First order coherent Boson states

Helv Phys Acta
Vol. 65 (1992)

0018-0238/92/070965-20\$1.50+0.20/0

First Order Coherent Boson States

Reinhard Honegger and Alfred Rieckers

Institut für Theoretische Physik, Universität Tübingen
W-7400 Tübingen, F.R.G.

(17. II. 1992)

Abstract

We analyze the first order coherent states (and various sub-classes) over an arbitrary C*-Weyl algebra. Their normally ordered characteristic functions are expressed in terms of an infinite, positive-definite matrix, which exhibits an additional positive-definiteness condition if the states are classical. In the latter case the matrix elements are identified as moments of probability measures on $\mathbb{C}$. By going over to measures on $b\mathbb{C}$, the Bohr compactification of $\mathbb{C}$, there arises a Bauer simplex of generalized classical coherent states. For macroscopic coherent states we give the GNS-representation, the central decomposition, and construct an extended Weyl formalism.

1 Introduction and preliminary results

A large part of quantum optics is based on peculiar many photon states which exhibit optical features. By means of the factorization of certain normally ordered correlation functions a state of the quantized radiation field is characterized to be optically coherent of some order [1] [2]. Using a smearing procedure in [3] [4] this coherence condition is tranferred into its operator algebraic version and takes account of the fact, that the correlation functions are in general distributions. The smeared field formalism allows a systematic study of both the Fock and non-Fock coherent states, independently of any special representation of the photonic Weyl algebra, whereas usually coherence is considered only in the Fock representation. Recently the fully coherent states (that is, coherent in all orders) have been investigated and completely classified in the non-Fock case [4]. In experimental situations, however, only the correlation functionals of order one and two are accessible [2]. Thus full coherence seems to be too strong a requirement.

In the present work we study in terms of the rigorous formalism of operator algebraic quantum mechanics the first order coherent states together with their sub-classes of n-th order coherent states. We start with the definition of n-th order coherence (where n is arbitrary in $\mathbb{N}\cup\{\infty\}$) for smeared Boson fields, demanding the factorization of the normally ordered expectation values up to degree n in terms of a linear form L on the testfunction space and strengthening infinite differentiability of [4] to analyticity. Their characteristic functions are expressed by an infinite, positive-definite matrix of complex coefficients. In general the conditions for this matrix have not been completely specified to lead to a state on the Weyl algebra. The more remarkable is the sufficient and necessary condition, which we express by a modified positive-definiteness condition on the coefficient matrix, to give a classical coherent state (Theorem 2.3). It is demonstrated that for unbounded L (with respect to the norm of the testfunction space) there are only classical coherent states, and that these are just those coherent states which are not given by a density operator in Fock space. Concerning the so-called P-representation of classical coherent states we derive a probability measure μ on $\mathbb{C}L$ (dropping henceforth the L) by expressing the mentioned matrix elements as mixed moments. If L is bounded it is given by a mode in the (norm closure of the) testfunction space. For unbounded L we consider it as a generalized mode. The measure μ of the P-representation describes then a global (position space independent) variation of the phase and amplitude for one (generalized) mode. The state space of the C*-Weyl algebra contains, however, also non-regular states. By using measures μ on $b\mathbb{C}$, the Bohr compactification of $\mathbb{C}$, we define classical, first order coherent states also for this extended class. This leads to the nice structure of a Bauer simplex and makes manifest a classical sub-theory for every linear form L.

For unbounded L we construct the GNS-representation of coherent states as a tensor product of the Fock representation and a classical field representation. The corresponding represented field operator decomposes additively into the Fock and classical part. The measure μ of the P-representation gives here not only the decomposition into extremal coherent states (with the same L) but also the central decomposition and a unique decomposition into pure states on the Weyl algebra. Since the unboundedness of L leads to a finite particle density in the infinite volume [5] we call the coherent states associated with such an L "macroscopic". The before mentioned results show, that for macroscopic coherent states the exact specification of the phase and amplitude of the classical field part makes the state pure as a quantum state.

For macroscopic coherent states L is not part of the original test function space but may added to it by a canonical procedure. This leads to an extended Weyl formalism, which we outline in the last section. Here we have still the approximability of the macroscopic classical field by the original represented quantized field, but we have also its independent variability, which corresponds to the direct preparation methods for the classical field.

Many results here parallel those for fully coherent states in [4], but the technical difficulties are more subtle here, in view of the unbounded measure spaces for the amplitude variations.

A certain redundancy in the set of correlation functionals for classical higher order coherent states is disclosed in [6]. In [7] the non-classical coherent states, which occur only in the case of a bounded linear form L, are supplemented.

Let us start our exposition with preliminary results concerning the Weyl algebra. By $\mathcal{W}(E)$ we denote the Weyl algebra over the pre-Hilbert space E with (right linear) scalar product $\langle . \mid . \rangle$. In applications E is called the one-boson testfunction space. $\mathcal{W}(E)$ is the unique C*-algebra generated by the (unitary) Weyl operators $W(f)$, $f \in E$, satisfying the Weyl relations ([8], Theorem 5.2.8)

$$W(f)\,W(g) = \exp\{-\tfrac{i}{2}\,\mathrm{Im}\langle f \mid g\rangle\}\,W(f+g)\,, \quad W(f)^* = W(-f) \quad \forall f,g \in E\,.$$

Let be $\mathcal{S}(\mathcal{W}(E))$ the state space of $\mathcal{W}(E)$, which is convex and weak*-compact. We denote by $\mathsf{C}(E)$ the convex set of functions $C : E \to \mathbb{C}$, with $C(0) = 1$ and for which the map $(f,g) \mapsto \exp\{\frac{i}{2}\,\mathrm{Im}\langle f \mid g\rangle\}C(g-f)$ constitutes a positive-definite kernel $E \times E \to \mathbb{C}$ ([10], cf. also [4] Appendix). From [4] it is known, that the map

$$C : \mathcal{S}(\mathcal{W}(E)) \longrightarrow \mathsf{C}(E)\,, \quad \omega \longmapsto C_\omega\,, \tag{1.1}$$

which maps each state $\omega \in \mathcal{S}(\mathcal{W}(E))$ onto its characteristic function

$$C_\omega : E \longrightarrow \mathbb{C}\,, \quad f \longmapsto C_\omega(f) := \langle \omega\,;\, W(f)\rangle\,,$$

is affine and bijective. The characteristic function C_F of the Fock state $\omega_F \in \mathcal{S}(\mathcal{W}(E))$ is given by $C_F(f) = \exp\{-\frac{1}{4}\|f\|^2\}$ $\forall f \in E$. Its GNS-representation $(\Pi_F, F_+(\overline{E}), \Omega_F)$ is given by the Bose-Fock space $F_+(\overline{E})$ over the completion $\overline{E}$ of E, the vacuum vector Ω_F and $\Pi_F(W(f)) = W_F(f)$ $\forall f \in E$, where $W_F(h)$, $h \in \overline{E}$, are the usual Fock-Weyl operators, [8] Section 5.2.

As a generalization of Glauber's P-representation with positive measures (cf. also [11] Section 8.2) we introduce the classical states $\mathcal{S}_{\rm cl}(\mathcal{W}(E))$ on $\mathcal{W}(E)$. Let be $\mathsf{P}(E)$ the convex set of positive-definite functions $P: E \to \mathbb{C}$ with $P(0) = 1$, [10]. Then (cf. [12] Section 3)

$$\mathcal{S}_{\rm cl}(\mathcal{W}(E)) := \{\omega \in \mathcal{S}(\mathcal{W}(E)) \mid C_\omega = C_F P_\omega \text{ with } P_\omega \in \mathsf{P}(E)\} .$$

Obviously, $\mathcal{S}_{\rm cl}(\mathcal{W}(E))$ is convex. If $P \in \mathsf{P}(E)$ then $C_F P \in \mathsf{C}(E)$ (cf. [4]) and hence the map

$$P : \mathcal{S}_{\rm cl}(\mathcal{W}(E)) \longrightarrow \mathsf{P}(E) , \quad \omega \longmapsto P_\omega \tag{1.2}$$

is affine and bijective. Using the Bochner theorem for the elements in $\mathsf{P}(E)$, one obtains the following result [12].

Proposition 1.1 *The set of classical states $\mathcal{S}_{\rm cl}(\mathcal{W}(E))$ is a Bauer simplex which is affinely homeomorphic to $M_+^1(\widehat{E})$, the set of regular probability Borel measures on the topological character group $\widehat{E}$ of the discrete additive group E.*

For a regular $\varphi \in \mathcal{S}(\mathcal{W}(E))$ the field, annihilation and creation operators $\Phi_\varphi(f)$, $a_\varphi(f)$ resp. $a_\varphi^*(f)$, $f \in E$, associated with the GNS-representation $(\Pi_\varphi, \mathcal{H}_\varphi, \Omega_\varphi)$ are defined in the usual manner ([8] Subsection 5.2.3, cf. also [3] or [4]). If $\varphi \in \mathcal{S}(\mathcal{W}(E))$ is of class C^{2m}, then the associated cyclic vector Ω_φ is contained in the domain of each polynomial of field operators with degree $\leq m$, in which case one commonly defines

$$\langle \varphi ; \Phi_\varphi(f_1) \cdots \Phi_\varphi(f_{2m}) \rangle := \langle \Phi_\varphi(f_m) \cdots \Phi_\varphi(f_1)\Omega_\varphi \mid \Phi_\varphi(f_{m+1}) \cdots \Phi_\varphi(f_{2m})\Omega_\varphi \rangle .$$

If $m = \infty$ all field correlations exists, which is desirable for the statistical interpretation. The C^∞ assumption was appropriate for the definition of a fully coherent state in [3] and [4], since in this case it implies analyticity (which means that $\mathbb{R} \ni t \mapsto C_\omega(tf)$ for each $f \in E$ is extensible to an analytic function in a neighbourhood of $t = 0$). This is different for finite order coherence, where analyticity has to be postulated.

Definition 1.2 *An analytic state $\omega \in \mathcal{S}(\mathcal{W}(E))$ is called coherent of n-th order or of degree $n \in \mathbb{N}$, if there exists a linear form $L : E \to \mathbb{C}$ so that*

$$\langle \omega ; a_\omega^*(f_1) \cdots a_\omega^*(f_m)\, a_\omega(g_1) \cdots a_\omega(g_m) \rangle = L(f_1) \cdots L(f_m)\, \overline{L(g_1)} \cdots \overline{L(g_m)}$$

for all $f_1, \ldots, f_m, g_1, \ldots, g_m \in E$ and each $1 \leq m \leq n$. The state ω is called fully-coherent or coherent of order ∞, if it is coherent in each order $n \in \mathbb{N}$. The set of all coherent states of degree $n \in \mathbb{N} \cup \{\infty\}$ with linear form $L : E \to \mathbb{C}$ is denoted by $S_L^{(n)}(\mathcal{W}(E))$.

Obviously $S_L^{(n)}(\mathcal{W}(E))$ is a convex subset of $\mathcal{S}(\mathcal{W}(E))$. The condition for $\omega \in S_L^{(n)}(\mathcal{W}(E))$ to be analytic is needed to recover from the normally ordered expectation values $\langle \omega ; a_\omega^*(f)^k\, a_\omega(f)^l \rangle$, $k, l \in \mathbb{N}_0$, the characteristic function $C_\omega(f)$ by means of the series expansions from [4] Lemma 1.1, which we recapitulate here.

Lemma 1.3 *Let $\varphi \in \mathcal{S}(\mathcal{W}(E))$. φ is analytic, if and only if for each $f \in E$ there is a function $N_\varphi(z_1, z_2; f)$, analytic in $U_f \times U_f \subseteq \mathbb{C}^2$ (U_f a neighborhood of the origin of $\mathbb{C}$), such that*

$$C_\varphi(zf) \;=\; C_F(zf)\, N_\varphi(z, \overline{z}; f) \quad \forall z \in U_f\,. \tag{1.3}$$

Especially φ is entire analytic, if and only if $N_\varphi(z_1, z_2; f)$ is entire analytic on $\mathbb{C}^2$ for every $f \in E$.

Moreover, the analyticity condition and (1.3) determine N_φ uniquely to have the form

$$\begin{aligned} N_\varphi(z_1, z_2; f) \;&=\; e^{\frac{1}{8}(z_1^2 - z_2^2 + 2z_1 z_2)\|f\|^2} \left\langle e^{-\frac{i}{2}(\overline{z_1}+\overline{z_2})\Phi_\varphi(f)}\Omega_\varphi \mid e^{-\frac{1}{2}(z_2 - z_1)\Phi_\varphi(if)}\Omega_\varphi \right\rangle & (1.4) \\ &=\; \sum_{k,l=0}^{\infty} \left(\frac{iz_1}{\sqrt{2}}\right)^k \left(\frac{iz_2}{\sqrt{2}}\right)^l \frac{1}{k!}\frac{1}{l!} \left\langle \varphi\, ;\, a_\varphi^*(f)^k a_\varphi(f)^l \right\rangle, \quad z_1, z_2 \in U_f. & (1.5) \end{aligned}$$

Clearly we have the following inclusions

$$\mathcal{S}_L^{(1)}(\mathcal{W}(E)) \supseteq \mathcal{S}_L^{(2)}(\mathcal{W}(E)) \supseteq \ldots \supseteq \mathcal{S}_L^{(\infty)}(\mathcal{W}(E))\,.$$

An equivalent formulation for $\omega \in \mathcal{S}(\mathcal{W}(E))$ being an element of $\mathcal{S}_L^{(n)}(\mathcal{W}(E))$ is given by the following

Lemma 1.4 *In terms of Definition 1.2 it holds:*

(a) The Fock state $\omega_F \in \mathcal{S}(\mathcal{W}(E))$ is fully-coherent with linear form $L \equiv 0$. Conversely, if $\langle \omega\, ;\, a_\omega^*(f)\, a_\omega(f)\rangle = 0\ \forall f \in E$ for an $\omega \in \mathcal{S}(\mathcal{W}(E))$ of class C^2, then $\omega = \omega_F$.

