Helv Phys Acta
Vol. 66 (1993)

0018-0238/93/050453-18$1.50+0.20/0

The Perturbative Construction of Symanzik's Improved Action for Φ_4^4 and QED_4

GEORG KELLER * **

MAX-PLANCK-INSTITUT FÜR PHYSIK
WERNER-HEISENBERG-INSTITUT
FÖHRINGER RING 6
D-8000 MÜNCHEN 40

(27. I. 1993)

Abstract : For the perturbative Euclidean massive Φ_4^4 and QED_4 (with a small photon mass) an explicit construction of Symanzik's improved action is presented. It is established rigorously that all the Green functions exhibit improved convergence as the momentum space UV cutoff is sent to infinity. These results are obtained by an application of the powerful yet technically simple flow equation method.

---

* Supported by the Swiss National Science Foundation

** Present address : Dept. of Mathematics, University of Virginia, Charlottesville VA 22903, USA

1. Introduction

One of the conventional methods to get approximate, numerical results for quantities associated to continuum field theories is to put the theory on a lattice and then perform numerical computations, using the powerful computers now available, to calculate the corresponding quantity. Putting aside the questions of finite volume effects and other types of systematical errors the problem then is how small the lattice spacing should be chosen to get reliable results.

If $< \cdot >_a$ denotes expectations computed with lattice spacing a, and if $< \cdot >_0$ stands for the continuum limit, then in general $| < O >_a - < O >_0 | \leq a \cdot Plog(a)$, where $Plog(a)$ is some polynomial in $log(a)$. It has been Symanzik's idea [1] that it may be possible to speed up the convergence to the continuum limit by adding to the lattice action a finite number of irrelevant terms defining thereby his "improved action". For example he claimed that by adding well-adjusted $\mathbb{Z}_2$-invariant local dimension 6 terms to the standard bare action of the $\mathbb{Z}_2$-invariant Φ_4^4 lattice theory it would be possible to get for all n-point functions $| < \phi(x_1) \cdots \phi(x_n) >_a^I - < \phi(x_1) \cdots \phi(x_n) >_0 | \leq a^4 \cdot Plog(a)$ instead of $| < \phi(x_1) \cdots \phi(x_n) >_a - < \phi(x_1) \cdots \phi(x_n) >_0 | \leq a^2 \cdot Plog(a)$, where the index I on $< \cdot >^I$ indicates that this expectation is computed with the help of the improved action. This sounds very interesting; however, a careful reading of ref. [1] reveals that it does not present a rigorous proof of the feasibility of the improvement program in Φ_4^4, but rather a (very convincing) plausibility argument (and explicit calculations to lowest orders). In gauge theories the situation is even worse since so far no one seems to have made an attempt to prove or disprove the implementability of Symanzik's program.

Although the mathematical status of Symanzik's improvement program was unclear various people took up his ideas and began to add specific irrelevant terms to the standard bare action of lattice gauge theories with the aim of improving the speed of convergence of some particular expectation values in lowest orders of perturbation theory [2-7]. Recently there has been renewed interest in the Symanzik improvement program in the computation of weak matrix elements (see e.g. [8]). Using Wilson's method for putting fermions on the lattice one finds cutoff effects of order a which are potentially rather large. The groups [8] have thus sought to reduce these effects systematically using the Wohlert-Sheikholeslami (clover) action [9] which adds to the Wilson term only one additional term of dimension 5 which is local in the fermion fields and hence easy to incorporate in the updating algorithms. The results seem rather encouraging.

The purpose of this paper is to outline a simple but nonetheless rigorous proof that Symanzik's improvement program works for the perturbative Euclidean massive Φ_4^4 (see also ref.[14]) and for the perturbative Euclidean QED_4 (with a massive photon), under the technically essential but philosphically unimportant condition that the lattice theory is replaced by the corresponding continuum theory with momentum space UV cutoff $\Lambda_0 \equiv a^{-1}$. After this paper had been finished I had been made aware of the work of ref.[14] where, using a method which is similar to the one employed in this paper, in this continuum setting Symanzik improvement had already been accomplished for Φ_4^4. However, the methods of [14] are still more complicated than the ones used in the present paper. Moreover, the more simplified treatment of Φ_4^4 presented in the first part of this paper is crucial in order to understand how to extend the proof to QED_4. This work thus offers a simplified proof of Symanzik improvement for Φ_4^4 and puts Symanzik's program for abelian gauge theories on a sounder basis.

To be somewhat more precise, let me briefly describe the main result for Φ_4^4. Let $\Lambda \in [0, \Lambda_0]$ be a scale parameter. Write $\mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_{n-1})$ for the connected amputated momentum space n-point Green function at r^{th} order ($r \geq 1$) in perturbation theory, with independent external momenta $p_1, \dots, p_{n-1}$, with UV cutoff Λ_0, and the internal momenta (of the Feynman diagrams contributing to it) integrated over the range $[\Lambda, \Lambda_0]$. Then it will be shown that for any given, fixed $N \geq 1$ the improved convergence bound

$$|\frac{d}{d\Lambda_0}\mathcal{L}_{r,n}^{\Lambda=0,\Lambda_0}(p_1, \dots, p_{n-1})| \leq \Lambda_0^{-2-N} \cdot Plog(\Lambda_0)$$

can be realized for all r, n by adding to the usual bare Φ_4^4 action suitable local irrelevant terms of dimension $\leq (4+N)$.

The method which will be employed to prove these results is an elaborated version of Polchinski's [10],[14] continuous renormalization group approach as formulated in [11] and extended to QED in [12]. In the present paper we do not go into all the mathematical details because these can be found in the aforementioned references (i.e. [11, 12]). Rather I have made an effort to give a readable account of the method of [11, 12] which hopefully conveys its inherent simplicity and gives convincing evidence that this method is indeed a very powerful formalism to investigate structural properties (e.g. perturbative renormalizability [11,12], renormalization of composite operators [13], ...) of perturbative field theories.

In the next Section the continuous renormalization group approach of [11] to the perturbative Euclidean massive Φ_4^4 will be reviewed and supplemented with the necessary ingredients to accommodate improvement terms. The main new result will be the definition of the improvement terms.

The proof of improved convergence for Φ_4^4 will be given in Section 3, and in Section 4 I will comment on how to deal with QED.

2. Construction of the Improved Action for Φ_4^4

2.1 The differential flow equations for the improved Green functions

Choose a number N, $N \in \mathbb{N}$, and keep it fixed in what follows; $\Lambda_0^{-N}(\Lambda_0 \equiv \text{UV cutoff})$ will be the factor by which we will increase the speed of convergence of the Green functions.

