass of the proton. The phenomenon is related to the fact that the Lagrangian of QCD contains a hidden scale in the form of a cutoff. If the cutoff is changed, the coupling constant must be adjusted accordingly. The manner in which a coupling constant reacts to a change in scale is determined by the β-function. For a theory to be conformally invariant, the constants occurring in the Lagrangian must not only be dimensionless, but their β-functions must vanish. Most renormalizeable field theories have nonzero β-functions and therefore break conformal symmetry. The WZNW-model is one of the known exceptions [1-7].

Let me start the discussion of this model with a glance at the two-dimensional nonlinear σ-model. In both of these theories, the field lives on a compact Lie group which for definiteness I identify with SU(N). Denoting the value of the field at the point x by the element $U(x)$ of this group, the euclidean action of the σ-model is of the form

$$S = \frac{1}{4g^2}\int d^2x \mathrm{Tr}(\partial_\mu U \partial_\mu U^{-1}) \qquad (1)$$

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where g is a dimensionless coupling constant. The model can be analyzed perturbatively by setting $U(x) = \exp ig\pi(x)$ and expanding in powers of g. [The field $\pi(x)$ is a hermitean, traceless matrix and can be decomposed into N^2-1 ordinary scalar fields by writing $\pi(x) = \sum_a \lambda_a \pi^a(x)$ where $\lambda_1, \lambda_2, \ldots$ are the Gell-Mann matrices.] The perturbative expansion of the action then starts with

$$S = \int d^2x \{ \frac{1}{4} \mathrm{Tr}(\partial_\mu \pi \partial_\mu \pi) + \frac{g^2}{48} \mathrm{Tr}([\partial_\mu \pi, \pi][\partial_\mu \pi, \pi]) + O(g^4) \} \tag{2}$$

The first term is the standard kinetic energy and describes the Gaussian fluctuations of a set of free, massless scalar fields while the terms proportional to $g^2, g^4 \ldots$ generate interactions among these fields.

For the scattering matrix to be finite, the coupling constant of the nonlinear σ-model must be renormalized. The logarithmic divergence occurring at one loop order is removed if the coupling constant is renormalized according to

$$g = g_0 + \frac{N g_0^3}{4\pi} \ln \frac{\mu_0}{\mu} + O(g_0^5) \tag{3}$$

Here μ is the renormalization point (scale of g) and μ_0 is the cutoff (scale of g_0). The scale-dependence of the coupling constant defines the β-function of the model,

$$\mu \frac{dg}{d\mu} = \beta(g) \tag{4}$$

Comparison with eq. (3) gives

$$\beta(g) = -\frac{N}{4\pi} g^3 + O(g^5) \tag{5}$$

In the nonlinear σ-model, the renormalized coupling constant decreases if the scale μ is taken larger - the model is asymptotically free and thus represents an interesting toy model for nonabelian gauge field theories in four dimensions, such as QCD. The differential equation (4) implies that for values of the scale μ where the renormalized coupling constant is small, g^2 runs in inverse proportion to the logarithm of the scale μ,

$$\frac{g^2}{4\pi} \simeq \frac{1}{N \ln \frac{\mu^2}{\Lambda_\sigma^2}} \tag{6}$$

where Λ_σ is the renormalization group invariant scale of the model and plays a role analogous to Λ_{QCD}.

Let us now consider the extension of the nonlinear σ-model which results if one supplements the action by a Wess-Zumino term,

$$S = \frac{1}{4g^2} \int_{\mathcal{M}} d^2x \mathrm{Tr}(\partial_\mu U \partial_\mu U^{-1}) + ik\Gamma_{\mathrm{WZ}}\{U\} \tag{7}$$

The WZ-term is not a polynomial in the field $U(x)$ and its derivatives. A closed expression can only be given in terms of a construction which treats the two-dimensional manifold $\mathcal{M}$ of interest as boundary of a three-dimensional manifold $\bar{\mathcal{M}}$. [In the context to be discussed below, $\mathcal{M}$ has the topology of a sphere and can be represented as a two-dimensional closed surface in flat three-dimensional space. In this case, $\bar{\mathcal{M}}$ can be identified with the three-dimensional region enclosed by the surface $\mathcal{M}$]. Explicitly, the expression reads

$$\Gamma_{\mathrm{WZ}}\{U\} = \frac{1}{12\pi} \int_{\bar{\mathcal{M}}} \mathrm{Tr}[(d\bar{U}\bar{U}^{-1})^3] \tag{8}$$

Where $\bar{U}(x^1, x^2, x^3) \in \mathrm{SU}(N)$ is an auxiliary field whose boundary values on $\mathcal{M}$ agree with the field $U(x)$ of interest and d is the exterior derivative. The integral (8) is unambiguously determined by the boundary values, except for an integer multiple of 2π. Accordingly, eq. (7) specifies an acceptable action, provided the coupling constant k in front of the WZ-term is an integer: if this is the case, the quantity $\exp(-S)$ only depends on U and is independent of the auxiliary field $\bar{U}$ which enters the above construction.