(b) For an analytic $\omega \in \mathcal{S}(\mathcal{W}(E))\backslash\{\omega_F\}$ and given linear form $L : E \longrightarrow \mathbb{C}$, $L \neq 0$, and given $n \in \mathbb{N} \cup \{\infty\}$ the following statements are equivalent:

(i) $\omega \in \mathcal{S}_L^{(n)}(\mathcal{W}(E))$;

(ii) $\langle \omega\, ;\, a_\omega^*(f)\, a_\omega(f)\rangle = |L(f)|^2\ \forall f \in E$, and for each $2 \le m \le n$ there is an $h \notin \ker(L)$ with $\langle \omega\, ;\, a_\omega^*(h)^m a_\omega(h)^m\rangle = |L(h)|^{2m}$.

PROOF: (a): The fact that ω_F is fully-coherent with linear form $L \equiv 0$ follows from [4]. Conversely, from $\|a_\omega(f)\,\Omega_\omega\|^2 = \langle \omega\, ;\, a_\omega^*(f)\, a_\omega(f)\rangle = 0$ we conclude $a_\omega(f)\,\Omega_\omega = 0\ \forall f \in E$. Now [4] Proposition 2.1 (b) implies $\omega = \omega_F$. (b): From $\langle \omega\, ;\, a_\omega^*(f)\, a_\omega(f)\rangle = |L(f)|^2\ \forall f \in E$ with the help of the polarization identity we get $\langle \omega\, ;\, a_\omega^*(f)\, a_\omega(g)\rangle = L(f)\,\overline{L(g)}$, which by Lemma 2.2 of [4] implies $\overline{L(g)}\, a_\omega(f)\,\Omega_\omega = \overline{L(f)}\, a_\omega(g)\,\Omega_\omega$ $\forall f, g \in E$. Consequently, for the above $h \notin \ker(L)$ we obtain

$$\begin{aligned} |L(h)|^{2m} \langle \omega\, ;\, a_\omega^*(f_1)\cdots a_\omega^*(f_m)\, a_\omega(g_1)\cdots a_\omega(g_m)\rangle \;&=\; \\ =\; L(f_1)\cdots L(f_m)\,\overline{L(g_1)}\cdots\overline{L(g_m)}\ &\langle \omega\, ;\, a_\omega^*(h)^m\, a_\omega(h)^m\rangle\,. \end{aligned} \tag{1.6}$$

and the assertion follows. ∎

2 Matrix representations and positive-definiteness

A matrix $c : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{C}, (k,l) \mapsto c(k,l)$ is defined to be *analytic*, if

$$\sum_{k,l=0}^{\infty} \frac{\delta^{k+l}}{k!\,l!}\, |c(k,l)| \;<\; \infty \qquad \text{for some } \delta > 0\,.$$

A matrix c constitutes a *positive-definite kernel* on $\mathbb{N}_0 \times \mathbb{N}_0$, if [10]

$$\sum_{k,l=0}^{N} \overline{\beta_k}\,\beta_l\, c(k,l) \;\geq\; 0 \qquad \forall \beta_1,\ldots,\beta_N \in \mathbb{C}\,, \quad \forall N \in \mathbb{N}\,. \tag{2.1}$$

Clearly, these expressions coincide with $(\beta|c\beta) \geq 0$, where $\beta = (\beta_1,\ldots,\beta_N,0,0,\ldots)$ and $(.|.)$ is the usual scalar product on the sequence space $l^2(\mathbb{N})$. Hence positive-definiteness of the kernel $c : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ is just positivity of the matrix c which acts (as usual) on the subspace of $l^2(\mathbb{N})$ consisting of finite vectors. Specifying (2.1) to vectors $\beta = (0,\ldots,0,\beta_k,0,\ldots,0,\beta_l,0,\ldots)$ we see that the matrix $\begin{pmatrix} c(k,k) & c(k,l) \\ c(l,k) & c(l,l) \end{pmatrix}$ must be positive and therefore selfadjoint with positive determinant, which implies

$$\overline{c(k,l)} \;=\; c(l,k)\,, \quad |c(k,l)|^2 \;\leq\; c(k,k)\,c(l,l)\,, \qquad \forall\, k,l \in \mathbb{N}_0\,. \tag{2.2}$$

For $n \in \mathbb{N}_0 \cup \{\infty\}$ let us define

$$\mathbb{M}^{(n)}_{\text{posker}} \;:=\; \{c : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C} \mid c \text{ analytic, satisfying (2.1)}, c(0,0) = \ldots = c(n,n) = 1\}\,. \tag{2.3}$$

The set $\mathbb{M}^{(n)}_{\text{posker}}$ is convex, and

$$\mathbb{M}^{(0)}_{\text{posker}} \supseteq \mathbb{M}^{(1)}_{\text{posker}} \supseteq \mathbb{M}^{(2)}_{\text{posker}} \supseteq \ldots \supseteq \mathbb{M}^{(\infty)}_{\text{posker}}\,.$$

Proposition 2.1 *Let be $n \in \mathbb{N} \cup \{\infty\}$ and $L : E \to \mathbb{C}$ a non-zero linear form. Then there exists a unique affine, injective map*

$$c : \mathcal{S}^{(n)}_L(\mathcal{W}(E)) \longrightarrow \mathbb{M}^{(0)}_{\text{posker}}\,, \quad \omega \longmapsto c_\omega$$

such that

$$C_\omega(f) \;=\; C_F(f) \sum_{k,l=0}^{\infty} \left(\frac{i}{\sqrt{2}}\right)^{k+l} \frac{1}{k!}\,\frac{1}{l!}\, L(f)^k\, \overline{L(f)}^{\,l}\, c_\omega(k,l)\,, \tag{2.4}$$

which converges absolutely for all $f \in E$ with $|L(f)| < \delta$ for some $\delta > 0$. Moreover,

$$c\Big(\mathcal{S}^{(n)}_L(\mathcal{W}(E))\Big) \;\subseteq\; \mathbb{M}^{(n)}_{\text{posker}}\,,$$

and for each $\omega \in \mathcal{S}^{(n)}_L(\mathcal{W}(E))$ one has

$$\langle \omega\,;\, a^*_\omega(f_1)\cdots a^*_\omega(f_k)\, a_\omega(g_1)\cdots a_\omega(g_l)\rangle \;=\; L(f_1)\cdots L(f_k)\,\overline{L(g_1)}\cdots\overline{L(g_l)}\, c_\omega(k,l) \tag{2.5}$$

for all $f_1,\ldots,f_k,g_1,\ldots,g_l \in E$ and each $k,l \in \mathbb{N}_0$.

PROOF: For $\omega \in \mathcal{S}^{(n)}_L(\mathcal{W}(E))$ and $h \in E$ with $L(h) = 1$ we put

$$c_\omega : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{C}\,, \quad (k,l) \longmapsto \langle \omega\,;\, a^*_\omega(h)^k a_\omega(h)^l\rangle\,. \tag{2.6}$$

Obviously c_ω is a positive-definite kernel, and $c_\omega(0,0) = \ldots = c_\omega(n,n) = 1$ by the coherence of ω of degree n. ω being analytic we insert (2.6) into equation (1.5) of Lemma 1.3 and obtain a $\delta > 0$ such that the double series

$$N_\omega(z_1,z_2;h) \;=\; \sum_{k,l=0}^{\infty} \left(\frac{iz_1}{\sqrt{2}}\right)^k \left(\frac{iz_2}{\sqrt{2}}\right)^l \frac{1}{k!}\,\frac{1}{l!}\, c_\omega(k,l)$$

converges absolutely for all $z_1, z_2 \in \mathbb{C}$ with $|z_1|, |z_2| < \delta$. Now let $f \in E$ with $|L(f)| < \delta$. Inserting (2.6) into [4] eq. (2.2) (resp. into eq. (1.6)) we find that (2.6) is independent of the special choice of h and occurs in every normally ordered expectation (2.5). By means of Lemma 1.3 and the fact that $t \in \mathbb{R} \mapsto \langle \omega ; W(tf) \rangle$ is actually analytic in an open strip around the real axis (cf. [8], p. 39) we arrive at the stated series expansion of C_ω.

The injectivity of the map c: Let there be another matrix $c'_\omega \in \mathbb{M}^{(0)}_{\text{posker}}$ for which the expansion of C_ω holds for some $\delta' > 0$. Then for arbitrary $f \in E$ we get

$$N_\omega(z_1, z_2; f) = \sum_{k,l=0}^{\infty} \left(\frac{iz_1}{\sqrt{2}}\right)^k \left(\frac{iz_2}{\sqrt{2}}\right)^l \frac{1}{k!} \frac{1}{l!} L(f)^k \overline{L(f)}^l c'_\omega(k.l)$$

for the analytic function associated with ω by Lemma 1.3 (where the series converges for $|z_1|.|z_2| < |L(f)|^{-1} \delta'$) from which follows $L(f)^k \overline{L(f)}^l c'_\omega(k,l) = \langle \omega ; a^*_\omega(f)^k a_\omega(f)^l \rangle \; \forall f \in E$. Hence with the above h and (2.6) we get $c'_\omega(k,l) = c_\omega(k,l) \; \forall k, l \in \mathbb{N}_0$. The affinity of the map c is immediate from (2.6). ■

The range of the map c of Proposition 2.1 is not all of $\mathbb{M}^{(n)}_{\text{posker}}$, a problem which in the case of Fock-normal $\omega \in \mathcal{S}^{(n)}_L(\mathcal{W}(E))$ seems to be related to the P-representation with non-positive measures. To investigate the range of $c : \mathcal{S}^{(n)}_L \mathcal{W}(E)) \to \mathbb{M}^{(0)}_{\text{posker}}$ we need a stronger form of positive definiteness. By means of the addition $(k,l) + (m,n) = (k+m, l+n)$, the neutral element $(0,0)$, and the involution $I(m,n) := (n,m)$, the set $\mathbb{N}_0 \times \mathbb{N}_0$ becomes an involutive semigroup, which we denote by $(\mathbb{N}_0 \times \mathbb{N}_0, I)$. According to [10], Definition 2.3, the matrix $c : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ is a *positive-definite function* on $(\mathbb{N}_0 \times \mathbb{N}_0, I)$, if for each $N \in \mathbb{N}$

$$\sum_{i,j=1}^{N} \overline{\alpha_i}\, \alpha_j \, c(I(p_i) + p_j) \geq 0 \quad \forall p_1, \ldots, p_N \in \mathbb{N}_0 \times \mathbb{N}_0 \; \forall \alpha_1, \ldots, \alpha_N \in \mathbb{C}\,. \tag{2.7}$$

By use of the natural indices of $\mathbb{N}_0 \times \mathbb{N}_0$ (2.7) is written as

$$\sum_{k,l,m,n=0}^{M} \overline{\alpha_{kl}}\, \alpha_{mn} \, c(l+m, k+n) \geq 0 \quad \forall \alpha_{ij} \in \mathbb{C} \text{ with } i,j \in \{0,1,\ldots,M\} \tag{2.8}$$

for each $M \in \mathbb{N}$. Similar to (2.3) we define for $n \in \mathbb{N}_0 \cup \{\infty\}$

$$\mathbb{M}^{(n)}_{\text{posfun}} := \{c : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C} \mid c \text{ analytic, satisfying (2.8)}, c(0,0) = \ldots = c(n,n) = 1\}\,. \tag{2.9}$$

Setting $\alpha_{kl} = 0$ for $l \neq 0$ and $\alpha_{k,0} = \overline{\beta_k}$ one regains from (2.8) the relation (2.1), and thus

$$\mathbb{M}^{(n)}_{\text{posfun}} \subseteq \mathbb{M}^{(n)}_{\text{posker}} \qquad \forall n \in \mathbb{N}_0 \cup \{\infty\}\,.$$

Clearly $\mathbb{M}^{(n)}_{\text{posfun}}$ is convex, and

$$\mathbb{M}^{(0)}_{\text{posfun}} \supseteq \mathbb{M}^{(1)}_{\text{posfun}} \supseteq \mathbb{M}^{(2)}_{\text{posfun}} \supseteq \ldots \supseteq \mathbb{M}^{(\infty)}_{\text{posfun}}\,.$$

In the following we often need a simple fact, which we here formulate more stringent than in [4].

