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Patterns of $U_q[SU(2)]$ symmetry breaking in Heisenberg quantum chains

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It is shown, in the example of the spin one-half Heisenberg chain, that judiciously chosen boundary terms break the quantum algebra $U_q[SU(2)]$ symmetry but leave some unexpected degeneracy patterns.

1 Introduction

It is by now known that quantum chains commuting with the generators of quantum algebras have interesting symmetry properties. The invariance under the quantum algebra fixes both the bulk interaction and the surface terms. In this talk we we would like to show that if we keep the bulk interaction given by the quantum algebra but change the boundary terms one can discover symmetries which are not those of the full quantum algebras. We have chosen the subject of this talk being aware of Henri Ruegg's interest in quantum groups and algebras [1] and hoping to arise his inquisitiveness in this new development especially since it will turn out that we are left with an open problem.

We consider the spin-1/2 Heisenberg chain defined by the Hamiltonian

$$H = -\frac{\gamma}{2\sin\pi\gamma}\left\{\sum_{j=1}^{N-1}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - \cos\pi\gamma\,\sigma_j^z\sigma_{j+1}^z\right) + A_s(\sigma_1^z + \sigma_N^z) + A_a(\sigma_1^z - \sigma_N^z)\right\} \quad , \tag{1.1}$$

where σ^x, σ^y and σ^z are Pauli matrices. The complex parameters γ, A_s and A_a describe the anisotropy and surface fields. For γ real $(0 \leq \gamma < 0)$, all the energy gaps vanish in the continuum limit $N \to \infty$ and the finite-size scaling spectra are known [2]. These results are based on the Bethe ansatz [3] and numerical methods. The overall factor in Eq. (1.1) was chosen in order to fix the sound velocity to one.

We observe that H commutes with

$$S^z = \tfrac{1}{2}\sum_{j=1}^{N}\sigma_j^z \tag{1.2}$$

and denote the eigenvalues of S^z by Q, they are integer (half-integer) numbers if the chain has even (odd) number of sites N. All the energy levels belonging to a given Q define the <u>sector</u> Q.

As will be seen in Sec. 3, the continuum limit has a simple expression if we parametrize A_s and A_a as follows

$$A_s = -\frac{\sin\pi\gamma\sin\pi\delta}{\cos\pi\delta+\cosh\pi b}\,, \qquad A_a = i\frac{\sin\pi\gamma\sinh\pi b}{\cos\pi\delta+\cosh\pi b} \tag{1.3}$$

In the limit $b \to \infty$ ($A_s = 0$, $A_a = i\sin\pi\gamma$), the Hamiltonian is $U_q[SU(2)]$ invariant with $q = -\exp(i\pi\gamma)$. This means that H commutes not only with S^z but also with the generators $S^\pm$ of the quantum algebra $U_q[SU(2)]$, where

$$\begin{aligned} S^\pm &= \sum_{j=1}^{N} S_j^\pm \\ S_j^\pm &= q^{\frac{1}{2}\sum_{r=1}^{j-1}\sigma_j^z}\,\frac{\sigma_j^\pm}{2}\,q^{-\frac{1}{2}\sum_{r=j+1}^{N}\sigma_j^z} \end{aligned} \tag{1.4}$$

The generators S^z and $S^\pm$ verify the relations

$$\left[S^z, S^\pm\right] = \pm S^\pm\,, \qquad \left[S^+, S^-\right] = [2S^z]_q\,, \tag{1.5}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}\,. \tag{1.6}$$

If the anisotropy parameter γ is not a rational number, i.e., q is generic, the spectrum of the Hamiltonian shows $(2J+1)$-dimensional multiplets like in the usual $SU(2)$ symmetric case ($\gamma = 0$). The eigenvalues of the Casimir operator

$$C_2 = S^-S^+ + [S^z + \tfrac{1}{2}]_q^2 \tag{1.7}$$

are $[J+\frac{1}{2}]_q^2$. If however $\gamma = \frac{u}{v}$ (u, v coprimes), the range of values for J is limited:

$$0 \le J \le \frac{v-1}{2}\,. \tag{1.8}$$

Moreover the whole spectrum cannot be split according to the irreducible representations (1.8), since there are representations which are not fully reducible. The eigenvalues of the Casimir operator might also coincide for different angular momenta J. If, as an example, one takes $\gamma = \frac{1}{6}$ we notice that for $J = 2$ and $J = 3$ one has $[\frac{5}{2}]_q^2 = [\frac{7}{2}]_q^2 = 3$.