In the perturbative expansion, the leading contribution to the WZ-term is of the form

$$\Gamma_{\mathrm{WZ}} = -i\frac{g^3}{12\pi} \int_{\mathcal{M}} d^2x \epsilon^{\mu\nu} \mathrm{Tr}(\partial_\mu \pi \partial_\nu \pi \pi) + O(g^5) \tag{9}$$

The corresponding vertices involve 3,5,... legs while the number of legs emanating from one of the vertices occurring in the ordinary nonlinear σ-model is always even. The interaction generated by the WZ-term modifies the β-function of the model. At one loop order, we now obtain

$$\beta(g) = -\frac{Ng^3}{4\pi}\left\{1 - \left(\frac{g^2k}{2\pi}\right)^2\right\} + \ldots \tag{10}$$

[In principle, the coupling constant in front of the WZ-term could also require renormalization and have its own β-function. Since k is however an integer, it is not a surprise to find that the one-loop graphs only renormalize g and leave k untouched]. Eq. (10) shows that the contribution to the β-function generated by the Wess-Zumino term is of opposite sign to the asymptotically free term discussed above - the WZ-interaction tends to shield the coupling. The two contributions cancel provided

$$g^2k = \pm 2\pi \tag{11}$$

If the two coupling constants are related in this manner, the β-function vanishes, at least to one loop.

2 WZNW-model

The relation (11) singles out a very special theory: the WZNW-model. Since a parity operation flips the sign of the WZ-term, we can take k to be a positive integer without loss of generality, such that the upper sign applies in eq. (11) which fixes the value of g^2 in terms of the integer k. The perturbative expansion in powers of g^2 amounts to a series in powers of $1/k$ and therefore requires k to be large.

Next, consider the Noether currents associated with the global symmetry $U(x) \to V_L U(x) V_R$ of the model

$$L_\mu = -\frac{i}{4g^2}\{\partial_\mu U U^{-1} + i\frac{g^2 k}{2\pi}\epsilon_{\mu\nu}\partial_\nu U U^{-1}\}$$

$$R_\mu = \frac{i}{4g^2}\{U^{-1}\partial_\mu U - i\frac{g^2 k}{2\pi}\epsilon_{\mu\nu}U^{-1}\partial_\nu U\} \tag{12}$$

Since the action is invariant, both L_μ and R_μ are conserved, irrespective of the values of g and k. The expressions (12) however show that if the relation (11) holds, these currents become self-dual,

$$\epsilon_{\mu\nu}L_\nu = -iL_\mu, \quad \epsilon_{\mu\nu}R_\nu = iR_\mu \tag{13}$$

such that, as an immediate consequence of current conservation, the left-handed current only depends on $x^1 - ix^2$ while R_μ only involves $x^1 + ix^2$. This shows that the WZNW-model is distinguished by very special properties. In the following I discuss some further aspects of this special model, drawing from work done together with M. Shifman - a more detailed account is in preparation [8]. The aim of our work was to understand the significance of the familiar "renormalization" of the coupling constant $k \to k + N$ which occurs in the Sugawara representation for the energy-momentum tensor, viewing the model from the point of view of the Feynman diagrams which go with it rather than from the more abstract algebraic point of view [9]. Also, we wanted to see how the conformal symmetry of the model manages to peacefully coexist with the conformal anomaly which arises if the two-dimensional space underlying the model is curved.

Quite independently of the conformal anomaly it is of interest to study the model on a curved manifold for two reasons. (i) In two dimensions, the infrared singularities associated with massless particles can significantly affect the properties of the theory. In particular, as shown by Wanders and his collaborators [10], the asymptotic states occurring in the Minkowski space version of two-dimensional field theories may develop a rich life of their own. The infrared singularities can be eliminated by putting the model into a spherical box, replacing euclidean space by a sphere. The radius of the sphere then plays the role of an infrared cutoff. (ii) The energy momentum tensor of the model, $\Theta_{\mu\nu}$ represents the response of the action to an infinitesimal deformation of the metric. If the model can be analyzed for a generic geometry, the metric plays the role of an external field coupled to the energy momentum tensor. It is then not necessary to construct this operator by means of point splitting or other regularization techniques - the Green functions of $\Theta_{\mu\nu}$ are obtained by simply taking derivatives of the partition function with respect to the metric [11].