Lemma 2.2 *If $L : E \to \mathbb{C}$ is unbounded, then there is for every $\alpha \in \mathbb{C}$ a sequence $(f^\alpha_n)_{n \in \mathbb{N}}$ in E with*

$$\lim_{n\to\infty} \|f^\alpha_n\| = 0\,, \quad \textit{and} \qquad L(f^\alpha_n) = \alpha \quad \forall n \in \mathbb{N}\,.$$

PROOF: Since L is unbounded, there is a sequence $(g_n)_{n\in\mathbb{N}}$ in E with $\|g_n\| = 1$ and $\lim_{n\to\infty} L(g_n) = \infty$. Without restriction in generality $L(g_n) \neq 0\ \forall n \in \mathbb{N}$, and we may define $f_n^\alpha := \frac{\alpha g_n}{L(g_n)}$. ■

Theorem 2.3 *Let be all as in Proposition 2.1. We have for each $n \in \mathbb{N} \cup \{\infty\}$:*

(a) $c\Big(\mathcal{S}_L^{(n)}(\mathcal{W}(E)) \cap \mathcal{S}_{\rm cl}(\mathcal{W}(E))\Big) \;=\; \mathbb{M}^{(n)}_{\rm posfun}.$

(b) Let the linear form $L : E \to \mathbb{C}$ be unbounded (with respect to the norm on E). Then

$$\mathcal{S}_L^{(n)}(\mathcal{W}(E)) \;\subseteq\; \mathcal{S}_{\rm cl}(\mathcal{W}(E))\,,$$

from which by (a) follows

$$c\Big(\mathcal{S}_L^{(n)}(\mathcal{W}(E))\Big) \;=\; \mathbb{M}^{(n)}_{\rm posfun}\,.$$

PROOF: (a) will be proved after Theorem 3.1. (b): Let $L : E \to \mathbb{C}$ be unbounded and $\omega \in \mathcal{S}_L^{(n)}(\mathcal{W}(E))$. (2.5) implies $c_\omega(k,l) = \Big\langle \omega\,;\, a_\omega^*\big(\tfrac{f}{L(f)}\big)^k\, a_\omega\big(\tfrac{f}{L(f)}\big)^l \Big\rangle\ \forall k,l \in \mathbb{N}_0\ \forall f \notin \ker(L)$. With the canonical commutation relations (CCR) we get for $\alpha_{ij} \in \mathbb{C}$, $0 \le i,j \le M$

$$\begin{aligned}
&\sum_{k,l,m,n=0}^{M} \overline{\alpha_{kl}}\,\alpha_{mn}\, c_\omega(l+m, k+n) \;= \\
&= \sum_{k,l,m,n=0}^{M} \overline{\alpha_{kl}}\,\alpha_{mn}\,\Big\langle \omega\,;\, a_\omega^*\Big(\tfrac{f}{L(f)}\Big)^{l+m} a_\omega\Big(\tfrac{f}{L(f)}\Big)^{k+n}\Big\rangle \\
&= \sum_{k,l,m,n=0}^{M} \overline{\alpha_{kl}}\,\alpha_{mn}\,\Big\langle \omega\,;\, a_\omega^*\Big(\tfrac{f}{L(f)}\Big)^{l} a_\omega\Big(\tfrac{f}{L(f)}\Big)^{k} a_\omega^*\Big(\tfrac{f}{L(f)}\Big)^{m} a_\omega\Big(\tfrac{f}{L(f)}\Big)^{n}\Big\rangle \;+ \\
&\qquad + \frac{\|f\|^2}{|L(f)|^2}\, P\Big(\tfrac{\|f\|}{|L(f)|}\Big) \\
&= \Big\langle \sum_{k,l=0}^{M} \alpha_{kl}\, a_\omega^*\Big(\tfrac{f}{L(f)}\Big)^{k} a_\omega\Big(\tfrac{f}{L(f)}\Big)^{l}\,\Omega_\omega \;\Big|\; \sum_{m,n=0}^{M} \alpha_{mn}\, a_\omega^*\Big(\tfrac{f}{L(f)}\Big)^{m} a_\omega\Big(\tfrac{f}{L(f)}\Big)^{n}\,\Omega_\omega \Big\rangle + \\
&\qquad + \frac{\|f\|^2}{|L(f)|^2}\, P\Big(\tfrac{\|f\|}{|L(f)|}\Big)\,,
\end{aligned}$$

where P is a complex polynomial, which one obtains by successive use of the CCR. e.g.

$$\begin{aligned}
&\Big\langle \omega\,;\, \Big(a_\omega^*\Big(\tfrac{f}{L(f)}\Big)^{l+m} a_\omega\Big(\tfrac{f}{L(f)}\Big)^{k+n}\Big)\Big\rangle \;= \\
&\overset{\rm CCR}{=} \Big\langle \omega\,;\, \Big(a_\omega^*\Big(\tfrac{f}{L(f)}\Big)^{l+m-1} a_\omega\Big(\tfrac{f}{L(f)}\Big)\, a_\omega^*\Big(\tfrac{f}{L(f)}\Big)\, a_\omega\Big(\tfrac{f}{L(f)}\Big)^{k+n-1}\Big)\Big\rangle \;- \\
&\qquad - \frac{\|f\|^2}{|L(f)|^2}\, c_\omega(l+m-1, k+n-1)
\end{aligned}$$

and so on... . Altogether

$$\begin{aligned}
&\sum_{k,l,m,n=0}^{M} \overline{\alpha_{kl}}\,\alpha_{mn}\, c_\omega(l+m, k+n) \;= \qquad\qquad (2.10)\\
&= \Big\| \sum_{k,l=0}^{M} \alpha_{kl}\, a_\omega^*\Big(\tfrac{f}{L(f)}\Big)^{k} a_\omega\Big(\tfrac{f}{L(f)}\Big)^{l}\,\Omega_\omega \Big\|^2 \;+\; \frac{\|f\|^2}{|L(f)|^2}\, P\Big(\tfrac{\|f\|}{|L(f)|}\Big) \qquad \forall f \notin \ker(L)\,.
\end{aligned}$$

Because L is unbounded by Lemma 2.2 there is a sequence $(f_n)_{n\in\mathbb{N}}$ in E with $\lim_{n\to\infty} \|f_n\| = 0$ and $\lim_{n\to\infty} |L(f_n)| = 1$. Since the left hand side of (2.10) is independent on the sequence $(f_n)_{n\in\mathbb{N}}$, the limit

$$\lim_{n\to\infty} \Big\| \sum_{k,l=0}^{M} \alpha_{kl}\, a_\omega^*\Big(\tfrac{f_n}{L(f_n)}\Big)^k a_\omega\Big(\tfrac{f_n}{L(f_n)}\Big)^l \Omega_\omega \Big\|^2 \geq 0$$

has to exist. Consequently (2.8) is valid, showing $c_\omega \in \mathbb{M}^{(n)}_{\text{posfun}}$. Let be $\mu_\omega \in M^1_+(\mathbb{C})$ the probability measure on $\mathbb{C}$ of Lemma A.1 (Appendix) associated with c_ω. From the expansion of Proposition 2.1 it follows $\omega \in \mathcal{S}_{\text{cl}}(\mathcal{W}(E))$ with $P_\omega(f) = \int_{\mathbb{C}} \exp\{i\sqrt{2}\,\mathrm{Re}(zL(f))\}\,\mathrm{d}\mu_\omega(z)$, $\forall f \in E$. ∎

If the linear form $L : E \to \mathbb{C}$ is bounded there are examples of pure states $\omega \in \mathcal{S}_L^{(\infty)}(\mathcal{W}(E))$ which are not classical, cf. [13] [5]. Similar to [4], Proposition 2.5, the boundedness resp. unboundedness of L has essential consequences concerning the Fock-normality of $\omega \in \mathcal{S}_L^{(n)}(\mathcal{W}(E))$.

Proposition 2.4 *$\omega \in \mathcal{S}_L^{(1)}(\mathcal{W}(E))$ is normal to the Fock representation, if and only if the linear form $L : E \to \mathbb{C}$ is bounded.*

PROOF: Immediate consequence of Theorem 2.3 and Proposition 3.4 below. ∎

3 Measures and simplices

For each $\alpha \in \mathbb{C}$ define the periodic function (which is in fact a character on the additive group $\mathbb{C}$)

$$\xi_\alpha : \mathbb{C} \longrightarrow \mathbb{C}\,, \quad z \longmapsto \exp\{i\sqrt{2}\,\mathrm{Re}(\alpha z)\}\,. \tag{3.1}$$

By Theorem 2.3 and Theorem A.1 (Appendix) each $\omega \in \mathcal{S}_L^{(n)}(\mathcal{W}(E)) \cap \mathcal{S}_{\text{cl}}(\mathcal{W}(E))$ corresponds to a unique $\mu_\omega \in M^1_+(\mathbb{C})$ (here $M^1_+(X)$ denotes the regular probability Borel measures on the topological space X) such that

$$P_\omega(f) = \mu_\omega(\xi_{L(f)}) \quad \forall f \in E\,. \tag{3.2}$$

If $w(z) \equiv 1$ and A denotes the normal operator $(A\xi)(z) = z\xi(z)$ $\forall z \in \mathbb{C}$, $\xi \in \mathrm{L}^2(\mathbb{C}, \mu_\omega)$, then it follows from the definitions

$$P_\omega(f) = \Big\langle w \mid \exp\{\tfrac{i}{\sqrt{2}}(L(f)A + \overline{L(f)}A^*)\}w \Big\rangle \quad \text{and} \quad \mathrm{d}\mu_\omega(z) = \langle w \mid \mathrm{d}G(z)w\rangle \tag{3.3}$$

with the spectral family $z \mapsto G(z)$ of A. Obviously $1 = c_\omega(k,k) = \int_{\mathbb{C}} |z|^{2k}\mathrm{d}\mu_\omega(z) = \|A^k w\|^2$ for all $0 \leq k \leq n$.

Starting from (3.2) we generalize the notion of a classical coherent state with a fixed non-zero linear form $L : E \to \mathbb{C}$. The functions $\{\xi_\alpha \mid \alpha \in \mathbb{C}\}$ generate the C*-algebra $\mathsf{AP}(\mathbb{C})$ of the continuous almost periodic functions on $\mathbb{C}$ (cf. Section 5). Each $\eta \in \mathsf{AP}(\mathbb{C})$ extends uniquely to a continuous function $\eta^{(b)} \in \mathcal{C}(b\mathbb{C})$ on the Bohr compactification $b\mathbb{C}$ of $\mathbb{C}$, in which sense $\mathsf{AP}(\mathbb{C}) \cong \{\eta^{(b)} \mid \eta \in \mathsf{AP}(\mathbb{C})\} = \mathcal{C}(b\mathbb{C})$ (cf. [17] (26.11), (33.18), (33.19), and (33.26)). Each $\nu \in M^1_+(\mathbb{C})$ gives a unique $\nu^{(b)} \in M^1_+(b\mathbb{C})$ with $\int_{\mathbb{C}} \eta(z)\,\mathrm{d}\nu(z) = \int_{b\mathbb{C}} \eta^{(b)}(z)\,\mathrm{d}\nu^{(b)}(z)$ $\forall \eta \in \mathsf{AP}(\mathbb{C})$. (We mention that the canonical embedding of $\mathbb{C}$ into $b\mathbb{C}$ is an open map, in which sense $\mathbb{C} \subset b\mathbb{C}$ and $M^1_+(\mathbb{C}) \subset M^1_+(b\mathbb{C})$.) Thus the state space of $\mathsf{AP}(\mathbb{C}) \cong \mathcal{C}(b\mathbb{C})$ is just $M^1_+(b\mathbb{C})$. Obviously, if $\rho \in M^1_+(b\mathbb{C})$, then $f \in E \mapsto \rho(\xi^{(b)}_{L(f)})$ is an element of $\mathsf{P}(E)$ and hence by (1.2) defines a classical state on $\mathcal{W}(E)$, which leads to the convex subset of $\mathcal{S}_{\text{cl}}(\mathcal{W}(E))$:

$$\mathcal{S}_L^{\text{cl}}(\mathcal{W}(E)) := \Big\{\omega \in \mathcal{S}_{\text{cl}}(\mathcal{W}(E)) \mid P_\omega(f) = \mu_\omega(\xi^{(b)}_{L(f)})\,\forall f \in E \text{ for some } \mu_\omega \in M^1_+(b\mathbb{C})\Big\}. \tag{3.4}$$

Theorem 3.1 *Let $L : E \to \mathbb{C}$ be a non-zero linear form. Then the map*

$$\mu : \mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E)) \longrightarrow M_+^1(b\mathbb{C})\,, \quad \omega \longmapsto \mu_\omega$$

is an affine homeomorphism with respect to the weak-topologies, showing $\mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))$ to be a Bauer simplex. Moreover for $\omega \in \mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))$ we have*

(a) ω is regular, if and only if $\mu_\omega \in M_+^1(\mathbb{C})$ (i.e. $\mathrm{supp}(\mu_\omega) \subseteq \mathbb{C}$).

(b) ω is of class C^{2m} for $m \in \mathbb{N}$, if and only if $\mu_\omega \in M_+^1(\mathbb{C})$ and $\int_{\mathbb{C}} |z|^{2m}\, \mathrm{d}\mu_\omega(z) < \infty$. In this case one has (cf. equation 2.5)

$$\begin{aligned}\langle \omega\,;\, a_\omega^*(f_1)\cdots a_\omega^*(f_k)\, a_\omega(g_1)\cdots a_\omega(g_l)\rangle &= \\ &= \left(\int_{\mathbb{C}} z^k \bar{z}^l\, \mathrm{d}\mu_\omega(z)\right) L(f_1)\cdots L(f_k)\,\overline{L(g_1)}\cdots\overline{L(g_l)}\end{aligned}$$

for all $f_1,\dots,f_k,g_1,\dots,g_l \in E$ and each $0 \le k,l \le m$.