We will show here [4] that if the parameter δ in Eq. (1.3) is tuned to the parameter γ through one of the relations

$$\delta = (2d+1)\gamma - 2d \tag{1.9}$$

the spectra show remarkable symmetry properties which we believe are related to some new algebraic structures. In Eq. (1.9) d is a positive integer or half-integer number. Note that the parameter b from Eq. (1.3) does not appear in Eq. (1.9) and since it will be shown in Sec. 2 that the pattern of degeneracies are also independent of b, we will rewrite the Hamiltonian

(1.1) in an $U_q[SU(2)]$ invariant part H_S and a symmetry breaking part H_{SB}:

$$H = H_S + H_{SB} \tag{1.10}$$

where

$$H_{SB} = \frac{\epsilon\gamma}{(\epsilon + e^{i\pi\delta})(\epsilon + e^{-i\pi\delta})}\left\{\left(\epsilon + ie^{-i\pi\delta}\right)\sigma_1^z - \left(\epsilon + ie^{i\pi\delta}\right)\sigma_N^z\right\} \tag{1.11}$$

and

$$\epsilon = e^{-\pi b}\ . \tag{1.12}$$

As we will show in the next Section, different choices of d in Eq. (1.9) will give different patterns of degeneracies in the spectra. The spectra themselves move smoothly from the $U_q[SU(2)]$ symmetric situation ($\epsilon = 0$) to the perturbed situation ($\epsilon \neq 0$).

2 Symmetry breaking patterns

In order to fix the ideas, let us consider the $\gamma = 0$ case in the Hamiltonian (1.1) ($A_s = A_a = 0$). One can easily convince oneselves that the Hamiltonian is $O(3)$ symmetric. Our knowledge of group theory would suggest to look for symmetry breakings along the subgroups $O(2)$ or $U(1)$. As we will see, in the quantum case ($\gamma \neq 0$), the symmetry breaking can show up in a very different way (notice that for $q = 1$ ($\gamma = 0$), H_{SB} vanishes and the whole phenomenon disappears!). We will first describe the patterns of symmetry breaking and in the next Section we will show how we got the idea to look for them. Let us first consider the case q generic.

<u>Rule 1</u>: All the energy levels in the sector $Q = d + M + 1$ are contained in the sector $Q = d - M$ (M is a non-negative integer or half-integer number.

<u>Rule 2</u>: If $d = 0$, all the energy levels in the sector $Q = -M$ which have no correspondent in the sector $Q + M + 1$ are ϵ-independent.

In order to illustrate these rules, in Figure 1 one considers two idealized examples. The case a) corresponds to $d = 0$. We assume that for $\epsilon = 0$ we have multiplets of spin $J = 0, 1, 2$ and 3. In this case the fact that all the levels in the sector $Q = 1$ ($M = 0$) are contained in the sector $Q = 0$, that all the levels in $Q = 2$ ($M = 1$) are contained in $Q = -1$ and that all the levels in $Q = 3$ ($M = 2$) are contained in $Q = -2$ is a consequence of $U_q[SU(2)]$ invariance. If we now take $\epsilon \neq 0$ (two values ϵ_1 and ϵ_2 are shown in the Figure), many levels change, but the symmetry pattern remains. Notice that some levels are independent of ϵ (Rule 2). The case b) corresponds to $d = 1/2$. Here we assume that we have $J = 1/2$, $3/2$ and $5/2$. According to Rule 1, the levels for $Q = 3/2$ ($M = 0$) are contained in $Q = 1/2$ and those in $Q = 5/2$ ($M = 1$) are contained in $Q = -1/2$. There are no levels which are ϵ-independent (Rule 2 does not apply).

We consider now the case when q is not generic: $\gamma = \frac{v}{u+v}$ (v and u coprimes). In this case Rules 1 and 2 apply again, but the sectors are defined modulo $n = u + v$ ($d \leq \frac{1}{2} + \frac{v}{u}$). In the next Section we will show how we discovered the symmetry breaking patterns.