We make the following Ansatz for the improved action of the Euclidean, massive, and (for the sake of simplicity of this presentation) also $\mathbb{Z}_2$-invariant Φ_4^4 theory:

$$S^{\Lambda,\Lambda_0;N}(\phi) := \frac{1}{2}\int d^4x\, d^4y\, \phi(x)(C_\Lambda^{\Lambda_0})^{-1}(x-y)\phi(y) + L^{\Lambda_0;N}(\phi) \quad . \tag{1}$$

The conventional kinetic term $\int d^4x\, \phi(x)(-\Box + m^2)\phi(x)$, $m^2 > 0$, has been replaced by the regularized version $\int d^4x\, d^4y\, \phi(x)(C_\Lambda^{\Lambda_0})^{-1}(x-y)\phi(y)$, where the regularization will be chosen in such a way that in the functional integral more or less just those components $\phi(p)$ of ϕ are integrated out which belong to momenta p with $|p| \in [\Lambda, \Lambda_0]$. The first term on the r.h.s. of (1) represents the 0^{th} order contribution to $S^{\Lambda,\Lambda_0;N}$, and $L^{\Lambda_0;N}$ comprises all terms which are of higher order in perturbation theory (i.e. ϕ^4 interaction vertex, counter terms, improvement terms). I wish to stress that we are going to improve convergence only for the nontrivial, i.e. at least 1^{st} order, Green functions, because the 0^{th} order 2-point function (see (2), (3) below) is given by a simple, explicitly known formula; moreover it is easily seen that the momentum space 0^{th} order 2-point function converges exponentially.

Let us discuss the regularization. We put

$$C_\Lambda^{\Lambda_0}(x-y) := \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2+m^2}\Big(R(\Lambda_0,p) - R(\Lambda,p)\Big) \quad , \tag{2}$$

where in principle one would like to set $R(\Lambda,p) = \chi_{[0,\Lambda]}(|p|)$, but due to the nondifferentiable nature of the characteristic function this choice would be a disaster for doing easy estimates. So for technical reasons it is much more convenient to require

$$R(\Lambda,p) := \left(1 - e^{-\Lambda/m}\right) \cdot K(\tfrac{p^2}{\Lambda^2}) \quad , \tag{3}$$

where $K(\frac{p^2}{\Lambda^2})$ is a smooth version of $\chi_{[0,\Lambda]}(|p|)$; namely one imposes that K should be smooth with $K(a) = 1$, for $0 \leq a \leq 1$, and $K(a) = 0$, for $4 \leq a$. The factor $(1 - e^{-\Lambda/m})$ guarantees that $R(\Lambda,p)$ converges to 0, as $\Lambda \to 0$, uniformly in p. As a result of (2), (3) we see that roughly speaking

$$C_\Lambda^{\Lambda_0}(p) \sim \begin{cases} 0 & , \quad |p| < \Lambda \text{ or } |p| > \Lambda_0 \\ (p^2 + m^2)^{-1} & , \quad |p| \in [\Lambda, \Lambda_0] \quad . \end{cases}$$

The bare interaction $L^{\Lambda_0;N}$ is split into the standard Φ_4^4 piece, G^{Λ_0}, and the improvement terms, $I^{\Lambda_0;N}$:

$$L^{\Lambda_0;N} = G^{\Lambda_0} + I^{\Lambda_0;N} \tag{4}$$

where

$$G^{\Lambda_0} := \delta m^2 \cdot \int d^4x\, \phi^2(x) \; - \; \delta Z \cdot \int d^4x\, \phi(x)\Box\phi(x) \; + \lambda \cdot \int d^4x\, \phi^4(x) \quad , \tag{5}$$

$$\begin{aligned} I^{\Lambda_0;N} \; := \; \int d^4x\, (&\text{local, Euclidean invariant, even polynomial in } \phi \\ &\text{and its derivatives, of dimension } \leq 4+N) \quad . \end{aligned} \tag{6}$$

Here the coefficients δm^2, δZ, λ and those present in $I^{\Lambda_0;N}$ are formal power series (fps) in a parameter which is called g; the interpretation of g heavily depends on the renormalization conditions; for instance one can impose renormalization conditions such that g may be identified with the renormalized coupling "constant". One of the important points is that δm^2, δZ, ... are supposed to be at least first order in g.

The major aim of this work is to show that δm^2, δZ, ... can be determined uniquely in such a way that

a) any renormalization condition on the mass, the wave function and the vertex can be realized;

b) improved convergence of order Λ_0^{-2-N} holds true for all the renormalized Green functions.

With our notation the generating functional, $Z^{\Lambda,\Lambda_0;N}(J)$, of the (unnormalized) Green functions with $|p| \in [\Lambda, \Lambda_0]$ integrated out (formally) reads

$$Z^{\Lambda,\Lambda_0;N}(J) \ := \ \int D\phi \ e^{-S^{\Lambda,\Lambda_0;N}(\phi) + \int d^4x\, \phi(x)J(x)} \quad . \tag{7}$$

The generating functional, $-log\, Z^{\Lambda,\Lambda_0;N}(J)$, of the corresponding connected Green functions is written as the sum of its 0^{th} order part, $-\frac{1}{2}\int d^4x\, d^4y\, J(x)C_\Lambda^{\Lambda_0}(x-y)\ J(y)$, and of the higher order remainder, $L^{\Lambda,\Lambda_0;N}$:

$$-log\, Z^{\Lambda,\Lambda_0;N}(J) \ = \ -\frac{1}{2}\int d^4x\ d^4y\ J(x)C_\Lambda^{\Lambda_0}(x-y)J(y) \ + \ L^{\Lambda,\Lambda_0;N}(\phi)\big|_{\phi = C_\Lambda^{\Lambda_0}J} \quad . \tag{8}$$

In other words, $L^{\Lambda,\Lambda_0;N}(\phi)$ is the generating functional of the nontrivial (i.e. at least 1^{st} order) connected amputated Green functions with momenta integrated out in the range $[\Lambda, \Lambda_0]$.

Expanding $L^{\Lambda,\Lambda_0;N}$ as a fps in g, and at each order in g as a polynomial in $\phi(p)$, the expansion "coefficients"

$$\mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N}(p_1,\dots,p_{n-1}) \quad , \quad r = 1,2,\dots \quad , \quad n \ = \ 2,4,6,\dots \quad , \tag{9}$$

are precisely the momentum space connected amputated n-point functions at perturbative order r, with independent external momenta $p_1,\dots,p_{n-1}$; the n^{th} external momentum, p_n, is fixed by momentum conservation as $p_n = -p_1 - p_2 - \dots - p_{n-1}$. The $\mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N}$ respect among others S_n (i.e. Bose) and $O(4)$ symmetry:

$$\mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N}(p_1,\dots,p_{n-1}) \ = \ \mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N}(p_{\pi(1)},\dots,p_{\pi(n-1)}) \ , \ \forall\pi \in S_n \quad , \tag{10}$$

and

$$\mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N}(Ap_1,\dots,Ap_{n-1}) \ = \ \mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N}(p_1,\dots,p_{n-1}) \ , \ \forall A \in O(4) \quad . \tag{11}$$