(c) ω is (entire-)analytic, if and only if $\mu_\omega \in M_+^1(\mathbb{C})$ and $\int_{\mathbb{C}} \mathrm{e}^{\gamma|z|}\, \mathrm{d}\mu_\omega(z) < \infty$ for some (all) $\gamma > 0$.

PROOF: Since $\{\xi_\alpha \mid \alpha \in \mathbb{C}\}$ is total in $\mathsf{AP}(\mathbb{C}) \cong C(b\mathbb{C})$ two measures $\rho, \nu \in M_+^1(b\mathbb{C})$ agree, if and only if $\rho(\xi_\alpha^{(b)}) = \nu(\xi_\alpha^{(b)})$ $\forall \alpha \in \mathbb{C}$, from which follows the bijectivity of the map μ by observing $L \neq 0$ and the bijectivity of the map C of (1.1). Clearly μ is affine. The continuity of μ, resp. μ^{-1}, follows from the fact that $\rho_i \stackrel{i\in I}{\to} \rho$ in the weak*-topology of $M_+^1(b\mathbb{C})$, if and only if $\rho_i(\xi_\alpha^{(b)}) \stackrel{i\in I}{\to} \rho(\xi_\alpha^{(b)})$ $\forall \alpha \in \mathbb{C}$, and $\omega_i \stackrel{i\in I}{\to} \omega$, if and only if $\langle \omega_i\,;\, W(f)\rangle \stackrel{i\in I}{\to} \langle \omega\,;\, W(f)\rangle$ $\forall f \in E$. Since $M_+^1(b\mathbb{C})$ is a Bauer simplex, the map μ^{-1} transfers this property to $\mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))$.

(a): ω is regular, if and only if $t \in \mathbb{R} \mapsto P_\omega(tf) = \mu_\omega(\xi_{tL(f)}^{(b)})$ is continuous for each $f \in E$. Now let $\widehat{\xi}_\alpha$ be the multiplication operator $f \mapsto \xi_\alpha^{(b)} f$ on $\mathsf{L}^2(b\mathbb{C}, \mu_\omega)$. Observing $\left|\xi_\gamma^{(b)}(z)\right| = 1$ $\forall z \in b\mathbb{C}$ and each $\gamma \in \mathbb{C}$ we obtain

$$\left\|\left(\widehat{\xi}_{t\alpha} - \widehat{\xi}_0\right)\xi_\gamma^{(b)}\right\|^2 = \int_{b\mathbb{C}} \left|\xi_{t\alpha}^{(b)} - 1\right|^2 \left|\xi_\gamma^{(b)}\right|^2 \mathrm{d}\mu_\omega = 2 - \mu_\omega(\xi_{-t\alpha}^{(b)}) - \mu_\omega(\xi_{t\alpha}^{(b)}) \stackrel{t\to 0}{\longrightarrow} 0\,.$$

$\mathsf{LH}\{\xi_\gamma^{(b)} \mid \gamma \in \mathbb{C}\}$ being dense in $\mathsf{L}^2(b\mathbb{C}, \mu_\omega)$ by an $\frac{\varepsilon}{3}$-argument the strong continuity of the unitary one-parameter group $\widehat{\xi}_{t\alpha}$, $t \in \mathbb{R}$, follows. Hence $(t_1, t_2) \in \mathbb{R}^2 \mapsto \widehat{\xi}_{t_1}\widehat{\xi}_{it_2} = \widehat{\xi}_{t_1+it_2}$ is strongly continuous, from which the continuity of $\alpha \in \mathbb{C} \mapsto \left\langle w \mid \widehat{\xi}_\alpha w\right\rangle = \mu_\omega(\xi_\alpha^{(b)}) =: F_\omega(\alpha)$ follows (here, $w(z) \equiv 1 \in \mathsf{L}^2(b\mathbb{C}, \mu_\omega)$). However, Bochner's theorem for the positive-definite, continuous function $F_\omega : \mathbb{C} \to \mathbb{C}$ ensures the existence of a measure $\widetilde{\mu}_\omega \in M_+^1(\mathbb{C})$ with $F_\omega(\alpha) = \widetilde{\mu}_\omega(\xi_\alpha)$ $\forall \alpha \in \mathbb{C}$. Consequently, by the above argument $\mu_\omega = \widetilde{\mu}_\omega \in M_+^1(\mathbb{C})$.

(b): By $(Af)(z) := zf(z)$ is defined the normal operator A in $\mathsf{L}^2(\mathbb{C}, \mu_\omega)$. Let $w(z) \equiv 1$. ω is C^{2m} if and only if $t \in \mathbb{R} \mapsto \mu_\omega(\xi_{t\alpha}) = \left\langle w \mid \exp\{\frac{i}{\sqrt{2}}t(\alpha A + \overline{\alpha}A^*)\}w\right\rangle$ is so for all $\alpha \in \mathbb{C}$. By the spectral calculus this is equivalent to $w \in \mathcal{D}((\alpha A + \overline{\alpha}A^*)^m)$ $\forall \alpha \in \mathbb{C}$. Especially for $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{2i}$ we have $w \in \mathcal{D}(S^m)$ and $w \in \mathcal{D}(T^m)$ for the selfadjoint operators $S = \frac{1}{2}(A + A^*)$ and $T = \frac{1}{2i}(A - A^*)$. Since $A = S + iT$ and $A^* = S - iT$ we obtain $w \in \mathcal{D}(A^m)$, from which by $\|A^m w\|^2 = \int_{\mathbb{C}} |z|^{2m}\, \mathrm{d}\mu_\omega(z)$ the assertion follows. The stated normally ordered expectation values follow by differentiation and with the CCR (cf. equation (2.5) and the Appendix).

(c): Similar to (b) one gets, that ω is (entire-)analytic, if and only if w is an (entire-)analytic vector for S and T, and hence for A, resp. $|A| = \sqrt{A^*A} = |A^*|$ (since $\|A^m w\|^2 = \int |z|^{2m}\, \mathrm{d}\mu_\omega(z) = \||A|^m z\|^2$). But that is $w \in \mathcal{D}(\mathrm{e}^{\delta|A|})$ for some (all) $\delta > 0$. ∎

As a consequence, the non-regular states in $\mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))$ are just given by the measures $\rho \in M_+^1(b\mathbb{C})\backslash M_+^1(\mathbb{C})$, that is $\rho(b\mathbb{C}\backslash\mathbb{C}) \neq 0$.

As a corollary we give the

PROOF OF THEOREM 2.3(a): First we show $\mathbb{M}^{(n)}_{\mathrm{posfun}} \subseteq c\Big(\mathcal{S}_L^{(n)}(\mathcal{W}(E)) \cap \mathcal{S}_{\mathrm{cl}}(\mathcal{W}(E))\Big)$. Let $c \in \mathbb{M}^{(n)}_{\mathrm{posfun}}$ and denote by $\rho \in M_+^1(\mathbb{C})$ the associated measure of Theorem A.1 of the appendix satisfying $c(k,l) = \int_{\mathbb{C}} \overline{z}^l z^k \mathrm{d}\rho(z)$ $\forall k,l \in \mathbb{N}_0$. Now define the state $\omega \in \mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))$ by $P_\omega(f) = \rho(\xi_{L(f)})$ $\forall f \in E$, that is $\rho = \mu_\omega$. By Theorem 3.1 (b) and (c) we have $\omega \in \mathcal{S}_L^{(n)}(\mathcal{W}(E))$ because of $1 = c(k,k) = \int |z|^{2k}\, \mathrm{d}\rho(z)$ $\forall 0 \leq k \leq n$. On the other hand, $\int \mathrm{e}^{\gamma|z|}\mathrm{d}\rho(z) < \infty$ for some $\gamma > 0$ (Theorem A.1) implies

$$\rho(\xi_\alpha) = \int_{\mathbb{C}} \exp\Big\{\tfrac{i}{\sqrt{2}}\alpha z\Big\} \exp\Big\{\tfrac{i}{\sqrt{2}}\overline{\alpha z}\Big\}\, \mathrm{d}\rho(z) = \sum_{k,l=0}^{\infty} \Big(\tfrac{i}{\sqrt{2}}\Big)^{k+l} \frac{\alpha^k}{k!}\,\frac{\overline{\alpha}^l}{l!}\, c(k,l)$$

for all $\alpha \in \mathbb{C}$ with $|\alpha| \leq 2^{-\frac{1}{2}}\gamma$, from which with Proposition 2.1 follows $c = c_\omega$.

Now we prove $c\Big(\mathcal{S}_L^{(n)}(\mathcal{W}(E)) \cap \mathcal{S}_{\mathrm{cl}}(\mathcal{W}(E))\Big) \subseteq \mathbb{M}^{(n)}_{\mathrm{posfun}}$. Assume $\omega \in \mathcal{S}_L^{(n)}(\mathcal{W}(E)) \cap \mathcal{S}_{\mathrm{cl}}(\mathcal{W}(E))$. According Proposition 2.1 for $u \in \mathbb{C}$ with $|u| < \delta$ we define the function

$$\Gamma_\omega(u) := \sum_{k,l=0}^{\infty} \left(\frac{i}{\sqrt{2}}\right)^{k+l} \frac{u^k}{k!}\,\frac{\overline{u}^l}{l!}\, c_\omega(k,l)\,, \qquad |u| < \delta\,.$$

Obviously $P_\omega(f) = \Gamma_\omega(L(f))$ for each $f \in E$ with $|L(f)| < \delta$. Since ω is analytic, for each $f \in E$ the map $t \in \mathbb{R} \mapsto C_\omega(tf)$ extends to a holomorphic function $z \mapsto C_\omega(z;f)$ in a strip around the real axis (cf. [8] p. 39). Hence $t \in \mathbb{R} \mapsto P_\omega(tf)$ extends to a holomorphic function $z \mapsto \exp\{\frac{z^2}{4}\|f\|^2\}C_\omega(z;f)$ in the same strip. Now let $\alpha \in \mathrm{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. For each $g \in E$ with $L(g) = \alpha$ the map $t \in]-\delta,\delta[\mapsto P_\omega(tg) = \Gamma_\omega(t\alpha)$ has a holomorphic extension $z \mapsto F_g(z)$ to such a strip (which depends on g). By the identity theorem for holomorphic functions these functions all agree on the real axis: $P_\omega(tg) = F_g(t) = F_h(t) = P_\omega(th)$ $\forall t \in \mathbb{R}$ and each $g,h \in E$ with $L(g) = \alpha = L(h)$. Doing so for each $\alpha \in \mathrm{T}$ we get an extension $\widetilde{\Gamma}_\omega : \mathbb{C} \to \mathbb{C}$ of Γ_ω to all of $\mathbb{C}$, such that $P_\omega(f) = \widetilde{\Gamma}_\omega(L(f))$ $\forall f \in E$. Since $P_\omega : E \to \mathbb{C}$ is positive-definite, so is $\widetilde{\Gamma}_\omega : \mathbb{C} \to \mathbb{C}$. By Bochner's theorem there is a measure $\widetilde{\mu}_\omega \in M_+^1(\widehat{\mathbb{C}})$ on the characters $\widehat{\mathbb{C}}$ of the discrete additive group $(\mathbb{C},+)$ such that $\widetilde{\Gamma}_\omega(u) = \int_{\widehat{\mathbb{C}}} \chi(u)\mathrm{d}\widetilde{\mu}_\omega(\chi)$ $\forall u \in \mathbb{C}$. Because $\xi_\alpha^{(b)} \mapsto (\chi \mapsto \chi(\alpha))$ defines an isomorphism between $C(b\mathbb{C})$ and $C(\widehat{\mathbb{C}})$ [18] we regard $\widetilde{\mu}_\omega$ as a probability measure on $b\mathbb{C}$, that is $\widetilde{\Gamma}_\omega(u) = \widetilde{\mu}_\omega(\xi_u^{(b)})$ $\forall u \in \mathbb{C}$. Thus $P_\omega(f) = \widetilde{\Gamma}_\omega(L(f)) = \widetilde{\mu}_\omega(\xi_{L(f)}^{(b)})$ $\forall f \in E$. ω being analytic, Theorem 3.1(c) implies $\widetilde{\mu}_\omega \in M_+^1(\mathbb{C})$ and the existence of a $\gamma > 0$ with $\int_{\mathbb{C}} \mathrm{e}^{\gamma|z|}\mathrm{d}\widetilde{\mu}_\omega(z) < \infty$. Consequently

$$P_\omega(f) = \widetilde{\mu}_\omega(\xi_{L(f)}) = \sum_{k,l=0}^{\infty} \left(\frac{i}{\sqrt{2}}\right)^{k+l} \frac{L(f)^k}{k!}\,\frac{\overline{L(f)}^l}{l!}\, \widetilde{c}_\omega(k,l)$$

with $\widetilde{c}_\omega(k,l) := \int_{\mathbb{C}} \overline{z}^l z^k \mathrm{d}\widetilde{\mu}_\omega(z)$. Now from Theorem A.1 and Proposition 2.1 follows $\widetilde{c}_\omega = c_\omega \in \mathbb{M}^{(n)}_{\mathrm{posfun}}$. ∎

Via the map μ the extreme boundary $\partial_e\mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))$ of the simplex $\mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))$ corresponds to the point measures on $b\mathbb{C}$. For $z \in b\mathbb{C}$ and non-zero linear form $L : E \to \mathbb{C}$ we denote by α_z^L the *-automorphism on $\mathcal{W}(E)$ which is given in terms of the character $f \in E \mapsto \xi^{(b)}_{L(f)}(z) \in \{u \in \mathbb{C} \mid |u| = 1\}$ by

$$\alpha_z^L(W(f)) = \xi^{(b)}_{L(f)}(z)\, W(f) \qquad \forall f \in E \tag{3.5}$$

(gauge transformation of the second kind). One easily checks the following results.