3 Effective action

The currents of the model can be analyzed in the same fashion, perturbing the system with external fields which couple to the currents. In the WZNW-model, the right- and left-handed currents separate like oil and water. I restrict myself to the left-handed sector - the right-handed one is a mere mirror image. Accordingly, I consider two external fields: the metric $\gamma_{\mu\nu}(x)$ of the manifold and an external left-handed field $l^\mu(x)$ [note that the operator $L_\mu = \frac{1}{2}\sum_a \lambda_a L^a_\mu$ defined in eq. (12) collects the $N^2 - 1$ left-handed currents belonging to $\mathrm{SU}(N)_L$ in a traceless hermitean matrix; $l^\mu(x) = \frac{1}{2}\sum_a \lambda_a l^\mu_a(x)$ is the corresponding collection of external fields]. In the presence of these fields, the action of the model takes the form

$$S\{U \mid l, \gamma\} = \frac{1}{4g^2}\int d^2x \sqrt{\gamma}\gamma^{\mu\nu}\mathrm{Tr}(\partial_\mu U \partial_\nu U^{-1}) + ik\Gamma_{\mathrm{WZ}}\{U\} + 2\int d^2x\sqrt{\gamma}\mathrm{Tr}(l^\mu L_\mu) \tag{14}$$

Note that the WZ-term does not involve the metric - the formula (8) applies irrespective of whether or not the manifold we are considering is curved. The metric does occur in the first term of eq. (14), however only in a combination which is invariant under conformal transformations ($\gamma_{\mu\nu} \to \lambda\gamma_{\mu\nu}$). This is why the model can be solved also on a curved manifold: in two dimensions, a suitable conformal transformation removes the curvature in the neighbourhood of any given point and therefore reduces the model to the flat space version, at least locally.

The partition function of the model is given by the functional integral

$$e^{-S_{\mathrm{eff}}\{l,\gamma\}} = \int [dU] e^{-S\{U|l,\gamma\}} \tag{15}$$

I refer to the logarithm of the partition function as the effective action of the model. It represents the generating functional of the Green functions associated with the operators L_μ and $\Theta_{\mu\nu}$. The connected part of the two-point-function $< L_\mu(x)L_\nu(y) >$, e.g., is given by the second derivative of $S_{\mathrm{eff}}\{l,\gamma\}$ with respect to the external field l^μ, while the expectation value $< \Theta_{\mu\nu}(x) >$ and the correlation function $< \Theta_{\mu\nu}(x)\Theta_{\rho\sigma}(y) >$ represent the first two derivatives with respect to the metric.

Quantum theory is not fully determined by the classical action. In addition, we need to specify the measure $[dU]$. This can be done along the same lines as in string theory [12]. One equips the space of field configurations with a metric, defining the distance d between two neighbouring field configurations $U(x), U(x) + dU(x)$ as

$$d^2 = \int d^2x\sqrt{\gamma}\mathrm{Tr}(dU dU^{-1}) \tag{16}$$

where dU^{-1} stands for $-U^{-1}dUU^{-1}$. The measure $[dU]$ is the volume element induced by this metric.

4 Classical fields and quantum fluctuations

In perturbation theory, the functional integral is expanded around a solution of the classical equation of motion which represents a stationary point of the action. The equation of motion belonging to the action (14) is of the form

$$\partial_\mu(\sqrt{\gamma}\gamma^{\mu\nu}R^{\mathrm{cl}}_\nu) = 0 \tag{17}$$

where R^{cl}_μ is the classical right-handed current. In the presence of the external fields $\gamma_{\mu\nu}$ and l_μ, the explicit expression reads

$$R_\mu^{\rm cl} = \frac{i}{4g^2}\{U_{\rm cl}^{-1}D_\mu U_{\rm cl} - i\frac{g^2k}{2\pi}\bar{\epsilon}_{\mu\nu}U_{\rm cl}^{-1}D^\nu U_{\rm cl}\} \tag{18}$$

The covariant derivative occurring here contains the external field,

$$D_\mu U = \partial_\mu U + il_\mu U \tag{19}$$

and $\bar{\epsilon}_{\mu\nu}$ stands for $\epsilon_{\mu\nu}\sqrt{\gamma}$.