Using the methods of [11] it can be shown that the $\mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N}$ satisfy an infinite set of coupled differential equations:

$$\partial_\Lambda \mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N}(p_1,\dots,p_{n-1}) \ = \ \mathcal{F}_{r,n}^{\Lambda,\Lambda_0;N}(p_1,\dots,p_{n-1}) \quad , \tag{12}$$

where $\partial_\Lambda \equiv \partial/\partial\Lambda$ and

$$\mathcal{F}^{\Lambda,\Lambda_0;N}_{r,n}(p_1,\dots,p_{n-1}) := -\binom{n+2}{2}\int \frac{d^4q}{(2\pi)^4}\,\frac{\partial_\Lambda R(\Lambda,q)}{q^2+m^2}\cdot \mathcal{L}^{\Lambda,\Lambda_0;N}_{r,n+2}(q,-q,p_1,\dots,p_{n-1})$$
$$+\sum_{\substack{r'+r''=r\\ n'+n''=n+2}} \frac{n'\cdot n''}{2}\left[\frac{\partial_\Lambda R(\Lambda,Q)}{Q^2+m^2}\cdot \mathcal{L}^{\Lambda,\Lambda_0;N}_{r',n'}(p_1,\dots,p_{n'-1})\right.$$
$$\left.\cdot\,\mathcal{L}^{\Lambda,\Lambda_0;N}_{r'',n''}(-Q,p_{n'},\dots,p_{n-1})\right]_{symm}. \tag{13}$$

$Q := -p_1 - \dots - p_{n'-1}$, $[\cdots]_{symm}$ denotes symmetrization with respect to the momenta $p_1,\dots,p_n$; notice that because of $r', r'' \geq 1$ the restriction $r' + r'' = r$ implies that $r', r'' < r$. Thus the r.h.s. of the differential flow equation (12) consists of a term which is linear in $\mathcal{L}$ ($\mathcal{L}$ being of the same order as, but having a larger number of external legs than, the differentiated $\mathcal{L}$ on the l.h.s.) and of a piece which is quadratic in $\mathcal{L}$ (the $\mathcal{L}$'s contributing to which being of strictly lower order than r).

It is easy to provide an heuristic motivation for the correctness of (12), (13): $\mathcal{L}^{\Lambda,\Lambda_0;N}_{r,n}$ can be written as a sum over (finitely many) connected amputated Feynman diagrams whose vertices stem from $L^{\Lambda_0;N}$ (see (4) - (6)) and whose propagators $C^{\Lambda_0}_{\Lambda}(p)$ (cf. (2)) are the only Λ-dependent quantities. Therefore the derivative ∂_Λ acting on such a Feynman diagram acts only on the lines of the diagram; if the line on which it acts is a 1PI line then this produces a contribution to the (1-loop graph type) linear part on the r.h.s. of (12), and otherwise we obtain a contribution to the (tree graph type) quadratic piece on the r.h.s. of (12).

2.2. The boundary conditions for the improved Green functions

The differential equation (12), (13) tells us how the connected amputated Green functions vary if we slightly change the range of the momenta over which we integrate. The basic strategy to prove perturbative renormalizability and improved convergence will be to get suitable estimates on the solution $\{\mathcal{L}^{\Lambda,\Lambda_0;N}_{r,n}\}$ of (12), (13). For this reason it is useful to investigate the boundary conditions which the flow of Green functions obeys, i.e. we are interested in the behaviour of $\{\mathcal{L}^{\Lambda,\Lambda_0;N}_{r,n}\}$ as $\Lambda = 0$ and $\Lambda = \Lambda_0$. Now, if $\Lambda = \Lambda_0$ then no momenta at all are integrated out, so obviously

$$\mathcal{L}_{r,n}^{\Lambda_0,\Lambda_0;N} = \mathcal{L}_{r,n}^{\Lambda_0;N} \quad ; \tag{14}$$

in analogy to (4) we write

$$\mathcal{L}_{r,n}^{\Lambda_0;N} \equiv \mathcal{G}_{r,n}^{\Lambda_0} + \mathcal{I}_{r,n}^{\Lambda_0;N} \quad . \tag{15}$$

On the other hand, if $\Lambda = 0$ then all momenta $\leq \Lambda_0$ are integrated out and we wish to impose the renormalization conditions reading

$$\mathcal{L}_{r,4}^{0,\Lambda_0;N}(p_i = 0) \stackrel{!}{=} \lambda_r^R \tag{16}$$

and

$$\mathcal{L}_{r,2}^{0,\Lambda_0;N}(0) \stackrel{!}{=} (\delta m^2)_r^R \quad , \quad \partial_{p_\mu}\partial_{p_\nu}\mathcal{L}_{r,2}^{0,\Lambda_0;N}(0) \stackrel{!}{=} 2\cdot\delta_{\mu,\nu}\cdot(\delta Z)_r^R \quad , \tag{17}$$

where the renormalization constants $\lambda_r^R, (\delta m^2)_r^R, (\delta Z)_r^R$ are supposed to be finite and independent of Λ_0, but otherwise they can be chosen at will. Probably the most standard choice of renormalization conditions is $(\delta m^2)_r^R = (\delta Z)_r^R = 0$ and $\lambda_r^R = (4!)^{-1}\cdot\delta_{r,1}$ so that m and g may be viewed as the renormalized mass and coupling "constant". Of course we will have to show that the general renormalization conditions (16), (17) can really be realized by adjusting the bare parameters $\delta m^2, \delta Z$ and λ appropriately. This will be done in a short while.

2.3. The definition of the improvement terms

To begin with, a few technical remarks need to be added. Notice, first, that we may assume without loss of generality that the UV cutoff Λ_0 belongs to the interval $[\Lambda_{0,\min},\infty)$, where $\Lambda_{0,\min}$ is some fixed positive number, e.g. $\Lambda_{0,\min} = 10^{12}m$. Next it is convenient to introduce yet another scale parameter, Λ_1; the only restriction Λ_1 has to obey is $0 < \Lambda_1 < \Lambda_{0,\min}$; the point is that we will establish good bounds on the renormalized Green functions $\mathcal{L}^{0,\Lambda_0;N}$ by relating these to $\mathcal{L}^{\Lambda_1,\Lambda_0;N}$, and these latter will be bounded by doing estimates on $\mathcal{L}^{\Lambda,\Lambda_0;N}$, $\Lambda \in [\Lambda_1,\Lambda_0]$, by using the differential flow equation (12) for $\Lambda \in [\Lambda_1,\Lambda_0]$; to fix ideas let us put $\Lambda_1 := \frac{1}{2}\Lambda_{0,\min}$. I wish to emphasize that there is no physics contained in Λ_1 and $\Lambda_{0,\min}$. Finally, for $k \in \mathbb{Z}$ the symbol τ^k is introduced; if

$k \geq 0$ then $\tau^k f(p_1, \ldots, p_{n-1})$ stands for the Taylor expansion of $f(p_1, \ldots, p_{n-1})$ around $p_1 = \cdots = p_{n-1} = 0$ up to and including the k^{th} order in $p_1, \ldots, p_{n-1}$, i.e.

$$\tau^k f(p_1, \ldots, p_{n-1}) \;:=\; \sum_{j=0}^{k} \tfrac{1}{j!}\,(\tfrac{d}{dt})^j\, f(tp_1, \ldots, tp_{n-1})\Big|_{t=0} \quad ; \tag{18}$$

if $k < 0$ we set $\tau^k f \;:=\; 0$.