Proposition 3.2 *(a)* $\partial_e\mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E)) = \{\omega_F \circ \alpha_z^L \mid z \in b\mathbb{C}\}$ *, i.e.* $\partial_e\mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))$ *is an orbit of the representation (3.5) for the group* $b\mathbb{C}$.

(b) *By the map μ of Theorem 3.1 (a) the convex set*

$$\mathcal{S}^{\rm cl}_{L,\rm reg}(\mathcal{W}(E)) := \{\omega \in \mathcal{S}^{\rm cl}_L(\mathcal{W}(E)) \mid \omega \text{ is regular}\}$$

is associated with the measures $M^1_+(\mathbb{C})$, showing $\mathcal{S}^{\rm cl}_{L,\rm reg}(\mathcal{W}(E))$ to be a simplex too, which however is not weak-compact. Its extreme boundary is given by $\partial_e\mathcal{S}^{\rm cl}_{L,\rm reg}(\mathcal{W}(E)) = \{\omega_F \circ \alpha^L_z \mid z \in \mathbb{C}\}$. Hence, by [4] Proposition 2.1 $\partial_e\mathcal{S}^{\rm cl}_{L,\rm reg}(\mathcal{W}(E))$ consists just of the the pure, quasifree, coherent states of degree ∞ associated with the linear forms zL, $z \in \mathbb{C}$. The boundary $\partial_e\mathcal{S}^{\rm cl}_{L,\rm reg}(\mathcal{W}(E))$ is an orbit of the representation (3.5) for the additive group $\mathbb{C}$.*

We now decompose the states $\mathcal{S}^{\rm cl}_L(\mathcal{W}(E))$ into the extreme ones. Such an integral decomposition is provided by the following Lemma.

Lemma 3.3 *The map $p_L : b\mathbb{C} \to \mathcal{S}(\mathcal{W}(E))$, $z \mapsto \omega_F \circ \alpha^L_z$ is continuous and its range $p_L(b\mathbb{C}) = \partial_e\mathcal{S}^{\rm cl}_L(\mathcal{W}(E))$ is a compact subset of $\mathcal{S}(\mathcal{W}(E))$ with respect to the weak*-toplogy.*

PROOF: $z \in b\mathbb{C} \mapsto \xi^{(b)}_{L(f)}(z)\,\langle\omega_F\,;\,W(f)\rangle$ is continuous for each $f \in E$, from which the continuity of p_L follows. ∎

We denote the image measure of $\mu_\omega \in M^1_+(b\mathbb{C})$ with respect to the map p_L by μ^L_ω,

$$\mu^L_\omega(B) := \mu_\omega(p_L^{-1}(B)) \qquad \text{for each Borel subset } B \subseteq \mathcal{S}(\mathcal{W}(E))\,.$$

Analoguously to the proof of [23] Lemma 3.3 one shows that μ^L_ω is a regular Borel measure, $\mu^L_\omega \in M^1_+(\mathcal{S}(\mathcal{W}(E)))$. By definition of $\omega \in \mathcal{S}^{\rm cl}_L(\mathcal{W}(E))$ and by (1.1) we have the integral decomposition

$$\omega = \int_{b\mathbb{C}} \omega_F \circ \alpha^L_z \,\mathrm{d}\mu_\omega(z) = \int_{\mathcal{S}(\mathcal{W}(E))} \varphi \,\mathrm{d}\mu^L_\omega(\varphi)\,. \tag{3.6}$$

Now we investigate in which cases the states of $\mathcal{S}^{\rm cl}_L(\mathcal{W}(E))$ are normal to the Fock representation.

Proposition 3.4 *Concerning the Fock normality we have the following results:*

(a) *Let $\omega \in \mathcal{S}(\mathcal{W}(E))$ be of class C^2 with $\langle\omega\,;\,a^*_\omega(f)\,a_\omega(f)\rangle = |L(f)|^2$ $\forall f \in E$ for some bounded linear form $L : E \to \mathbb{C}$. Then ω is normal to the Fock representation.*

(b) *Let $L : E \to \mathbb{C}$ be a non-zero linear form. Then $\omega \in \mathcal{S}^{\rm cl}_L(\mathcal{W}(E))\backslash\{\omega_F\}$ is normal to the Fock representation, if and only if L is bounded and $\mu_\omega \in M^1_+(\mathbb{C})$. (For arbitrary L the Fock state ω_F always belongs to the point measure δ_0 at $z = 0$.)*

PROOF: Since here the states are not supposed to be analytic, we cannot do the proof analoguously to [4] Proposition 2.5.

(a) It exists an $h \in \overline{E}$ with $L(f) = \langle h \mid f\rangle$ $\forall f \in E$. With the notation of [8] Theorem 5.2.14 and the orthogonal projections P_F from $\overline{E}$ onto the finite dimensonal subspaces $F \subseteq \overline{E}$ we obtain

$$\begin{aligned} n_{\omega,F}(\Omega_\omega) &= \sum_i n_{\omega,f_i}(\Omega_\omega) = \sum_i \|a_\omega(f_i)\,\Omega_\omega\|^2 = \sum_i \langle\omega\,;\,a^*_\omega(f_i)\,a_\omega(f_i)\rangle \\ &= \sum_i |L(f_i)|^2 = \sum_i |\langle h \mid f_i\rangle|^2 = \|P_F h\|^2 \le \|h\|^2 \end{aligned}$$

for each finite dimensional subspace $F \subseteq \overline{E}$. Thus $\Omega_\omega \in \mathcal{D}(n_\omega)$ and from [8] Theorem 5.2.14 follows ω being Fock-normal.

(b): Since each Fock-normal state is regular ([8] Proposition 5.2.4(4)), Theorem 3.1(a) gives $\mu_\omega \in M_+^1(\mathbb{C})$.

Let be $\omega \in \mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))\setminus\{\omega_F\}$ Fock-normal, but assume L unbounded. Then with the sequences $(f_n^\alpha)_{n\in\mathbb{N}}$, $\alpha \in \mathbb{C}$, of Lemma 2.2, we get $\lim_{n\to\infty} C_\omega(f_n^\alpha) = \mu_\omega(\xi_\alpha)$. On the other hand, if ϱ_ω is a density operator in the Fock space $F_+(\overline{E})$, then by [8], Proposition 5.2.4(4), one has $\lim_{n\to\infty} C_\omega(f_n^\alpha) = \lim_{n\to\infty} \mathrm{tr}_{F_+}(\varrho_\omega W_F(f_n^\alpha)) = 1$, $\forall\alpha \in \mathbb{C}$. Consequently $\mu_\omega(\xi_\alpha) = 1$, $\forall \alpha \in \mathbb{C}$. However, the map $\alpha \in \mathbb{C} \mapsto \mu_\omega(\xi_\alpha)$ is the Fourier transform of the measure $\mu_\omega \in M_+^1(\mathbb{C})$. By Bochner's theorem, [19], Theorem IX.9, the Fourier transformation gives a one-to-one correspondence between positive measures and positive-definite functions on the additive group $\mathbb{C} \cong \mathbb{R}^2$, from which it follows $\mu_\omega = \delta_0$, that is $\omega = \omega_F$, a contradiction.

Conversely, assume L bounded and $\mu_\omega \in M_+^1(\mathbb{C})$. Then there is an $h \in \overline{E}$ with $L(f) = \langle h \mid f\rangle\ \forall f \in E$. With the Weyl relations and $\langle \Omega_F \mid W_F(f)\Omega_F\rangle = C_F(f)$, $\forall f \in E$, one easily checks, that the density operator in $F_+(\overline{E})$ associated with ω is given by

$$\int_{\mathbb{C}} |W_F(-i\overline{z}h)\Omega_F\rangle\langle W_F(-i\overline{z}h)\Omega_F|\, d\mu_\omega(z)\,,$$

showing ω to be Fock-normal (cf. [11] Section 8.2). ∎

4 GNS–representation and central decomposition

In this section we assume a fixed, but arbitrary, **unbounded** linear form $L : E \to \mathbb{C}$. We treat the GNS-representation $(\Pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ and central decomposition of $\omega \in \mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))$.

As in [4] let $\overline{\mathcal{K}} = \overline{E} \oplus \mathbb{C}$ with scalar product $\langle f \oplus \alpha \mid g \oplus \beta\rangle = \langle f \mid g\rangle + \overline{\alpha}\beta$, and embed E therein by $\lambda : E \to \overline{\mathcal{K}}$, $f \mapsto f \oplus L(f)$. From the unboundedness of L it follows that $\lambda(E)$ is dense in $\overline{\mathcal{K}}$, [4] Section 3. The continuous extension $\sigma_{\mathcal{K}}$ of the symplectic form $\mathrm{Im}\langle . \mid .\rangle$ from E to $\overline{\mathcal{K}}$ is given by $\sigma_{\mathcal{K}}(f\oplus\alpha, g\oplus\beta) = \mathrm{Im}\langle f \mid g\rangle\ \ \forall f\oplus\alpha, g\oplus\beta \in \overline{\mathcal{K}}$, which obviously possesses a non-trivial null space. By $\mathcal{L}(\mathcal{H})$ we denote the bounded operators on the Hilbert space $\mathcal{H}$.

Theorem 4.1 *For $\omega \in \mathcal{S}_L^{\mathrm{cl}}(\mathcal{W}(E))$ the GNS-representation is given by*

$$\mathcal{H}_\omega = F_+(\overline{E}) \otimes \mathsf{L}^2(b\mathbb{C}, \mu_\omega)\,, \qquad \Omega_\omega = \Omega_F \otimes w\,, \qquad \Pi_\omega(W(f)) = W_\omega(\lambda(f)) \quad \forall f \in E\,,$$

where μ_ω is introduced in Theorem 3.1 and where $w(z) \equiv 1$, $z \in b\mathbb{C}$, and $W_\omega(g\oplus\beta) := W_F(g) \otimes \xi_\beta^{(b)}$ $\forall g\oplus\beta \in \overline{\mathcal{K}}$. For the associated von Neumann algebra we have

$$\mathcal{M}_\omega = \Pi_\omega(\mathcal{W}(E))'' = \mathcal{L}(F_+(\overline{E}))\overline{\otimes}\mathsf{L}^\infty(b\mathbb{C}, \mu_\omega)\,,$$

where the elements of $\mathsf{L}^\infty(b\mathbb{C},\mu_\omega)$ act on $\mathsf{L}^2(b\mathbb{C},\mu_\omega)$ as multiplication operators. The symbol $\overline{\otimes}$ denotes the W^*-tensor product [21]. Further, the commutant $\mathcal{M}_\omega'$ and the center $\mathcal{Z}_\omega = \mathcal{M}_\omega \cap \mathcal{M}_\omega'$ are both equal to

$$\mathcal{Z}_\omega = \mathcal{M}_\omega' = \mathbb{1}_{F_+} \otimes \mathsf{L}^\infty(b\mathbb{C}, \mu_\omega)\,.$$

Moreover

(a) $\langle \Omega_\omega \mid W_\omega(f\oplus\alpha)\,\Omega_\omega\rangle = C_F(f)\,\mu_\omega(\xi_\alpha^{(b)})\quad \forall f\oplus\alpha \in \overline{\mathcal{K}}.$

(b) $W_\omega(\psi) \in \mathcal{M}_\omega \quad \forall \psi \in \overline{\mathcal{K}}$.

(c) *Let $\omega \in \mathcal{S}^{\rm cl}_{L,\rm reg}(\mathcal{W}(E))$. Then the map $W_\omega(.) : \overline{\mathcal{K}} \longrightarrow \mathcal{M}_\omega$, $\psi \mapsto W_\omega(\psi)$ is continuous with respect to the norm of $\overline{\mathcal{K}}$ and the strong operator topology on $\mathcal{M}_\omega$. As a consequence, if the selfadjoint operators $\Psi_\omega(\psi)$, $\psi \in \overline{\mathcal{K}}$, generate the unitary groups $\{W_\omega(t\psi) \mid t \in \mathbb{R}\}$, the map*

$$\psi = f \oplus \alpha \in \overline{\mathcal{K}} = \overline{E} \oplus \mathbb{C} \longmapsto \Phi_F(f) \otimes \mathbb{1}_{\mathsf{L}^2} + \mathbb{1}_F \otimes \Phi_{\rm cl}(\alpha)$$

is continuous in the strong resolvent sense. Here $\Phi_F(f)$, $f \in \overline{E}$, denote the Fock field operators, and $\Phi_{\rm cl}(\alpha) := \exp\{\frac{i}{\sqrt{2}}(\alpha A + \overline{\alpha}A^*)\}$, $\alpha \in \mathbb{C}$, with the normal operator $(A\xi)(z) = z\xi(z)$ $\forall z \in \mathbb{C}$, $\xi \in \mathsf{L}^2(\mathbb{C}, \mu_\omega)$, are the field operators associated with the classical Weyl operators $W_{\rm cl}(\alpha) = \xi_\alpha$ (cf. (3.3)).

Moreover, since $\Psi_\omega(\lambda(f)) = \Phi_\omega(f)$ with the above embedding $\lambda : E \longrightarrow \overline{\mathcal{K}}$, the extended field operators $\Psi_\omega(\psi)$, $\psi \in \overline{\mathcal{K}}$, are approximable by the original ones in the strong resolvent sense.