In analyzing the properties of the path integral, it is convenient to change the variables of integration, setting

$$U(x) = U_{\rm cl}(x)U_q(x) \tag{20}$$

where $U_{\rm cl}$ is the classical solution while U_q describes the quantum fluctuations. Inserting this decomposition in the action, we obtain

$$\begin{aligned} S\{U_{\rm cl}U_q \mid l,\gamma\} &= S\{U_{\rm cl} \mid l,\gamma\} + S\{U_q \mid 0,\gamma\} \\ &+ 2i\int d^2x\sqrt{\gamma}\gamma^{\mu\nu}\{U_q^{-1}\partial_\mu U_q R_\nu^{\rm cl}\} \end{aligned} \tag{21}$$

The first term is the classical action which depends on the external field $l^\mu(x)$ both explicitly and implicitly, through the classical solution. The second term is the action of the quantum fluctuations for $l^\mu = 0$. The decomposition (21) shows that the quantum fluctuations feel the external field only through the classical current $R_\mu^{\rm cl}$ which occurs in the last term.

At this point, the peculiar properties of the WZNW-model become evident. The explicit expression for $R_\mu^{\rm cl}$ given in eq. (18) shows that for $g^2k = 2\pi$, this current is self-dual. The equation of motion requires its divergence to vanish. For a self-dual vector, this however implies that the curl vanishes, too. On a simply connected, compact manifold such as a sphere, these two requirements lead to $R_\mu^{\rm cl} = 0$. In other words, for the WZNW-model, the equation of motion is equivalent to the Polyakov-Wiegman equation [13]

$$D_\mu U_{\rm cl} = i\bar{\epsilon}_{\mu\nu}D^\nu U_{\rm cl} \tag{22}$$

As an immediate consequence, the third term in eq. (21) vanishes - the quantum fluctuations do not notice the presence of the external field l^μ at all! The action depends on this field exclusively through the classical part. Transforming the variables of integration from U to U_q and using the fact that the metric specified in eq. (16) is invariant under left translations on the group, one concludes that the Jacobian is equal to one, $[dU] = [dU_q]$. The effective action therefore inherits the property of the classical action emphasized above,

$$S_{\rm eff}\{l,\gamma\} = S_{\rm cl}\{l,\gamma\} + S_q\{\gamma\} \tag{23}$$

It depends on the external field l^μ exclusively through the first term which stands for the classical action $S\{U_{\rm cl} \mid l,\gamma\}$ while the contribution generated by the quantum fluctuations is given by

$$e^{-S_q\{\gamma\}} = \int [dU]e^{-S\{U|0,\gamma\}} \tag{24}$$

and only depends on the geometry of the manifold.

The above discussion is formal and two points require a more careful analysis:

(i) The equation of motion does not fix the classical solution uniquely. If $U_{cl}(x)$ is a solution, so is $U'_{cl}(x) = U_{cl}(x)V$ where V is a constant matrix; conversely, any two solutions are related in this manner. In the perturbative analysis of the model, this ambiguity in the classical solution corresponds to the occurrence of zero modes.

(ii) In the presence of a WZ-term, the action is complex. Accordingly, the solutions of the classical equation of motion - or of the Polyakov-Wiegman equation - are not unitary, i.e. do not belong to SU(N), but to the complex extension of this group.

Both of these problems are discussed in detail in [8] where it is shown that the basic property (23) of the effective action indeed holds true to any finite order of perturbation theory. In the following, I briefly discuss some consequences of this result. For a more detailed account, I refer you to [8].