Let us discuss now the construction of the improvement terms $\{\mathcal{I}_{r,n}^{\Lambda_0;N}\}$. Keeping in mind that the Green functions $\mathcal{L}^{0,\Lambda_0;N}$, computed with the help of the $\mathcal{I}^{\Lambda_0;N}$'s, should be closer to the continuum limit, $\mathcal{L}^{0,\infty}$, than the unimproved $\mathcal{L}^{0,\Lambda_0}$'s one expects that the $\mathcal{I}^{\Lambda_0;N}$'s should contain some information on the theories with larger UV cutoff, i.e. on $\mathcal{L}^{0,\Lambda_0'}$'s with $\Lambda_0' \geq \Lambda_0$. This intuition is made more precise in our formula

$$\mathcal{I}_{r,n}^{\Lambda_0;N}(p_1, \ldots, p_{n-1}) \;:=\; -\int_{\Lambda_0}^{\infty} d\Lambda_0'\;\partial_{\Lambda_0'} \int_{\Lambda_1}^{\Lambda_0'} d\Lambda'\,\tau^{4+N-n} \mathcal{F}_{r,n}^{\Lambda',\Lambda_0';N}(p_1, \ldots, p_{n-1}) \quad , \tag{19}$$

$$\big(\Lambda_0 \,\in\, [\Lambda_{0,\min}, \infty) \quad , \quad r \geq 1\big)$$

($\mathcal{F}$ is defined in (13)) which, as I will demonstrate in a moment, gives a recursive definition of the improvement terms. Equation (19) has not been obtained by trial and error, but rather (19) seems to be the only natural thing to do once one attempts to prove our improved convergence Theorem (see Section 3).

It is evident that (19) implies that the dimension of $\mathcal{I}_{r,n}^{\Lambda_0;N}$ does not exceed $4+N$, as required; in particular $\mathcal{I}_{r,n}^{\Lambda_0;N} \;\equiv\; 0$ if $n > 4+N$. Since $\mathcal{G}_{r,n}^{\Lambda_0} \;\equiv\; 0$ for $n > 4$, and because $N \geq 1$, (15) shows that also $\mathcal{L}_{r,n}^{\Lambda_0;N} \;\equiv\; 0$, if $n > 4+N$. Let us prove now that (16), (17) and (19) provide us also with a unique definition, respecting all the required symmetries, of the remaining improvement and bare interaction vertices, i.e. of $\{\mathcal{I}_{r,n}^{\Lambda_0;N}, \mathcal{L}_{r,n}^{\Lambda_0;N} \;:\; n \leq 4+N\}$. The proof will be carried out using induction in (r,n).

Induction hypothesis: For $\Lambda_0 \geq \Lambda_{0,\min}$, and for all pairs (r',n') with either ($r' < r$ and $n' \geq 1$) or ($r' = r$ and $n' > n$) the $\mathcal{I}_{r',n'}^{\Lambda_0;N}$ and $\mathcal{L}_{r',n'}^{\Lambda_0;N}$ are uniquely defined $O(4)$- and $S_{n'}$-symmetric polynomials in $p_1, \ldots, p_{n'-1}$; moreover, for $n' =$ odd we have $\mathcal{I}_{r',n'}^{\Lambda_0;N} = \mathcal{L}_{r',n'}^{\Lambda_0;N} = 0$.

Induction step (i.e. prove the same properties for $\mathcal{I}_{r,n}^{\Lambda_0;N}$ and $\mathcal{L}_{r,n}^{\Lambda_0;N}$): The Green function $\mathcal{L}_{r'',n''}^{\Lambda,\Lambda_0;N}$ is a sum over connected amputated Feynman diagrams whose vertices are of the type $\mathcal{L}_{r''',n'''}^{\Lambda_0,\Lambda_0;N}$, where either ($r''' < r''$ and $n''' \geq 1$) or ($r''' = r''$ and $n''' \geq n''$);

thus the induction hypothesis says that all the Green functions $\mathcal{L}_{r',n'}^{\Lambda,\Lambda_0;N}$, (r',n') as in the induction hypothesis, are uniquely determined (and have all the required symmetries). Comparing (19) to (13) we thus see that by the induction hypothesis the r.h.s. of (19) is indeed well-defined, and that $\mathcal{I}_{r,n}^{\Lambda_0;N} = 0$, if $n =$ odd; obviously $\mathcal{I}_{r,n}^{\Lambda_0;N}$ is a polynomial in $p_1, \cdots, p_{n-1}$. The $O(4)$-invariance of $R(\Lambda, Q)$, $R(\Lambda, q)$ and the induction hypothesis imply $O(4)$-invariance of $\mathcal{F}_{r,n}^{\Lambda',\Lambda_0';N}$, and the induction hypothesis also says that $\mathcal{F}_{r,n}^{\Lambda',\Lambda_0';N}$ is S_n-symmetric; using (18) this means that $\mathcal{I}_{r,n}^{\Lambda_0;N}$ inherits these symmetries. Now, if $n > 4$ we have $\mathcal{L}_{r,n}^{\Lambda_0;N} = \mathcal{I}_{r,n}^{\Lambda_0;N}$, so in this case the induction step is completed. However, if $n \leq 4$ the renormalization conditions (16), (17) enter the game and we encounter the following situation: By the induction hypothesis all the $\mathcal{L}_{r',n'}^{\Lambda_0;N}$, and by the induction step also $\mathcal{I}_{r,n}^{\Lambda_0;N}$, are known; these vertices, together with the counter term $\mathcal{G}_{r,n}^{\Lambda_0}$, determine the Green function $\mathcal{L}_{r,n}^{0,\Lambda_0;N}$, thus $\mathcal{G}_{r,n}^{\Lambda_0}$ can be adjusted uniquely in such a way that (16), if $n = 4$, or (17), if $n = 2$, is satisfied. This finishes the induction step for $n \leq 4$.