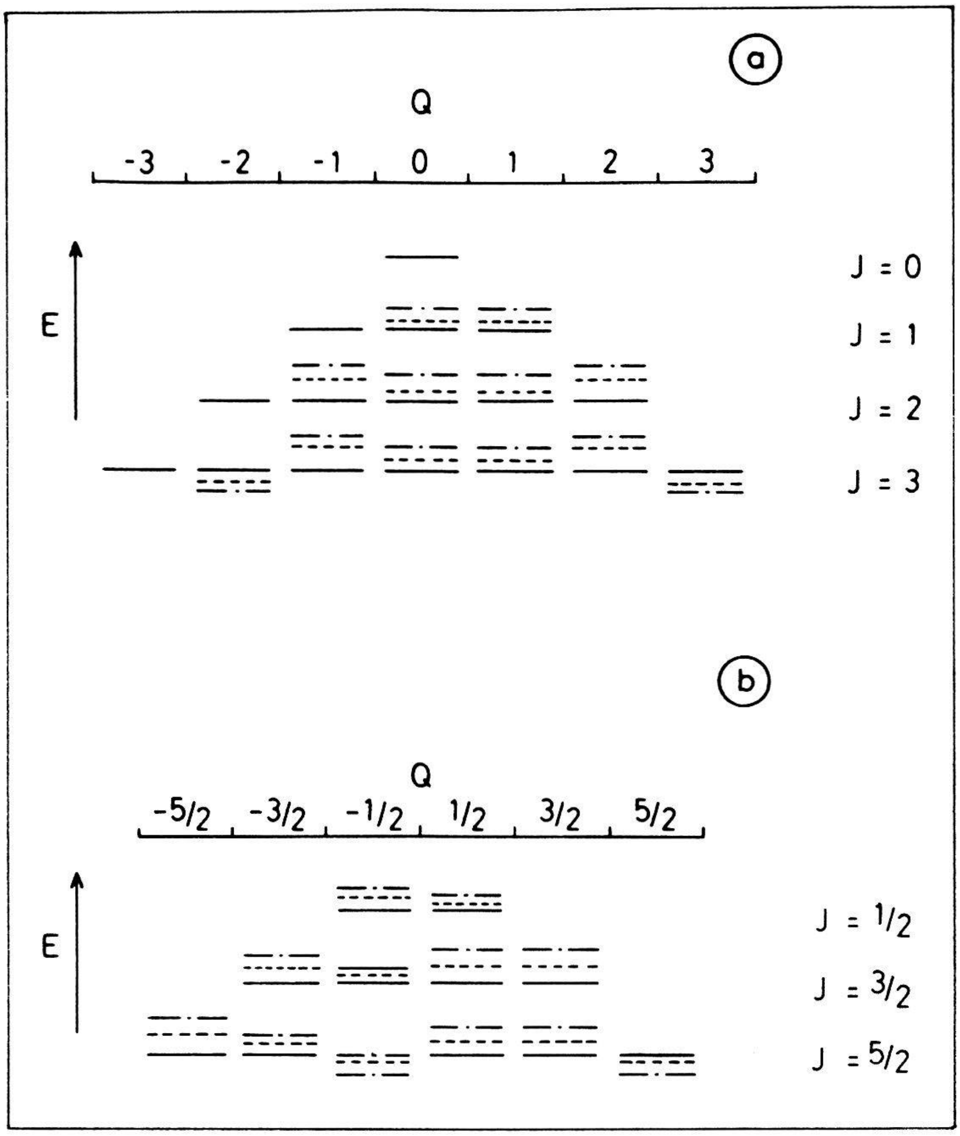

Figure 1: Symmetry breaking patterns. a) corresponds to $d = 0$. The solid lines represent the $U_q[SU(2)]$ symmetric case ($\epsilon = 0$), the other levels to two values ϵ_1 and ϵ_2 in H_{SB}. Notice that some levels are ϵ-independent. b) corresponds to $d = 1/2$. Here all levels are ϵ-dependent.