5 Current correlation functions

The correlation functions of the left-handed current are given by the derivatives of the effective action with respect to the external field $l^\mu(x)$. In these dervivatives, the term $S_q\{\gamma\}$ which stems from the quantum fluctuations drops out, because it does not depend on $l^\mu(x)$ - the current correlation functions are the derivatives of the classical action. In Feynman graph language, the classical action represents the tree graph approximation. Accordingly, the current correlation functions are given by the corresponding tree graphs. To calculate the correlation function $< L_\mu(x)L_\nu(y) >$, e.g., it suffices to work out the classical action to second order in the external field - a simple exercise in classical field theory. Alternatively, one may evaluate the corresponding tree graph which describes the exchange of a massless particle between the two currents - the result is of course the same. The most remarkable property of the resulting explicit expression for $< L_\mu(x)L_\nu(y) >$ is that it is invariant under a conformal deformation of the metric, $\gamma_{\mu\nu}(x) \rightarrow \lambda(x)\gamma_{\mu\nu}(x)$. In fact, this property can be established without performing an explicit calculation. It suffices to observe that the classical action of the model is invariant under the above conformal transformation of the metric if $l_\mu(x)$ and $U_{cl}(x)$ are held fixed. This immediately implies that the transformation does not modify the solution of the classical equation of motion. Hence the classical action and its derivatives with respect to $l_\mu(x)$ retain their values - the n-point functions $< L_{\mu_1}(x_1)\ldots L_{\mu_n}(x_n) >$ are invariant under conformal deformations of the geometry. This shows that in these correlation functions, the conformal anomaly does not leave any trace whatsoever. One may choose the coordinate frame in such a manner that, on a given coordinate patch, the metric differs from the euclidean one only by a conformal factor, $\gamma_{\mu\nu}(x) = \delta_{\mu\nu} \exp \omega(x)$. In this frame, the current correlation functions coincide with the well-known flat space expressions.

Note that the coupling constants g, k occurring in the tree graph contributions to the current correlation functions are the bare ones. The absence of loop contributions shows that in the WZNW-model a renormalization of these couplings does not occur. In the context of the more general model which results if the coupling constant g is not a priori tied to the value of the integer k, the β-function of g has a zero at the point $g^2 = 2\pi/k$.

6 Effective action of the quantum fluctuations

In contrast to the correlation functions of the current, those of the energy momentum tensor do receive contributions from graphs containing loops, because the term $S_q\{\gamma\}$ which represents the effective action of these graphs depends on the metric. Remarkably, for the WZNW-model, the effective action of the quantum fluctuations can be worked out explicitly: $S_q\{\gamma\}$ is proportional to the effective action of a free massless scalar field. In [8] this property is established to all orders of perturbation theory, using an indirect argument based on the expansion of the operator product $L_\mu(x)L_\nu(y)$ at short distances. This derivation is interesting by itself, because it shows why the coefficients occurring in the Sugawara representation of the energy momentum tensor are renormalized ($k \to k+N$), despite the fact that the coupling constant as such does not undergo renormalization.

In the following, I instead consider the more straightforward appraoch to the problem which perturbation theory immediately indicates and briefly discuss the evaluation of the Feynman graphs occurring at one- and two-loop order. In these graphs, the coupling constants g and k only enter as overall factors and we may just as well treat them as independent, specializing to the WZNW-model only at the end of the calculation. The relevant functional integral is specified in eq. (24). Setting $U(x) = \exp ig\pi(x)$, the expansion of the action in powers of g starts with the kinetic term which describes N^2-1 free massless scalars. Accordingly, the functional integral reduces to a Gaussian integral whose value can be expressed in terms of the determinant associated with the Laplacian,

$$\Delta = \frac{1}{\sqrt{\gamma}}\partial_\mu \sqrt{\gamma}\gamma^{\mu\nu}\partial_\nu \tag{25}$$

Actually, on a compact manifold this determinant vanishes, because the Laplacian admits a normalizeable eigenfunction with eigenvalue zero. The occurrence of this zero mode is related to the nonuniqueness of the classical solution mentioned above. Since the action associated with space-independent fluctuations, $U(x) = \text{const.}$, vanishes, they cannot be analyzed in terms of a perturbative expansion around $U(x) = 1$, but must be treated collectively. Accordingly, we set $U(x) = U_0 \exp ig\pi(x)$ where $\pi(x)$ lives on the space orthogonal to the zero mode, $\int dx \sqrt{\gamma}\pi(x) = 0$, while the constant matrix $U_0 \in \mathrm{SU}(N)$ is a collective variable. Now the expansion in powers of g makes sense and the leading contribution to $S_q\{\gamma\}$ becomes

$$S_q\{\gamma\} = (N^2-1)D\{\gamma\}\{1 + O(g^2, g^6k^2)\} \tag{26}$$

where $D\{\gamma\}$ stand for

$$D\{\gamma\} = \frac{1}{2}\{\ln \det{}'(-\Delta) - \ln V\} \tag{27}$$

The first term stems from the integral over the Gaussian fluctuations in the nonzero modes, the prime indicating that the zero mode contribution to the determinant is to be omitted. The second term originates in the integral over the collective variable and involves the volume $V = \int d^2x \sqrt{\gamma}$ of the manifold.