It remains to be verified that with this kind of induction scheme we really cover all pairs (r, n), $r \geq 1$, $n \geq 1$. But this is easy to understand: First of all notice that for ($r = 1$, $n = 4 + N$) the induction hypothesis is trivially fulfilled, so we may start our inductive process at ($r = 1$, $n = 4 + N$). After the first induction step we are lead to treat ($r = 1$, $n = 4 + N - 1$), afterwards ($r = 1$, $n = 4 + N - 2$), and so on, until we have reached and dealt with ($r = 1$, $n = 1$). But now the induction hypothesis is automatically true for ($r = 2$, $n = 4 + N$), and so we proceed with the bare vertices at 2^{nd} order in g. After this we move to 3^{rd} order in g, and so forth. This finally proves the claim.

3. Improved Convergence for Φ_4^4

In order to state the subsequent results we need to introduce a suitable measure of the size of $\partial_p^w \mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N}$, where $\partial_p^w \equiv \partial_{p_1}^{w_1} \cdots \partial_{p_{n-1}}^{w_{n-1}}$ is a momentum derivative of order $|w| := \sum_{j=1}^{n-1} |w_j|$, $|w_j| := \sum_{\mu=1}^{4} w_{j,\mu}$, $w_j \equiv (w_{j,1}, \cdots, w_{j,4})$, $w_{j,\mu} \in \mathbb{N}_0$. Namely, we define $\| \partial^z \mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N} \|_{(a,b)}$ to be the maximum over all multiindices w with $|w| = z$ and over all $(p_1, \cdots, p_{n-1})$ with $|p_j| \leq \max\{2a, 2b\}$, $1 \leq j \leq n-1$, of $|\partial_p^w \mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N}(p_1, \cdots, p_{n-1})|$.

Choose some η, $0 \leq \eta < \infty$, and keep it fixed in what follows. η will indicate the range of the momenta $p_1, \dots, p_{n-1}$ within which (i.e. for $|p_j| \leq 2\eta$, $1 \leq j \leq n-1$) the improved convergence of $\mathcal{L}_{r,n}^{0,\Lambda_0;N}$ will be shown.

Theorem: *For the general renormalization conditions (16), (17), for any fixed η, $0 \leq \eta < \infty$, all $z \geq 0$, all Λ_0 with $\Lambda_0 \geq \Lambda_{0,\min}$ and all $\Lambda \in [\Lambda_1, \Lambda_0]$ we have the bounds*

$$\| \partial^z \mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N} \|_{(\Lambda,\eta)} \leq \Lambda^{4-n-z} \cdot Plog(\tfrac{\Lambda}{\Lambda_1}) \tag{20}$$

$$\| \partial^z \partial_{\Lambda_0} \mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N} \|_{(\Lambda,\eta)} \leq \Lambda_0^{-2-N} \cdot \Lambda^{5+N-n-z} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \quad , \tag{21}$$

where $Plog(\cdot)$ stands for some polynomial in $log(\cdot)$ whose coefficients depend neither on Λ nor on Λ_0 but which may depend on $\eta, \Lambda_1, N, r, n, z, \dots$.

Before proceeding to the proof of our Theorem I wish to point out some of its immediate and important consequences.

Corollary 1:

$$\| \partial_{\Lambda_0} \mathcal{L}_{r,n}^{0,\Lambda_0;N} \|_{(0,\eta)} \leq \Lambda_0^{-2-N} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \quad . \tag{22}$$

Proof: It follows from [11] that $\mathcal{L}_{r,n}^{0,\Lambda_0;N}$ may be written as a sum over connected amputated Feynman diagrams whose vertices are of the type $\mathcal{L}_{r',n'}^{\Lambda_1,\Lambda_0;N}$ (instead of the conventional $\mathcal{L}_{r',n'}^{\Lambda_0;N}$) and whose propagators are given by $(p^2+m^2)^{-1} \cdot R(\Lambda_1, p)$ (instead of $(p^2+m^2)^{-1} \cdot R(\Lambda_0, p)$). Thus each internal line of such a Feynman diagram carries a momentum p with $|p| \leq 2\Lambda_1$, and the derivative ∂_{Λ_0} acts just on the vertices $\mathcal{L}_{r',n'}^{\Lambda_1,\Lambda_0;N}$. A short moment's thought, employing (20) and (21) at $\Lambda = \Lambda_1$ and $z = 0$, leads to (22). ■

Corollary 2: *The renormalized Green functions $\mathcal{L}_{r,n}^{0,\Lambda_0;N}(p_1, \dots, p_{n-1})$ converge as $\Lambda_0 \to \infty$, if $|p_j| \leq 2\eta$, $1 \leq j \leq n-1$. In fact we have the improved convergence bound*

$$|\mathcal{L}_{r,n}^{0,\Lambda_0;N}(p_1, \dots, p_{n-1}) - \mathcal{L}_{r,n}^{0,\infty;N}(p_1, \dots, p_{n-1})| \leq \Lambda_0^{-1-N} \cdot Plog(\tfrac{\Lambda_0}{\Lambda_1}) \quad , \tag{23}$$

if $|p_j| \leq 2\eta$, $1 \leq j \leq n-1$.

Proof: (23) is a straightforward consequence of (22) (Cauchy sequences). ∎

Corollary 3: *In the limit* $\Lambda_0 \to \infty$ *we find*

$$\mathcal{L}^{0,\infty;N}_{r,n}(p_1,\dots,p_{n-1}) = \mathcal{L}^{0,\infty}_{r,n}(p_1,\dots,p_{n-1}) \quad , \tag{24}$$

if $|p_j| \leq 2\eta$, $1 \leq j \leq n-1$, *where* $\{\mathcal{L}^{\Lambda,\Lambda_0}_{r,n}\}$ *is the set of standard, i.e. unimproved, connected amputated* Φ^4_4 *Green functions defined using the same regularization (2), (3) and the same renormalization conditions (16), (17).*

Proof: See Proposition 6 in ref. [11]. ∎

Let me finally remark that the bound (23) is not the best one obtainable with our methods. Indeed, in order to derive (21) it is *not* crucial that the theory under consideration is actually $\mathbb{Z}_2$-symmetric; but we could exploit this symmetry to end up with an additional factor $\Lambda \cdot \Lambda_0^{-1}$ on the r.h.s. of (21) and so we would get an extra Λ_0^{-1} on the r.h.s. of (23).

Proof of Theorem: The proof is by induction, and the induction scheme is (in principle) the one which was employed to discuss equation (19). That scheme can be applied also in the present situation because there is for each r a number $n(r)$ such that $\mathcal{L}^{\Lambda,\Lambda_0;N}_{r,n} \equiv 0$, if $n > n(r)$; it is easy to verify that one may set $n(r) := (2+N)r + 2$.

So choose (r,n). The induction hypothesis is that the inequalities (20), (21) hold for all $\| \partial^z(\partial_{\Lambda_0})\mathcal{L}^{\Lambda,\Lambda_0;N}_{r',n'} \|_{(\Lambda,\eta)}$ with $z \geq 0$, $\Lambda_0 \geq \Lambda_{0,\min}$ and either ($r' < r$, $n' \geq 1$) or ($r' = r$, $n' > n$).