3 The continuum limit and discussion

Let us denote by $E_{Q;i}(N)$ the i'th level of the Hamiltonian (1.1) with N sites in the sector of charge Q and boundary conditions (1.3), $i = 0$ being the lowest one. We define the scaled energy gaps

$$F_{\overline{Q};i}(N) = \frac{N}{\pi}\left(E_{\overline{Q}+d;i}(N) - E_{d;0}(N)\right) , \tag{3.1}$$

and the finite-size scaling partition functions:

$$\mathcal{F}_{\overline{Q}}(z) = \lim_{N\to\infty} \sum_i z^{F_{\overline{Q};i}(N)} . \tag{3.2}$$

These partition functions have a simple expression:

$$\mathcal{F}_{\overline{Q}}(z) = z^{\Delta_{\overline{Q}}}\,\Pi_V(z) \tag{3.3}$$

where

$$\Pi_V(z) = \prod_{n=1}^{\infty}(1-z^n)^{-1} \tag{3.4}$$

$$\Delta_{\overline{Q}} = (d+\phi)^2(1-\gamma)\left[\left(\frac{\overline{Q}}{d+\phi}+1\right)^2 - 1\right] \tag{3.5}$$

$$\phi = (\pi-\pi\gamma)^{-1}\arctan\left(\tan\frac{\pi\delta}{2}\right) . \tag{3.6}$$

Note that the continuum limit is independent of the parameter b which appears in the boundary terms A_s and A_a of Eq. (1.3). We denote

$$\gamma = \frac{1}{m+1} \tag{3.7}$$

and use Eq. (1.9) to get:

$$\Delta_{\overline{Q}} = \frac{(2\overline{Q}m+1)^2-1}{4m(m+1)} . \tag{3.8}$$

We now notice that using Eqs. (3.3) and (3.8) we have

$$\chi_{1,s}(z) = \mathcal{F}_{\frac{1-s}{2}}(z) - \mathcal{F}_{\frac{1+s}{2}}(z) \qquad (s=1,2,\ldots) \tag{3.9}$$

where $\chi_{r,s}(z)$ is the character of the irreducible representation of highest weight $\Delta_{r,s}$ of the Virasoro algebra with central charge

$$c = 1 - \frac{6}{m(m+1)} , \qquad \Delta_{r,s} = \frac{[(m+1)r-ms]^2-1}{4m(m+1)} \tag{3.10}$$

(m irrational). Since the power expansions in z of the functions $\chi_{r,s}(z)$ have positive coefficients, we have shown that in the finite-size scaling limit, Rule 1 of Sec. 2 is valid. If $m=\frac{u}{v}$, u, v coprimes, we consider the partition functions:

$$\mathcal{G}_p(z) = \sum_{\alpha=-\infty}^{\infty} \mathcal{F}_{p+n\alpha}(z) \qquad (p=0,\ldots,n-1) \tag{3.11}$$

with $n=u+v$ and get

$$\chi_{1,s}(z) = \mathcal{G}_{\frac{1-s}{2}}(z) - \mathcal{G}_{\frac{1+s}{2}}(z) \qquad (s=1,\ldots,n-1) \tag{3.12}$$

This extends Rule 1 to the case of non-generic q (γ rational) again in the continuum limit. Our main observation is that Rule 1 does not work only in the continuum case but also in the discrete one (finite lattices). This observation is based on numerous numerical tests but we have not yet an algebraic explanation. The origin of Rule 2 (applying only to $d=0$) of Sec. 2 related to the existence of frozen (ϵ-independent) levels, is certainly related to the observation that if in each sector we consider only the frozen levels and count their number in a chain of N sites, we get precisely the dimensions of the irreducible representations of a Temperley-Lieb algebra

$$\begin{aligned} U_j U_{j+1} U_j &= U_j \qquad (j = 1, \ldots, N) \\ [U_i, U_j] &= 0 \qquad (|i-j| \geq 2) \\ U_j^2 &= \sqrt{4\cos^2(\pi\gamma)}\, U_j \end{aligned} \tag{3.13}$$

In the course of our attempts to understand the algebraic structure of Rule 1, we have discovered that the Hamiltonian

$$H = \alpha(P_1 - P_{N-1}P_N) + \beta(P_N - P_1P_2) + \gamma\left(\sum_{i=2}^{N-1} P_i - \sum_{i=2}^{N-2} P_iP_{i+1}\right) \tag{3.14}$$

where

$$P = \frac{1+\sigma^z}{2} \tag{3.15}$$

and α, β and γ are parameters also satisfies Rule 1 with $d = 0$. If $\alpha = \beta = \gamma$ Rule 1 is satisfied for any d ! May be this example will help the reader find an algebraic explanation of the degeneracy patterns observed in Sec. 2.

We want to conclude this talk with the remark, that the symmetry breaking patterns observed for the spin one-half chains will certainly also be found for higher spin $U_q[SU(2)]$ Hamiltonians [5] and for other quantum algebras.

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