PROOF: Clearly the so defined $(\Pi_\omega, \mathcal{H}_\omega)$ gives a representation of $\mathcal{W}(E)$. Now we use the sequences $(f_n^\alpha)_{n\in\mathbb{N}}$, $\alpha \in \mathbb{C}$, in E of Lemma 2.2. Hence by [8] Proposition 5.2.4(4) $\text{s-}\lim_{n\to\infty} W_\omega(\lambda(f_n^\alpha)) = \mathbb{1}_{F_+} \otimes \xi_\alpha^{(b)}$ is an element of $\mathcal{M}_\omega$ for all $\alpha \in \mathbb{C}$. Consequently $W_\omega(\lambda(f))\left[\mathbb{1}_{F_+} \otimes \xi^{(b)}_{-L(f)}\right] = W_F(f) \otimes \mathbb{1}_{\mathsf{L}^2} \in \mathcal{M}_\omega$ $\forall f \in E$. Thus by [8] Proposition 5.2.4(4) $W_F(f) \otimes \mathbb{1}_{\mathsf{L}^2} \in \mathcal{M}_\omega$ $\forall f \in \overline{E}$. Consequently $W_\omega(g \oplus \beta) = \left[W_F(g) \otimes \mathbb{1}_{\mathsf{L}^2}\right]\left[\mathbb{1}_{F_+} \otimes \xi_\beta^{(b)}\right] \in \mathcal{M}_\omega$ $\forall g \oplus \beta \in \overline{\mathcal{K}}$. Now observe $\mathsf{LH}\{W_F(f) \mid f \in E\}'' = \mathcal{L}(F_+(\overline{E}))$ by [8] Proposition 5.2.4(3) and that $\mathsf{LH}\{\xi_\alpha^{(b)} \mid \alpha \in \mathbb{C}\}$ is dense in $\mathsf{AP}(\mathbb{C}) \cong C(b\mathbb{C})$ from which together with [21], Theorem III-1.2 (regard $\mu_\omega \in M_+^1(b\mathbb{C})$ as a state on $C(b\mathbb{C})$), follows $\mathsf{LH}\{\xi_\alpha^{(b)} \mid \alpha \in \mathbb{C}\}'' = \mathsf{L}^\infty(b\mathbb{C}, \mu_\omega)$, where the elements of $\mathsf{LH}\{\xi_\alpha^{(b)} \mid \alpha \in \mathbb{C}\}$ are considered as multiplication operators on $\mathsf{L}^2(b\mathbb{C}, \mu_\omega)$ and the bicommutant is taken with respect to $\mathcal{L}(\mathsf{L}^2(b\mathbb{C}, \mu_\omega))$. The cyclicity of Ω_ω now is immediate.

To prove (c) observe that $\omega \in \mathcal{S}_L^{\rm cl}(\mathcal{W}(E))$ to be regular is equivalent to $\mu_\omega \in M_+^1(\mathbb{C})$, and use [8] Proposition 5.2.4(4) and Lebesgue's dominated convergence theorem. The rest of the proof is easily checked. ■

Theorem 4.2 *For each $\omega \in \mathcal{S}_L^{\rm cl}(\mathcal{W}(E))$ the measure $\mu_\omega^L \in M_+^1(\mathcal{S}(\mathcal{W}(E)))$ is its central measure and (3.6) its central decomposition.*

PROOF: For a given map $h : b\mathbb{C} \longrightarrow \mathbb{C}$ define the function $\widehat{h} : \mathcal{S}(\mathcal{W}(E)) \longrightarrow \mathbb{C}$ by setting $\widehat{h}(\omega_F \circ \alpha_z^L) := h(z)$ and $\widehat{h}(\varphi) := 0$ for $\varphi \notin \partial_e \mathcal{S}_L^{\rm cl}(\mathcal{W}(E))$. Analoguously to the proof of [4] Proposition 3.5 we obtain $\kappa_\omega(\widehat{\xi}_\alpha^{(b)}) = W_\omega(0 \oplus \alpha)$ $\forall \alpha \in \mathbb{C}$ for the Tomita map $\kappa_\omega : \mathsf{L}^\infty(\mathcal{S}(\mathcal{W}(E)), \mu_\omega^L) \longrightarrow \mathcal{M}'_\omega = \mathcal{Z}_\omega$ (cf. [8] Lemma 4.1.21). Clearly $\mathsf{LH}\{\xi_\alpha^{(b)} \mid \alpha \in \mathbb{C}\}$ is dense in $\mathsf{L}^\infty(b\mathbb{C}, \mu_\omega)$ with respect to the $\sigma(\mathsf{L}^\infty, \mathsf{L}^1)$-topology (use [21] Theorem III-1.2 for the state $\mu_\omega \in M_+^1(b\mathbb{C})$ on $C(b\mathbb{C})$), which implies $\mathsf{LH}\left\{\widehat{\xi}_\alpha^{(b)} \mid \alpha \in \mathbb{C}\right\}$ to be $\sigma(\mathsf{L}^\infty, \mathsf{L}^1)$-dense in $\mathsf{L}^\infty(\mathcal{S}(\mathcal{W}(E)), \mu_\omega^L)$. Now the assertion follows from [8] Lemma 4.1.21. ■

Proposition 4.3 *The following assertions are valid:*

(a) *For each $\omega \in \mathcal{S}_L^{\rm cl}(\mathcal{W}(E))$ there is a unique maximal measure $\mu_\omega^{\max} \in M_+^1(\mathcal{S}(\mathcal{W}(E))$ with $\omega = \int_{\mathcal{S}(\mathcal{W}(E))} \varphi \, d\mu_\omega^{\max}(\varphi)$, namely the central measure $\mu_\omega^{\max} = \mu_\omega^L$ of ω.*

(b) *The states in $\mathcal{S}_L^{\rm cl}(\mathcal{W}(E))$ constitute a weak*-compact face of $\mathcal{S}(\mathcal{W}(E))$.*

(c) The regular states $\mathcal{S}^{\rm cl}_{L,\rm reg}(\mathcal{W}(E))$ constitute a face, which is not weak-compact.*

(d) The set of first order coherent states $\mathcal{S}^{(1)}_L(\mathcal{W}(E))$ is not a face of $\mathcal{S}(\mathcal{W}(E))$.

PROOF: (a) follows from $\mathcal{M}'_\omega = \mathcal{Z}_\omega$ and [21] Lemma IV–6.26.

(b): Let $\omega = \lambda\varphi_1 + (1-\lambda)\varphi_2$ be an arbitrary convex decomposition of a $\omega \in \mathcal{S}^{\rm cl}_L(\mathcal{W}(E))$ with $0 < \lambda < 1$ and $\varphi_1, \varphi_2 \in \mathcal{S}(W(E))$. By [9] Proposition 4.1.3 there exist maximal measures $\mu_1, \mu_2 \in M^1_+(\mathcal{S}(W(E)))$ with $\varphi_k = \int_{\mathcal{S}(W(E))} \omega' \, d\mu_k(\omega')$ for $k \in \{1,2\}$. Then by [9] Proposition 4.1.14 $\mu := \lambda\mu_1 + (1-\lambda)\mu_2$ is a maximal measure decomposing ω. By (a) $\mu = \mu^L_\omega$ is the central measure of ω. Since μ^L_ω is concentrated on $\partial_e \mathcal{S}^{\rm cl}_L(\mathcal{W}(E))$ and $0 < \lambda < 1$ so has to be each μ_k, from which follows $\varphi_k \in \mathcal{S}^{\rm cl}_L(\mathcal{W}(E))$, $k \in \{1,2\}$.

(c): Since the regular states on $\mathcal{W}(E)$ constitute a folium (i.e. a norm-closed split face [20]) in $\mathcal{S}(\mathcal{W}(E))$, they especially define a face. Hence the assertion follows from (b).

(d): If $z_1, z_2 \in \mathbb{C}$ with $\lambda|z_1|^2 + (1-\lambda)|z_2|^2 = 1$ and $|z_1| \neq 1 \neq |z_2|$ for some $0 < \lambda < 1$ and φ_1, φ_2 are the states in $\mathcal{S}^{\rm cl}_L(\mathcal{W}(E))$ associated with the point measures δ_{z_1} resp. δ_{z_2} (Theorem 3.1), then $\varphi := \lambda\varphi_1 + (1-\lambda)\varphi_2 \in \mathcal{S}^{(1)}_L(\mathcal{W}(E))$, but φ_1 and φ_2 are not elements of $\mathcal{S}^{(1)}_L(\mathcal{W}(E))$. We have $\varphi_k \in \mathcal{S}^{(\infty)}_{z_k L}(\mathcal{W}(E))$ for $k = 1, 2$. ■

5 Extended Weyl formalism

Let throughout this Section the linear form $L : E \to \mathbb{C}$ be **unbounded** and $\overline{\mathcal{K}}$ be the completion of E with respect to the scalar product $\langle f \mid g\rangle_L = \langle f \mid g\rangle + \overline{L(f)}L(g)$ on E. Hence the embedding $\lambda : E \to \overline{\mathcal{K}}$ gives rise to an extension of the one-particle space E to $\overline{\mathcal{K}} = \overline{E} \oplus \mathbb{C}$ as has been described at the beginning of Section 4. This suggests an extension of the original Weyl algebra $\mathcal{W}(E)$ to $\overline{\Delta(\overline{\mathcal{K}}, \sigma_{\mathcal{K}})}$, where $\sigma_{\mathcal{K}}$ is the previously defined continuous extension of the symplectic form $\mathrm{Im}\langle . \mid .\rangle$ on E. However $\sigma_{\mathcal{K}}$ is a degenerate symplectic form with kernel $\ker(\sigma_{\mathcal{K}}) = 0 \oplus \mathbb{C}$. $\overline{\Delta(\overline{\mathcal{K}}, \sigma_{\mathcal{K}})}$ is defined in [22] (cf. also [4] Section 3 and the Appendix of [23]). As in [22] the Weyl operators in $\overline{\Delta(\overline{\mathcal{K}}, \sigma_{\mathcal{K}})}$ are denoted by δ_ψ, $\psi \in \overline{\mathcal{K}}$. Because of the null space of $\sigma_{\mathcal{K}}$ the extended Weyl algebra $\overline{\Delta(\overline{\mathcal{K}}, \sigma_{\mathcal{K}})}$ decomposes into a tensor product

$$\overline{\Delta(\overline{\mathcal{K}}, \sigma_{\mathcal{K}})} \cong \mathcal{W}(\overline{E}) \otimes \overline{\Delta(\mathbb{C})} =: \mathcal{W}(\overline{\mathcal{K}}, \sigma_{\mathcal{K}}) \,, \tag{5.1}$$

where by [23] Theorem A.1 (Appendix) the abelian C*-algebra $\overline{\Delta(\mathbb{C})} := \overline{\Delta(\mathbb{C}, 0)}$ is *-isomorphic to the algebra of continuous functions $\mathcal{C}(\widehat{\mathbb{C}})$ on the character group $\widehat{\mathbb{C}}$ of the discrete additive group $(\mathbb{C}, +)$. But $\widehat{\mathbb{C}}$ is homeomorphic to $b\mathbb{C}$ [18], and hence

$$\overline{\Delta(\mathbb{C})} \cong \mathcal{C}(\widehat{\mathbb{C}}) \cong \mathcal{C}(b\mathbb{C}) \cong \mathsf{AP}(\mathbb{C}) \,. \tag{5.2}$$

The extended Weyl algebra (5.1) may be considered as the field algebra for macroscopic coherent states. It exhibits a classical part which belongs to two macroscopic degrees of freedom constituting $\mathbb{C}$. Choosing polar coordinates in $\mathbb{C}$ one has a phase and an amplitude as classical observables, where the Bohr compactification $b\mathbb{C}$ allows for the divergent amplitudes. In this Boson field algebra the new classical coordinates are independent from the original Weyl algebra $\mathcal{W}(E)$, whereas in the GNS-representation of $\mathcal{W}(E)$ over a macroscopic $\omega \in \mathcal{S}^{\rm cl}_L(\mathcal{W}(E))$ the Fock and the classical parts are coupled (cf. e.g. $\Pi(W(f)) = W_\omega(\lambda(f))$, Theorem 4.1). Due to these classical coordinates the C*-algebra of (5.1) is not simple and has non-faithful representations.

The state space $\mathcal{S}(\mathcal{W}(\overline{\mathcal{K}},\sigma_{\mathcal{K}}))$ contains especially the extension $\widetilde{\omega}$ of each $\omega \in \mathcal{S}_L^{\rm cl}(\mathcal{W}(E))$ to the product state

$$\widetilde{\omega} :\cong \omega_F \otimes \mu_\omega \quad \text{on} \quad \overline{\Delta(\overline{\mathcal{K}},\sigma_{\mathcal{K}})} \cong \mathcal{W}(\overline{E}) \otimes \mathsf{AP}(\mathbb{C}) \,, \tag{5.3}$$

where here ω_F means the Fock vacuum state on $\mathcal{W}(\overline{E})$ introduced after (1.1), and μ_ω is defined in Theorem 3.1. It is seen directly that the GNS-representation of $\widetilde{\omega}$ is the tensor product of the GNS-representations of $\mathcal{W}(\overline{E})$ over ω_F and of $\mathsf{AP}(\mathbb{C})$ over μ_ω in the Hilbert space $\mathcal{H}_{\widetilde{\omega}} = \mathcal{H}_\omega$ with cyclic vector $\Omega_{\widetilde{\omega}} = \Omega_\omega$, and representation $\Pi_{\widetilde{\omega}}(\delta_\psi) = W_\omega(\psi)\ \forall \psi \in \overline{\mathcal{K}}$. Obviously $\Pi_{\widetilde{\omega}}(\mathcal{W}(\overline{\mathcal{K}},\sigma_{\mathcal{K}}))'' =: \mathcal{M}_{\widetilde{\omega}} = \mathcal{M}_\omega$ (cf. Theorem 4.1). The GNS-representation is faithful, if and only if $\mathsf{supp}(\mu_\omega) = b\mathbb{C}$. If $\mathsf{supp}(\mu_\omega)$ is not the whole of $b\mathbb{C}$ the preparation of ω (resp. $\widetilde{\omega}$) reduces the range of the classical coordinates correspondingly.