In the remainder of this Section I will sketch the induction step. Remember that each time the symbol Plog($\cdot$) appears it stands in general for a new polynomial in log($\cdot$).

I: Some preparations

Employing the induction hypothesis on (13) using (20) it is not difficult to check that for $\Lambda \in [\Lambda_1, \Lambda_0]$

$$\| \partial^z \mathcal{F}_{r,n}^{\Lambda,\Lambda_0;N} \|_{(\Lambda,\eta)} \quad \leq \quad \Lambda^{3-n-z} \cdot \mathrm{Plog}(\tfrac{\Lambda}{\Lambda_1}) \quad , \tag{25}$$

and similarly (13), (20) and (21) yield (as usual for $\Lambda \in [\Lambda_1, \Lambda_0]$)

$$\| \partial^z \partial_{\Lambda_0} \mathcal{F}_{r,n}^{\Lambda,\Lambda_0;N} \|_{(\Lambda,\eta)} \quad \leq \quad \Lambda_0^{-2-N} \cdot \Lambda^{4+N-n-z} \cdot \mathrm{Plog}(\tfrac{\Lambda_0}{\Lambda_1}) \quad . \tag{26}$$

The bound (25) implies that e.g. for $\Lambda_0 \geq \Lambda_{0,\min}$

$$\| \partial^z \tau^{4+N-n} \mathcal{F}_{r,n}^{\Lambda_0',\Lambda_0';N} \|_{(\Lambda_0,\eta)} \quad \leq \quad \sum_{j=z}^{4+N-n} \Lambda_0^{j-z} \cdot (\Lambda_0')^{3-n-j} \cdot \mathrm{Plog}(\tfrac{\Lambda_0'}{\Lambda_1}) \quad , \tag{27}$$

whereas (26) can be applied to prove that e.g. for $\Lambda_0 \geq \Lambda_1$

$$\| \partial^z \tau^{4+N-n} \partial_{\Lambda_0'} \mathcal{F}_{r,n}^{\Lambda',\Lambda_0';N} \|_{(\Lambda_0,\eta)} \quad \leq \quad \sum_{j=z}^{4+N-n} \Lambda_0^{j-z} (\Lambda')^{4+N-n-j} \cdot (\Lambda_0')^{-2-N} \cdot \mathrm{Plog}(\tfrac{\Lambda_0'}{\Lambda_1}) \quad . \tag{28}$$

Taylor's remainder formula

$$(1-\tau^k) f(p_1, \dots, p_{n-1}) \quad = \quad \int_0^1 dt \, \frac{(1-t)^k}{k!} \, (\tfrac{d}{dt})^{k+1} f(tp_1, \dots, tp_{n-1}) \tag{29}$$

and (25) can be used to show that for $\Lambda_1 \leq \Lambda \leq \Lambda_0$ and $4+N-n \geq 0$

$$\| \partial^z (1-\tau^{4+N-n}) \mathcal{F}_{r,n}^{\Lambda_0,\Lambda_0;N} \|_{(\Lambda,\eta)} \quad \leq \quad \Lambda_0^{-2-N} \cdot \Lambda^{5+N-n-z} \cdot \mathrm{Plog}(\tfrac{\Lambda_0}{\Lambda_1}) \quad , \tag{30}$$

but for $4+N-n < 0$ the inequality (30) follows directly from (25) and from the observation that $\Lambda_0^{3-n-z} = \Lambda_0^{-2-N} \cdot \Lambda_0^{5+N-n-z} \leq \Lambda_0^{-2-N} \cdot \Lambda^{5+N-n-z}$. Finally, by (26) (and (29)) we obtain for $\Lambda_1 \leq \Lambda \leq \Lambda'$

$$\| \partial^z (1-\tau^{4+N-n}) \partial_{\Lambda_0} \mathcal{F}_{r,n}^{\Lambda',\Lambda_0;N} \|_{(\Lambda,\eta)} \quad \leq \quad \Lambda_0^{-2-N} \cdot \Lambda^{5+N-n-z} \cdot (\Lambda')^{-1} \cdot \mathrm{Plog}(\tfrac{\Lambda_0}{\Lambda_1}) \quad . \tag{31}$$

Notice also that for any $\Lambda \in [\Lambda_1, \Lambda_0]$ and $n+z \geq 5$

$$\Lambda_0^{4-n-z} \cdot \mathrm{Plog}(\tfrac{\Lambda_0}{\Lambda_1}) \quad \leq \quad \Lambda^{4-n-z} \cdot \mathrm{Plog}(\tfrac{\Lambda}{\Lambda_1}) \quad . \tag{32}$$

II: Induction step for (20)

(a) z with $z+n \geq 5+N$, $\Lambda = \Lambda_0$: Because for any w with $|w| = z$ we have $\partial_p^w \mathcal{L}_{r,n}^{\Lambda_0,\Lambda_0;N} \equiv 0$ (there are at most dimension $\leq 4+N$ terms present in the bare interaction) we certainly can write

$$\| \partial^z \mathcal{L}_{r,n}^{\Lambda_0,\Lambda_0;N} \|_{(\Lambda_0,\eta)} \quad \leq \quad \Lambda^{4-n-z} \cdot \mathrm{Plog}(\tfrac{\Lambda}{\Lambda_1}) \quad , \tag{33}$$

where $\Lambda \in [\Lambda_1, \Lambda_0]$.

(b) z with $4+N \geq z+n \geq 5$, $\Lambda = \Lambda_0$: For $|w| = z$ we evidently have $\partial_p^w \mathcal{G}_{r,n}^{\Lambda_0} \equiv 0$; so by (14), (15) and (19)

$$\begin{aligned}\| \partial^z \mathcal{L}_{r,n}^{\Lambda_0,\Lambda_0;N} \|_{(\Lambda_0,\eta)} \quad \leq \quad & \int_{\Lambda_0}^{\infty} d\Lambda_0' \ \Big\{ \| \partial^z \tau^{4+N-n} \mathcal{F}_{r,n}^{\Lambda_0',\Lambda_0';N} \|_{(\Lambda_0,\eta)} \\ & + \int_{\Lambda_1}^{\Lambda_0'} d\Lambda' \, \| \partial^z \tau^{4+N-n} \partial_{\Lambda_0'} \mathcal{F}_{r,n}^{\Lambda',\Lambda_0';N} \|_{(\Lambda_0,\eta)} \Big\} \quad ,\end{aligned}$$

and applying (27), (28) we can perform the integrals; using (32) we obtain the bound (33) also in the present case.