As is well-known, the determinant of the Laplacian contains a conformal anomaly: if the metric is subject to the infinitesimal conformal deformation $\delta\gamma_{\mu\nu} = \delta\omega\gamma_{\mu\nu}$, the functional

$D\{\gamma\}$ changes according to

$$\delta D\{\gamma\} = -\frac{1}{48\pi}\int d^2x\sqrt{\gamma}R\delta\omega \tag{28}$$

where R is the scalar curvature of the manifold. The effective action inherits this property - in contrast to the classical action, the contribution generated by the quantum fluctuations is not invariant under conformal transformations of the metric.

It is very instructive to pursue the perturbative evaluation of the effective action to two loops, because, at that order, the measure starts generating nontrivial contributions which do not occur in the flat space version of the model. The evaluation is discussed in detail in appendix C of [8] where it is shown that, even if the coupling constants do not obey the relation characteristic of the WZNW-model, the two-loop graphs merely generate a finite multiplicative renormalization of the one-loop result:

$$S_q\{\gamma\} = (N^2-1)D\{\gamma\}\left[1-\frac{Ng^2}{4\pi}\left\{3-\left(\frac{g^2k}{2\pi}\right)^2\right\}+\ldots\right] \tag{29}$$

The first term in the curly bracket stems from the interaction generated by the ordinary nonlinear σ-model ($k=0$) and from the contribution due to the measure mentioned above, while the second term arises from a graph containing two WZ-vertices. If the coupling constants are related by $g^2k = 2\pi$, the renormalization factor occurring in eq. (29) takes the form $[1-N/k+\ldots]$.

The perturbative evaluation of the effective action is instructive, but it is quite cumbersome and cannot be carried to arbitrary order. The analysis of the operator product expansion which I referred to above does not suffer from this limitation and it can be used to determine the higher order contributions to $S_q\{\gamma\}$ without actually performing the functional integral. It turns out that the higher order terms merely generate further multiplicative renormalizations of the one-loop result which sum up to the geometric series $(1+N/k)^{-1} = 1-N/k+\ldots$ The effective action of the WZNW-model can therefore be given in closed form:

$$S_{\text{eff}}\{l,\gamma\} = S_{\text{cl}}\{l,\gamma\} + (N^2-1)\frac{k}{k+N}D\{\gamma\} \tag{30}$$

where $D\{\gamma\}$ is related to the determinant of the Laplacian in the manner specified above. The coefficient in front of this term is the central charge of the Virasoro algebra - barely a surprise.

7 Conformal anomaly

The explicit expression for the effective action given in the last section shows that an infinitesimal conformal deformation of the metric generates the change

$$\delta S_{\text{eff}} = \delta S_q = -\frac{1}{48\pi}(N^2-1)\frac{1}{N+k}\int d^2x\sqrt{\gamma}R\delta\omega \tag{31}$$

The response of the effective action to a variation of the metric represents the expectation value of the energy momentum tensor. Eq. (31) shows that for the particular variation $\delta\gamma_{\mu\nu} = \delta\omega\gamma_{\mu\nu}$, the response is independet of the external field l^μ. The relation

$$\Theta^\mu_\mu = \frac{N^2-1}{24\pi}\frac{k}{N+k}R \tag{32}$$

therefore holds as an operator identity. In particular, in correlation functions of the type $< \Theta^\mu_\mu L_{\alpha_1} L_{\alpha_2} \ldots >$ only the disconnected part which is proportional to $R < L_{\alpha_1} L_{\alpha_2} \ldots >$ contributes - this is why the conformal anomaly does not afflict the conformal invariance of the current correlation functions. In fact, the same reasoning also applies to the correlation functions of the energy momentum tensor. Consider, e.g., the connected part of the two-point-function $< \Theta_{\mu\nu}(x)\Theta_{\rho\sigma}(y) >$, given by the second derivative of $S_q\{\gamma\}$ with respect to the metric. As long as the two points x, y do not coincide, a conformal deformation of the metric leaves this quantity invariant, because the expression (31) for the change in the effective action only involves the metric at the point where the conformal deformation occurs. Hence, disregarding contact terms, the connected part of $< \Theta_{\mu\nu}\Theta_{\rho\sigma} >$ is conformally invariant. Similarly, correlation functions involving the trace of the energy momentum tensor do not have a connected part. This shows that in a model, such as the one considered here, where the metric is an external field, the conformal anomaly does not seriously distort conformal symmetry.

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