We have here a nice illustration of the formalism in [22]. For given $\omega \in \mathcal{S}_L^{\rm cl}(\mathcal{W}(E))$ the GNS-representation $\Pi_{\widetilde{\omega}}$ defines a C*-seminorm on $\Delta(\overline{\mathcal{K}},\sigma_{\mathcal{K}})$ (that is the *-algebra linearly generated by the Weyl operators δ_ψ, $\psi \in \overline{\mathcal{K}}$), the completion of which is $\mathcal{W}(\overline{E}) \otimes \mathcal{C}(s_\omega)$, where $s_\omega := \mathsf{supp}(\mu_\omega)$. By [22] this algebra can also be obtained as the quotient of $\mathcal{W}(\overline{\mathcal{K}},\sigma_{\mathcal{K}})$ with the closed *-ideal $\mathcal{W}(\overline{E}) \otimes \mathcal{C}_\infty(b\mathbb{C}\setminus s_\omega)$, the kernel of $\Pi_{\widetilde{\omega}}$ (here $\mathcal{C}_\infty(X)$ denotes the continuous functions on X vanishing at infinity).

The appropriateness of $\mathcal{W}(\overline{\mathcal{K}},\sigma_{\mathcal{K}})$ should be discussed with reference to the generalized first order coherent states in $\mathcal{S}_L^{\rm cl}(\mathcal{W}(E))$, among which those are distinguished, in which the classical variables range over all of $b\mathbb{C}$. These give via the GNS-representation described above faithful representations of $\mathcal{W}(\overline{\mathcal{K}},\sigma_{\mathcal{K}})$ and make explicit the structure of a Boson algebra with microscopic modes $\overline{E}$ and one classical field mode L.

6 Discussion

In our analysis of the n–th order coherent states on a C*-Weyl algebra, which are sub-classes of the first order coherent states, the first general result is Proposition 2.1. It gives a general form for the characteristic function $f \in E \mapsto C_\omega(f)$ of a coherent state ω by means of a series expansion (2.4) in powers of $L(f)^k$ and $\overline{L(f)}^l$, where $L : E \to \mathbb{C}$ is the linear form of the coherence condition Definition 1.2. That is, the test functions $f \in E$ enter the normally ordered characteristic function $P_\omega(f) = C_\omega(f)/C_F(f)$ only via $L(f)$. If L is bounded with respect to the norm on E, there is a $h \in \overline{E}^{\|\cdot\|}$ with $L(f) = \langle h \mid f\rangle\ \forall f \in E$, and only the component of f in the one-mode space $\mathbb{C}h$ contributes to $P_\omega(f)$. The mode h corresponds to the mode b in formula (2.15) of [13], where the dnsity operator of a first order coherent state in Fock space is discussed. If L is unbounded then the complement of its null space in E is always of infinite dimensions and the associated coherent state is non-Fock (Proposition 2.4). If we call L a macroscopic mode, then Proposition 2.1 gives the one-mode structure of $P_\omega(f)$ and of the normally ordered correlations (2.5). The latter are dealt with also in [24], but only with equally many creation and annihilation operators. The formula (2.4) is a complete characterization of all first order coherent states (where analyticity is required), if one knows that the matrix $c_\omega(k,l)$ has the property to make the series times $C_F(f)$ a characteristic function of a state on $\mathcal{W}(E)$. For this $c_\omega \in \mathbb{M}^{(1)}_{\rm posker}$ (cf. (2.3)) is necessary but not sufficient.

The decisive progress is Theorem 2.3(a) which says that $c_\omega \in \mathbb{M}^{(n)}_{\rm posfun}$ (cf. (2.9)) is for every L necessary and sufficient to make (2.4) a characteristic function of a classical n–th order coherent state. For L unbounded Theorem 2.3(b) tells us, that by this condition all coherent states are exhausted since all of them are classical.

The condition $c_\omega \in \mathbb{M}^{(n)}_{\text{posfun}}$, which according to (2.8) is essentially positive definiteness over a semigroup of entire valued tuples, seems not to have been known in the literature. That it expresses a canonical mathematical structure is demonstrated by Theorem A.1 of the Appendix, which connects it with the Hamburger type moment problem for probability measures on $\mathbb{C}$. This, in turn, supplements the spectral theory for normal (not necessarily bounded) operators in a Hilbert space.

In our context the measure $\mu_\omega \in M^1_+(\mathbb{C})$, which is determined by $c_\omega \in \mathbb{M}^{(n)}_{\text{posfun}}$, provides — after having been transferred to a regular probability measure μ^L_ω on the state space $\mathcal{S}(\mathcal{W}(E))$ on $\mathcal{W}(E)$ — a decomposition of a classical first order coherent state ω into pure coherent states (of infinite coherence order). The classical states in $\mathcal{S}^{(1)}_L(\mathcal{W}(E))$ are in this way shown to be affine isomorphic to the analytic measures in $M^1_+(\mathbb{C})$ with fixed first moment. By the Bohr compactification, which extends $\mathbb{C}$ to the compact topological group $b\mathbb{C}$, one gets the Bauer simplex $M^1_+(b\mathbb{C})$, which is transferred to the state space as $\mathcal{S}^{\text{cl}}_L(\mathcal{W}(E))$ (Theorem 3.1). $\mathcal{S}^{\text{cl}}_L(\mathcal{W}(E))$ is an extrapolation of the classical first order coherent states and contains also non-regular states. In spite of being not connected with a field operator, the non-regular states arise in physics, e.g. by a gauge constraint [25], [26]. Here one sees from Proposition 3.2, that the non-regular states in the extremal boundary $\partial_e\mathcal{S}^{\text{cl}}_L(\mathcal{W}(E))$ of $\mathcal{S}^{\text{cl}}_L(\mathcal{W}(E))$ are obtained from a regular one in $\partial_e\mathcal{S}^{\text{cl}}_L(\mathcal{W}(E))$ by a gauge transformation of the second kind in terms of a non-continuous character on $\mathbb{C}$. The use of the compact set $\mathcal{S}^{\text{cl}}_L(\mathcal{W}(E))$ has technical advantages, also for dealing with the (regular, even analytic) classical states in $\mathcal{S}^{(1)}_L(\mathcal{W}(E))$.

If L is unbounded the GNS-representation for all $\omega \in \mathcal{S}^{\text{cl}}_L(\mathcal{W}(E))$ can be constructed by an extension of the methods of [27], here even in the case of non-regular states (Theorem 4.1). The center of the GNS–von Neumann algebra is thereby identified as $\mathsf{L}^\infty(b\mathbb{C},\nu_\omega)$ and signifies in the case of a regular ω the arise of the classical smeared field $\sqrt{2}\,\mathrm{Re}(zL(f))$, where z is distributed over $\mathbb{C}$ according to the statistics μ_ω. By determining the image of the Tomita map, which is associated with the transferred measure μ^L_ω, the latter is in Theorem 4.2 identified as the central measure of $\omega \in \mathcal{S}^{\text{cl}}_L(\mathcal{W}(E))$ (also if ω is not regular). That means, that the specification of the central (optical) variables "phase" and "amplitude" of the classical field leads to the purification of the generalized first order coherent states as a quantum state.

In the extended Weyl formalism of Section 5 the classical field becomes independent from the Fock space Bosons. In this sense one has the final stage in deriving a classical field from the collective ordering condition for a state. The ordering condition for first order coherent states seems rather similar to "off diagonal long range order" (cf. [28], [29]). It expresses a *collective* ordering, however, only if the involved linear form L is unbounded. Only in this case one obtains in a photon theory the genuine optical features like the classical phase and amplitude.

Acknowledgements

The authors are indebted to J. Hertle for useful discussions and one of us (R.H.) is grateful to the Deutsche Forschungsgemeinschaft for financial support.

Appendix

We construct here the Kolmogorov decomposition (cf. [10] and the appendix of [4]) of the positive-definite function $c \in \mathbb{M}^{(0)}_{\text{posfun}}$ on the involutive semigroup $(\mathbb{N}_0 \times \mathbb{N}_0, I)$ explicitely (cf. formula (2.9).

Theorem A.1 *For a matrix $c : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ with $c(0,0) = 1$ the following assertions are equivalent:*

(i) $c \in \mathbb{M}^{(0)}_{\text{posfun}}$;

(ii) there is a $\mu \in M^1_+(\mathbb{C})$ with $\int_{\mathbb{C}} \exp\{\gamma |z|\}\, \mathrm{d}\mu(z) < \infty$ for some $\gamma > 0$ (such a measure is called analytic*) and $c(k,l) = \int_{\mathbb{C}} \overline{z}^l z^k \, \mathrm{d}\mu(z)$ for all $k, l \in \mathbb{N}_0$;*

(iii) there exists a Hilbert space $\mathcal{H}$, a normal operator A acting in $\mathcal{H}$, and a vector $w \in \mathcal{H}$, $\|w\| = 1$, which is analytic for A (and hence for A^* and $|A| = \sqrt{AA^*}$), such that $c(k,l) = \langle A^l w \mid A^k w \rangle$ $\forall k, l \in \mathbb{N}_0$.

If one of these conditions is fulfilled, then in addition we have:

(a) The measure $\mu \in M^1_+(\mathbb{C})$ of (ii) is unique.

(b) $\mathcal{H}$, A, w in (iii) may be choosen so that $\mathcal{H} = \overline{\mathsf{LH}}\{A^{*m} A^n w \mid m, n \in \mathbb{N}_0\}$, where $\overline{\mathsf{LH}}$ denotes the closure of the linear hull. In this case the (cyclic) representation $(\mathcal{H}, A, w)$ associated with c is unique up to unitary equivalence.

PROOF: The proof is an extension of the Hamburger moment problem from $\mathbb{R}$ to $\mathbb{C}$ (cf. [14] Theorem X.4, [15] Chapter X, § 7).

(ii) $\Rightarrow$ (i): Because of $\int_{\mathbb{C}} \exp\{\gamma |z|\}\, \mathrm{d}\mu(z) < \infty$ all the integrals $c(k,l) = \int_{\mathbb{C}} \overline{z}^l z^k \, \mathrm{d}\mu(z)$ exist. Consequently for $\alpha_{ij} \in \mathbb{C}$ with $0 \le i, j \le M$

$$\begin{aligned}
\sum_{k,l,m,n=0}^{M} \overline{\alpha_{kl}}\, \alpha_{mn}\, c(l+m, k+n) &= \int_{\mathbb{C}} \Big(\sum_{k,l=0}^{M} \overline{\alpha_{kl}}\, \overline{z}^k z^l \Big) \Big(\sum_{m,n=0}^{M} \alpha_{mn}\, \overline{z}^n z^m \Big) \, \mathrm{d}\mu(z) \\
&= \int_{\mathbb{C}} \Big| \sum_{m,n=0}^{M} \alpha_{mn}\, \overline{z}^n z^m \Big|^2 \mathrm{d}\mu(z) \ \ge\ 0,
\end{aligned}$$

giving equation (2.8). Moreover we obtain for $\delta := \gamma/2$

$$\sum_{k,l=0}^{\infty} \delta^{k+l} \frac{1}{k!} \frac{1}{l!} |c(k,l)| \le \sum_{k,l=0}^{\infty} \delta^{k+l} \frac{1}{k!} \frac{1}{l!} \int_{\mathbb{C}} |z|^{k+l} \, \mathrm{d}\mu(z) = \int_{\mathbb{C}} \exp\{2\delta |z|\} \, \mathrm{d}\mu(z) < \infty .$$

(i) $\Rightarrow$ (iii): Let be $\mathcal{P}$ the set of all polynomials p, $p(z, \overline{z}) = \sum_{k,l=0}^{M} \alpha_{kl}\, z^k \overline{z}^l$, $(z \in \mathbb{C})$, with complex coefficients α_{kl}. For given $c \in \mathbb{M}^{(0)}_{\text{posfun}}$ we define on $\mathcal{P}$ the sesquilinearform $\langle . \mid . \rangle$

$$\Big\langle \sum_{k,l=0}^{M} \alpha_{kl}\, z^k \overline{z}^l \ \Big|\ \sum_{m,n=0}^{N} \beta_{mn}\, z^m \overline{z}^n \Big\rangle := \sum_{k,l=0}^{M} \sum_{m,n=0}^{N} \overline{\alpha_{kl}}\, \beta_{mn}\, c(l+m, k+n) . \tag{A.1}$$

which is positive by (2.8), i.e. $\langle p \mid p\rangle \geq 0 \ \forall p \in \mathcal{P}$. If $\mathcal{Q} := \{p \in \mathcal{P} \mid \langle p \mid p\rangle = 0\}$, then let be $\mathcal{H}$ the completion of the quotient space $\mathcal{P}/\mathcal{Q}$ with respect to $\langle . \mid . \rangle$.