(c) z with $z+n \geq 5$, $\Lambda \in [\Lambda_1, \Lambda_0]$: Acting with ∂_p^w, $|w| = z$, on (12) and integrating it from Λ up to Λ_0 gives, upon taking norms,

$$\| \partial^z \mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N} \|_{(\Lambda,\eta)} \quad \leq \quad \| \partial^z \mathcal{L}_{r,n}^{\Lambda_0,\Lambda_0;N} \|_{(\Lambda_0,\eta)} \quad + \quad \int_{\Lambda}^{\Lambda_0} d\Lambda' \, \| \partial^z \mathcal{F}_{r,n}^{\Lambda',\Lambda_0;N} \|_{(\Lambda',\eta)} \quad ;$$

(33) together with (25), (32) lead to (20).

(d) z with $z+n \leq 4$; as an illustration the case $n = 4$, $z = 0$ ($n = 2$, $0 \leq z \leq 2$ is treated similarly [11]): Remember that $\mathcal{L}_{r,4}^{0,\Lambda_0;N}$ is a sum over Feynman diagrams whose vertices are of the form $\mathcal{L}_{r',n'}^{\Lambda_1,\Lambda_0;N}$ and whose propagators look like $(p^2+m^2)^{-1} \cdot R(\Lambda_1, p)$; so, using the renormalization condition (16) and the induction hypothesis for $\{\mathcal{L}_{r',n'}^{\Lambda_1,\Lambda_0;N}\}$ it is easily seen that

$$|\mathcal{L}_{r,4}^{\Lambda_1,\Lambda_0;N}(p_i = 0)| \quad \leq \quad \mathrm{const} \quad , \tag{34}$$

where "const" stands for some Λ_0-independent number. Integrating (12) from Λ_1 up to Λ, at $p_1 = p_2 = p_3 = 0$, using (34) and (25) we arrive at

$$|\mathcal{L}_{r,4}^{\Lambda,\Lambda_0;N}(0)| \quad \leq \quad \mathrm{Plog}(\tfrac{\Lambda}{\Lambda_1}) \quad . \tag{35}$$

Applying (29) with $k = 0$ on $f = \mathcal{L}_{r,4}^{\Lambda,\Lambda_0;N}$, using the already established bound (20) for $\| \partial^1 \mathcal{L}_{r,4}^{\Lambda,\Lambda_0;N} \|_{(\Lambda,\eta)}$, we end up with (20) for $n = 4$, $z = 0$.

III: Induction step for (21)

(a) z with $z+n \geq 5$, and $\Lambda \in [\Lambda_1, \Lambda_0]$: Integrate (12) from Λ up to Λ_0, apply $\partial_{\Lambda_0}\partial_p^w$, $|w| = z$, on it, and use (14), (15), (19) to get

$$\begin{aligned} -\partial_p^w \partial_{\Lambda_0} \mathcal{L}_{r,n}^{\Lambda,\Lambda_0;N} &= \partial_p^w \left(1 - \tau^{4+N-n}\right) \mathcal{F}_{r,n}^{\Lambda_0,\Lambda_0;N} \\ &+ \int_{\Lambda}^{\Lambda_0} d\Lambda' \; \partial_p^w \left(1 - \tau^{4+N-n}\right) \partial_{\Lambda_0} \mathcal{F}_{r,n}^{\Lambda',\Lambda_0;N} \\ &- \int_{\Lambda_1}^{\Lambda} d\Lambda' \; \partial_p^w \tau^{4+N-n} \partial_{\Lambda_0} \mathcal{F}_{r,n}^{\Lambda',\Lambda_0;N} \quad . \end{aligned}$$

This equation together with (30), (31) and (28) yields the desired bound.

(b) z with $z+n \leq 4$, as an example $n=4$, $z=0$: The same considerations as in part II (d) of our proof, using the fact that $\partial_{\Lambda_0} \mathcal{L}_{r,4}^{0,\Lambda_0;N}(0) = 0$, show that

$$|\partial_{\Lambda_0} \mathcal{L}_{r,4}^{\Lambda_1,\Lambda_0;N}(0)| \quad \leq \quad \Lambda_0^{-2-N} \cdot \mathrm{Plog}(\tfrac{\Lambda_0}{\Lambda_1}) \quad .$$

Integrating $\partial_{\Lambda_0}(12)$ from Λ_1 up to Λ and applying (26) we obtain

$$|\partial_{\Lambda_0} \mathcal{L}_{r,4}^{\Lambda,\Lambda_0;N}(0)| \quad \leq \quad \Lambda_0^{-2-N} \cdot \Lambda^{1+N} \cdot \mathrm{Plog}(\tfrac{\Lambda_0}{\Lambda_1}) \quad ,$$

and a final application of (29) yields the claim. ■

4. Symanzik Improvement for QED

Let me begin by briefly summarizing the main features of the treatment of ref. [12], where the differential flow equation method (some principles of which have been presented in Sections 2 and 3) has been utilized to give a rather simple proof of the perturbative renormalizability of the 4-dimensional Euclidean QED (with a massive photon).

Because the momentum space regularization (3) violates the local $U(1)$ invariance the starting point of [12] is the fermion-photon theory with bare interaction Lagrangian

$$\begin{aligned} L^{\Lambda_0} &= G^{\Lambda_0} \quad , \\ G^{\Lambda_0} &= \int d^4x \left\{ \tfrac{z_3}{4} F_{\mu\nu}^2 \; + \; \tfrac{\delta\lambda}{2} (\partial A)^2 \; + \; \tfrac{\delta\mu^2}{2} A^2 \; + \; z_4 (A^2)^2 \right. \\ &\qquad \left. - \, z_2 \bar{\Psi} i \partial\!\!\!/ \Psi \; + \; \delta m \bar{\Psi}\Psi \; + \; e(1+z_1) \bar{\Psi} A\!\!\!/ \Psi \right\} \quad , \end{aligned} \tag{36}$$

where the bare parameters $z_1, \ldots, z_4$, $\delta\lambda$, δm and $\delta\mu^2$ are fps in the coupling e depending on the renormalization conditions to be chosen; the regularized propagators of the massive fermion and massive photon are given by formulae which are analogous to (2) where, however, $(p^2+m^2)^{-1}$ has to be replaced by

$$(\not{p}+m)^{-1} \quad , \quad \text{for the fermion} \quad (m>0) \tag{37}$$

or by

$$\frac{\delta_{\alpha\beta}}{p^2+\mu^2}+\frac{1-\lambda}{\lambda}\frac{p_\alpha p_\beta}{(p^2+\frac{\mu^2}{\lambda})(p^2+\mu^2)} \quad , \quad \text{for the photon} \quad (\mu^2>0) \quad . \tag{38}$$

In (38) λ is the nonzero gauge fixing parameter. The fermion-photon theory defined by (36) - (38) exhibits $O(4)$- and charge conjugation-invariance, and this fact implies that the counter terms (36) are indeed sufficient to account for all possible divergent Feynman diagrams. And now it is an easy matter to apply the differential flow equation method to show that for any renormalization conditions the fermion-photon theory (36) - (38) is perturbatively renormalizable. In a second step one investigates the validity of the QED Ward-Identities (WI) in the theory (36) - (38), and one can employ the flow equation method once more to show that if one imposes QED-type renormalization conditions on the theory (36) - (38) then the QED WI become fulfilled in the limit $\Lambda_0 \to \infty$.

After this condensed review of [12] let me sketch the realization of Symanzik improvement for the Euclidean QED_4 with a massive photon. The theory under consideration will be a fermion-photon theory with regularized propagators (37), (38) as before, but the bare interaction Lagrangian, $L^{\Lambda_0;N}$, will differ from (36) by the addition of irrelevant terms:

$$L^{\Lambda_0;N} \quad = \quad G^{\Lambda_0} \; + \; I^{\Lambda_0;N} \quad , \tag{39}$$

with G^{Λ_0} of the form (36), whereas $I^{\Lambda_0;N}$ is constructed inductively according to the definition (compare to (19))

$$\begin{aligned}&\left(\mathcal{I}^{\Lambda_0;N}_{r;k,2n}(p_1,\ldots,p_{k+2n-1})\right)_{\mu_1,\ldots,\mu_k,i_1,\ldots,i_n,j_1,\ldots,j_n}\\ &\quad := -\int_{\Lambda_0}^{\infty} d\Lambda_0'\;\partial_{\Lambda_0'}\int_{\Lambda_1}^{\Lambda_0'} d\Lambda'\left(\tau^{4+N-k-3n}\mathcal{F}^{\Lambda',\Lambda_0';N}_{r;k,2n}(p_1,\ldots,p_{k+2n-1})\right)_{\mu_1,\ldots,j_n} \quad ,\end{aligned} \tag{40}$$

where $k = \#$ external photons, $2n = \#$ external fermions, $\mu_1, \ldots$ are photon vector indices, $i_1, \ldots$ are spinor indices of $\bar{\Psi}$'s and $j_1, \ldots$ are spinor indices of Ψ's; as usual $r =$ order of perturbation theory. $\mathcal{F}^{\Lambda',\Lambda_0';N}_{r;k,2n}$ is determined by formula (15) of ref. [12], i.e. by an expression similar to (13). Inductively it can be seen that $I^{\Lambda_0;N}$, and thus $L^{\Lambda_0;N}$ as well, is invariant under $O(4)$ and charge conjugation. Thus the methods of Sections 2 and 3 can be applied again to prove the

Theorem: *For any renormalization conditions and any fixed η, $0 \leq \eta < \infty$, and all $\Lambda_0 \geq \Lambda_{0,\min}$ we find*

$$|\partial_{\Lambda_0}\mathcal{L}^{0,\Lambda_0;N}_{r;k,2n}(p_1,\dots,p_{k+2n-1})_{\mu_1,\dots,j_n}| \quad \leq \quad \Lambda_0^{-2-N}\cdot \mathit{Plog}(\tfrac{\Lambda_0}{\Lambda_1}) \quad , \tag{41}$$

if $|p_1| \leq 2\eta,\dots,|p_{k+2n-1}| \leq 2\eta$.

Because the $dim \geq 5$ contributions to $L^{\Lambda_0;N}$ can be shown to be irrelevant, i.e. they vanish sufficiently fast as $\Lambda_0 \to \infty$, the improved Green functions $\mathcal{L}^{0,\Lambda_0;N}$ converge to the corresponding unimproved ones, $\mathcal{L}^{0,\Lambda_0}$, if $\Lambda_0 \to \infty$. This observation together with (41) yields once more the explicit improved convergence estimate

$$|\mathcal{L}^{0,\Lambda_0;N}_{r;k,2n}(p_1,\dots)_{\mu_1,\dots} - \mathcal{L}^{0,\infty}_{r;k,2n}(p_1,\dots)_{\mu_1,\dots}| \leq \Lambda_0^{-1-N}\cdot \mathrm{Plog}(\tfrac{\Lambda_0}{\Lambda_1}) \quad , \tag{42}$$

if $|p_1| \leq 2\eta,\dots$, for any renormalization conditions on the fermion-photon theory.

Now, if we restrict ourselves to QED-type renormalization conditions (for more precision on this see [12]) we know that

a) improved convergence (42) still holds;

b) the cutoff-removed renormalized Green functions $\{\mathcal{L}^{0,\infty}_{r;k,2n}\}$, by (42) coinciding with $\{\mathcal{L}^{0,\infty;N}_{r;k,2n}\}$, obey the QED WI [12].

These remarks finish the proof that what we have achieved is improved convergence for the Green functions of Euclidean QED_4 (with a massive photon).

Acknowledgement

I am deeply indebted to P. Weisz for numerous helpful conversations on Symanzik's improvement program and for his critical reading of the manuscript.

References

[1] K.Symanzik : *Some topics in quantum field theory*, in: Mathematical Problems in Theoretical Physics, p. 47; eds. R. Schrader et al.; Springer, Berlin, 1982 (Lecture Notes in Physics 153).
K.Symanzik , *Nucl.Phys.* **B 226** (1983) 187, 205.

[2] P.Weisz , *Nucl.Phys.* **B 212** (1983) 1.

[3] P.Weisz, R.Wohlert , *Nucl.Phys.* **B 236** (1984) 397.

[4] G.Curci, P.Menotti, G.Paffuti , *Phys.Lett.* **130 B** (1983) 205.

[5] M.Lüscher : *Improved Lattice Gauge Theories*, in: Critical Phenomena, Random Systems, Gauge Theories (Les Houches 1984), p. 359;
eds. K.Osterwalder, R.Stora; Elsevier Science Publishers B.V., 1986.

[6] M.Lüscher, P.Weisz , *Commun.Math.Phys.* **97** (1985) 59.

[7] M.Lüscher, P.Weisz , *Nucl.Phys.* **B 266** (1986) 309.

[8] P.Mackenzie : *Charmed Quarks on the Lattice*, talk at Lattice 91, Japan.
G.Martinelli : *Weak Interaction Matrix Elements*, talk at Lattice 91, Japan.

[9] B.Sheikholeslami, R.Wohlert , *Nucl.Phys.* **B 259** (1985) 572.

[10] J.Polchinski , *Nucl.Phys.* **B 231** (1984) 269.

[11] G.Keller, C.Kopper, M.Salmhofer , *Helv.Phys.Acta* **65** (1992) 32.

[12] G.Keller, C.Kopper , *Phys.Lett.* **B 273** (1991) 323.
G.Keller, C.Kopper : preprint in preparation.

[13] G.Keller, C.Kopper : *Perturbative Renormalization of Composite Operators via Flow Equations I*, MPI-PTh/91-79 (1991), Commun.Math.Phys. (to appear).

[14] C.Wieczerkowski , *Commun.Math.Phys.* **120** (1988) 149.