On $\mathcal{P}$ we now define the two operators B and C as the multiplications with z resp. $\overline{z}$

$$(Bp)(z,\overline{z}) := z\,p(z,\overline{z}), \qquad (Cp)(z,\overline{z}) := \overline{z}\,p(z,\overline{z}) \qquad \forall z \in \mathbb{C} \ \ \forall p \in \mathcal{P}.$$

By direct calculations from (A.1) follows $\langle p \mid Bq\rangle = \langle Cp \mid q\rangle \ \forall p,q \in \mathcal{P}$, which with the Cauchy–Schwarz inequality implies $\langle Bp \mid Bp\rangle^2 = |\langle CBp \mid p\rangle|^2 \leq \langle CBp \mid CBp\rangle \langle p \mid p\rangle$, showing $B(\mathcal{Q}) \subseteq \mathcal{Q}$, and similar $C(\mathcal{Q}) \subseteq \mathcal{Q}$. Hence the operators B and C can be transfered to $\mathcal{P}/\mathcal{Q}$ and now become operators $\widehat{B}$ resp. $\widehat{C}$ acting in the Hilbert space $\mathcal{H}$ with domains $\mathcal{D}(\widehat{B}) = \mathcal{D}(\widehat{C}) = \mathcal{P}/\mathcal{Q}$.

Because of $\sum_{k,l=0}^{\infty} \delta^{k+l} \frac{1}{k!}\frac{1}{l!}|c(k,l)| < \infty$ for some $\delta > 0$ there is an $\alpha > 0$ with $|c(k,l)| \leq \alpha^2 \left(\frac{1}{\delta}\right)^{k+l} k!\,l!$ $\forall k,l \in \mathbb{N}_0$, which with (A.1) implies

$$\begin{aligned}\frac{\left\|\widehat{B}^n z^k \overline{z}^l\right\|}{n!} &= \frac{\sqrt{c(k+l+n,k+l+n)}}{n!} \leq \alpha\left(\frac{1}{\delta}\right)^{k+l+n} \frac{(k+l+n)!}{n!} \\ &\leq \alpha\left(\frac{2}{\delta}\right)^{k+l+n} (k+l)! \qquad \forall k,l,n \in \mathbb{N}_0 .\end{aligned}$$

Consequently $\sum_{n=0}^{\infty} \frac{1}{n!}\left\|\widehat{B}^n z^k \overline{z}^l\right\| t^n < \infty$ for each $t \in [0,\delta/2[$, showing the element $z \in \mathbb{C} \mapsto z^k\overline{z}^l$ of $\mathcal{P}/\mathcal{Q} \subseteq \mathcal{H}$ to be an analytic vector for $\widehat{B}$ for each $k,l \in \mathbb{N}_0$. Hence by the triangle inequality each $\psi \in \mathcal{P}/\mathcal{Q}$ is an analytic vector for $\widehat{B}$ ($\widehat{C}$ analoguous). Now one easily ensures, that $\mathcal{P}/\mathcal{Q}$ consists of analytic vectors for the symmetric operators $\frac{1}{2}\left(\widehat{B}+\widehat{C}\right)$ and $\frac{1}{2i}\left(\widehat{B}-\widehat{C}\right)$, from which with Nelson's theorem follows the selfadjointness of their closures, S resp. T. With series expansions on the analytic vectors $\mathcal{P}/\mathcal{Q}$ one checks $e^{isS}e^{itT} = e^{itT}e^{isS} \ \forall s,t \in \mathbb{R}$ showing S, T to commute in the sense that the associated spectral projections commute. Thus $A := S + iT$ is a normal operator in $\mathcal{H}$ extending $\widehat{B}$, resp. $A^* = S - iT$ extending $\widehat{C}$.

Let $w \in \mathcal{H}$ be given by the representative polynomial $w(z) = 1 \ \forall z \in \mathbb{C}$, then (A.1) implies $\|w\|^2 = c(0,0) = 1$, from which also follows $\langle A^l w \mid A^k w\rangle = \langle z^l \mid z^k\rangle = c(k,l) \ \forall k,l \in \mathbb{N}_0$. Further $\mathrm{LH}\{A^{*m}A^n w \mid m,n \in \mathbb{N}_0\} = \mathcal{P}/\mathcal{Q}$, which by construction is dense in $\mathcal{H}$ and also proves the first part of (b). (The uniqueness statement in (b) is immediate.)

(iii) $\Rightarrow$ (ii): With the spectral family $G(z)$, $z \in \mathbb{C}$, of A define $\mu \in M_+^1(\mathbb{C})$ by $d\mu(z) = \langle w \mid dG(z)w\rangle$. Then $c(k,l) = \langle A^l w \mid A^k w\rangle = \int_{\mathbb{C}} \overline{z}^l z^k \, d\mu(z) \ \forall k,l \in \mathbb{N}_0$, especially $1 = c(0,0)$.

Since w is an analytic vector for A so it is also for $|A|$ (since $\|A^n\xi\| = \sqrt{\langle A^n\xi \mid A^n\xi\rangle} = \sqrt{\langle \xi \mid (A^*A)^n\xi\rangle}$ $= \|\,|A|^n\,\xi\|$). Thus with Beppo–Levi's theorem ([16] Corollary 2.4.2)

$$\begin{aligned}\int_{\mathbb{C}} \exp\{\gamma|z|\}\,d\mu(z) &= \sum_{n=0}^{\infty} \frac{\gamma^n}{n!}\int_{\mathbb{C}} |z|^n\,d\mu(z) = \sum_{n=0}^{\infty}\frac{\gamma^n}{n!}\langle w \mid |A|^n w\rangle \\ &\leq \sum_{n=0}^{\infty}\frac{\gamma^n}{n!}\|\,|A|^n w\| < \infty \qquad \text{for some } \gamma > 0 .\end{aligned}$$

(a): Let be $\mu,\varrho \in M_+(\mathbb{C})$ with $\int_{\mathbb{C}} e^{\gamma|z|}\,d\mu(z) < \infty$ and $\int_{\mathbb{C}} e^{\gamma|z|}\,d\varrho(z) < \infty$ for some $\gamma > 0$ and

$$\int_{\mathbb{C}} \overline{z}^l z^k \,d\mu(z) = \int_{\mathbb{C}} \overline{z}^l z^k \,d\varrho(z) \qquad \forall k,l \in \mathbb{N}_0 . \tag{A.2}$$

Define $\widehat{\mu},\widehat{\varrho} \in M_+(\mathbb{C})$ by $d\widehat{\mu}(z) = e^{\gamma|z|}\,d\mu(z)$ and $d\widehat{\varrho}(z) = e^{\gamma|z|}\,d\varrho(z)$. (A.2) now implies

$$\int_{\mathbb{C}} p(z,\overline{z})\,e^{-\gamma|z|}\,d\widehat{\mu}(z) = \int_{\mathbb{C}} p(z,\overline{z})\,e^{-\gamma|z|}\,d\widehat{\varrho}(z) \qquad \forall p \in \mathcal{P}. \tag{A.3}$$

Let $\widetilde{\mathcal{P}}$ be the vector space consisting of functions $z \in \mathbb{C} \mapsto p(z, \bar{z}) e^{-\gamma|z|}$ with $p \in \mathcal{P}$. Since $\lim_{|z| \to \infty} f(z) = 0$ for each $f \in \widetilde{\mathcal{P}}$ the set $\widetilde{\mathcal{P}}$ is a subspace of $C_0(\mathbb{C})$, the continuous functions on $\mathbb{C}$ vanishing at infinity. By the Stone–Weierstraß theorem $\widetilde{\mathcal{P}}$ is dense in $C_0(\mathbb{C})$ (use the one-point compactification of $\mathbb{C}$), from which with (A.3) follows $\widehat{\mu}(f) = \widehat{\varrho}(f)\ \forall f \in C_0(\mathbb{C})$, that is $\widehat{\mu} = \widehat{\varrho}$ resp. $\mu = \varrho$. ■

References

[1] Glauber, R.J.: *The Quantum Theory of Optical Coherence.* Phys. Rev. **130**, 2529 – 2539 (1963). *Coherent and Incoherent States of the Radiation Field.* Phys. Rev. **131**, 2766 – 2788 (1963). In de Witt, C., Blandin, A., Cohen–Tannoudji, C. (eds.) "Quantum Optics and Electronics". Proceedings, Les Houches 1964. New York: Gordon and Breach 1965.

[2] Loudon, R.: *The Quantum Theory of Light.* Oxford: Clarendon Press, 1979.

[3] Honegger, R., Rapp, A.: *General Glauber Coherent States on the Weyl Algebra and their Phase Integrals.* Physica A **167**, 945 – 961 (1990).

[4] Honegger, R., Rieckers, A.: *The General Form of non-Fock Coherent Boson States.* Publ. RIMS Kyoto Univ. **26**, 397 – 417 (1990).

[5] Rieckers, A.: *Macroscopic Coherent States of the Quantized Electromagnetic Field.* In Barut, A.O. (ed.) "New Frontiers in QED and Quantum Optics". Proceedings, Istanbul 1989. New York: Plenum 1990.

[6] Honegger, R., Rieckers, A.: *On higher order coherent states on the Weyl algebra.* To appear in Lett. Math. Phys. (1992).

[7] Honegger, R.: *The general form of microscopic coherent boson states.* Tübingen, preprint (1992).

[8] Bratteli, O., Robinson, D.W.: *Operator Algebras and Quantum Statistical Mechanics, Vol. II.* Berlin, Heidelberg, New York: Springer 1981.

[9] Bratteli, O., Robinson, D.W.: *Operator Algebras and Quantum Statistical Mechanics, Vol. I.* Berlin, Heidelberg, New York: Springer 1979.

[10] Evans, D.E., Lewis, J.T.: *Dilations of Irreversible Evolutions in Algebraic Quantum Theory.* Dublin: Communications of the Dublin Institute for Advanced Studies, Series A (Theoretical Physics), No. 24, 1977.

[11] Davies, E.B.: *Quantum Theory of Open Systems.* London, New York, San Francisco: Academic Press 1976.

[12] Rieckers, A.: *Classical States of Quantized Boson Fields.* In Niederle, J., Fischer, J. (eds.) "Selected Topics in Quantum Field Theory and Mathematical Physics". Proceedings, Liblice 1989. Singapor, New Jersey, London, Hong Kong: World Scientific 1990.

[13] Titulaer, U.M., Glauber, R.J.: *Density Operators for Coherent Fields..* Phys. Rev. **145**, 1041 – 1050 (1966).

[14] Reed, M., Simon,B.: *Methods of Modern Mathematical Physics II; Fourier Analysis, Self-Adjointness.* New York, London, Toronto, Sydney, San Francisco: Academic Press 1975.

[15] Conway, J.B.: *A Course in Functional Analysis.* New York, Berlin, Heidelberg, Tokyo: Springer 1985.

[16] Cohn, D.L.: *Measure Theory.* Boston, Basel, Stuttgart: Birkhäuser 1980.

[17] Hewitt, E., Ross, K.A.: *Abstract Harmonic Analysis I, II.* New York, Berlin, Heidelberg, Tokyo: Springer 1963, 1970.

[18] Hertle, J.: Private communications of work. In preparation, Tübingen.

[19] Reed, M., Simon,B.: *Methods of Modern Mathematical Physics I; Functional Analysis.* New York, London, Toronto, Sydney, San Francisco: Academic Press 1980.

[20] Haag, R., Kadison, R.V., Kastler, D.: *Nets of C^*-algebras and classification of states.* Comm. Math. Phys. **16**, 81 (1970).

[21] Takesaki, M.: *Theory of Operator Algebras, Vol. I.* Berlin, Heidelberg, New York: Springer 1979.

[22] Manuceau, J., Sirugue, M., Testard, D., Verbeure, A.: *The Smallest C^*-Algebra for the Canonical Commutation Relations.* Commun. math. Phys. **32**, 231 - 243 (1973).

[23] Honegger, R.: *Decomposition of Positive Sesquilinear Forms and the Central Decomposition of Gauge-Invariant Quasi-Free States on the Weyl Algebra.* Z. Naturforsch. **45a**, 17 - 28 (1990).

[24] Titulaer, U.M., Glauber, R.J.: *Density Operators for Coherent Fields..* Phys. Rev. **145**, 1041 - 1050 (1966).

[25] Hurst, C.A.: *Quantum Theory of the Free Electromagnetic Field.* In Gruber, B., Leuczewski, R. (eds.) "Symmetries in Science II". New York: Plenum 1986.

[26] Grundling, H., Hurst, C.A.: *A note on regular states and supplementary conditions.* Lett. Math. Phys. **15**, 205 - 212 (1988).

[27] Araki, H., Woods, E.J.: *Representations of the Canonical Commutation Relations Describing a Nonrelativistic Infinite Free Bose Gas.* J. Math. Phys. **4**, 637 - 662 (1963).

[28] Penrose, O., Onsager, L.: *Bose-Einstein Condensation and Liquid Helium.* Phys. Rev. **104**, 576 (1956).

[29] Yang, C.N.: *Concept of Off-Diagonal Long-Range Order and the Quantum Phases of Liquid He and of Superconducters.* Rev. Mod. Phys. **34**, 694 (1962).