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Quantum Logic Requires Weak Modularity

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Abstract. The spirit of quantum logic cannot be extended to systems, such as separated quantum entities, which do not satisfy the axiom of weak modularity. This is because the implication relation has no definite equational characterisation in such systems.

1 Introduction

There are many different approaches to the foundations of quantum mechanics. Here we are interested in just two, namely the axiomatic approach of Piron and Aerts and the quantum logic approach. Loosely speaking, the former is based on what one can do to the system, namely the (not necessarily ideal) experiments that one can perform. The latter, however, is concerned with what one can say about the system and to what extent one can treat the resulting algebraic structure as a logic.

Our aim is to show that the quantum logic approach suffers sever difficulties if one tries to apply it to systems of separated quantum entities. These systems, first discussed by Aerts in 1982, violate what is known as weak (or ortho-) modularity. Weak modularity was first postulated as a way of trying to recover the usual Hilbert space formalism of quantum mechanics and is a weaker version of the property of modularity considered by Birkhoff and von Neumann in their pioneering work. Here we show that no further weaking of the set of axioms is possible if one wants to consider the physical system in terms of a genuine logic. Hence no quantum logical interpretation of systems of separated quantum entities is possible.

2 Properties

In this section we will briefly outline the "Geneva school" approach to the axiomatic foundations of quantum mechanics. More details can be found, for example, in Aerts [1982] and Piron [1990]. We start with the idea that the essential feature of the physical system which we wish to model is what we can do to that system, that is the experiemental projects that we can perform. There is no loss in generality in just considering eperimental projects with two possible outcomes "yes" and "no"; we simply choose the result desired in a given experiment and assign the response "yes" if we obtain it. For example we may assign the result "yes" if there is a darkening of a photographic plate in a certain pre-defined region and "no" otherwise.

The approach followed here is explicitly realistic, the elements of reality being those experimental projects which we are certain would give "yes" in a given state *if* we were to perform them. Such projects are called certain. It is then natural to regard two experimental projects as equivalent if they are certain for exactly the same system. The corresponding equivalence classes are then the properties of the system. A property is called actual if the corresponding experimental projects are certain, otherwise it is called potential. This leads to a canonical partial order on the set of properties; $a < b$ if b is actual whenever a is actual. It is this implication relation that will allow us to model the physical system mathematically. Note that the set of properties is a complete lattice under this partial order.

We can define an orthogonality relation on the set Σ of states of the system in the following manner; $\mathcal{E}_1 \perp \mathcal{E}_2$ if there exists an experimental project α for which the result "yes" is certain in the state $\mathcal{E}_1$ and impossible in the state $\mathcal{E}_2$. That is orthogonal states are those that can be separated by an experiment. If we then postulate three physical axioms[1] we find that the states are just the atoms of the lattice of propositions (that is the minimal non-zero propositions) and that each property is represented by a biorthogonal set $\mathcal{A}$ of atoms ($(\mathcal{A}^\perp)^\perp = \mathcal{A}$, where $\mathcal{A}^\perp$ is the set of states orthogonal to all states in $\mathcal{A}$). We note that $\perp$ is an orthocomplementation, $(\mathcal{A}^\perp)^\perp = \mathcal{A}$, $\mathcal{A} \cap \mathcal{A}^\perp = \emptyset$ and $\mathcal{A}_1 \subseteq \mathcal{A}_2$ implies that $\mathcal{A}_2^\perp \subseteq \mathcal{A}_1^\perp$, so that the lattice of properties is in fact an ortholattice.

Many systems of interest, such as a single particle, satisfy two further axioms, known as weak modularity and the covering law. An ortholattice is called weakly modular if $a < b$ implies that $a \vee (b \wedge a') = b$. A weakly modular lattice is often called an orthomodular lattice. On the other hand a lattice satisfies the covering law if whenever p is an atom and $a \wedge p = 0$ then $a < b < a \vee p$ implies that $b = a$ or $b = a \vee p$. Such systems have been called entities by Aerts [1982]. These five axioms have been shown to be (relatively) consistent by the explicit construction of a model in which the experimental projects are represented by certain operators on a Hilbert space [Cattaneo and Nisticó 1991].

The last two axioms are much less intuitive than the first three, however they provide a fundamental representation theorem for systems such as the single particle. In fact we find

[1] We do not have space to discuss these axioms in detail, they can be found in Piron [1990]. Physically they correspond to the following two tenets; (1) if a property becomes actual another must pass into potentiality, (2) every property is the inverse of another.

that all entities, as long as their property lattice is big enough, can be described in terms of a set of generalised Hilbert spaces indexed by the set of minimal classical properties [Piron 1976]. Note that a property $\mathcal{A}$ is classical if for any state either $\mathcal{A}$ or $\mathcal{A}^\perp$ is actual. The minimal classical properties are often called superselection rules. This motivates the habitual use of Hilbert spaces to describe at least simple systems in quantum mechanics, as well as allowing rigorous descriptions of two-body systems and the Mössbauer effect, where the centre of mass of the system must be treated as a superselection rule and not as a quantum variable.

It is a mistake however to think that all physical systems must satisfy the last two axioms. For example, Aerts [1982] has clearly shown that a composite of two separated quantum systems cannot satisfy either axiom. In particular, one cannot describe two separated quantum systems by the tensor product of the corresponding Hilbert spaces, even with the addition of superselection rules. Hence Bell's inequalities will be violated whenever the system is broken into separated systems; that this can occur in classical as well as quantum mechanics has been shown by Aerts [1985,1991].

3 Propositions

We now turn to quantum logic. The difference in philosophy between this approach and that of the "Geneva School" is that, while Piron and Aerts start with the idea of what one can do to the system, in quantum logic the emphasis becomes what one can *say* about the system. More explicitly, the set of *propositions* that can be made about the system is considered to be a weakly modular ortholattice $\mathcal{L}$. As the lattice operations $\wedge$, $\vee$ and $'$ are at first sight reminiscent of the logical connectives "and", "or" and "not" one is then tempted to try to treat this lattice as a kind of logical structure, a so-called quantum logic[2]. On such a logic a state is defined to be a generalised probability measure, that is a map $s : \mathcal{L} \to [0,1]$ such that $s(1) = 1$ and if $\{a_i\}$ is a set of mutually orthogonal propositions $s(\vee_i a_i) = \sum_i s(a_i)$ [Pták and Pulmannová 1991].

Of course not all realisations of such structures will represent physical systems in the sense we have used here. For example Greechie [1971] has constructed weakly modular ortholattices which do not have any states. This in itself is not an objection to the use of quantum logics as we merely seek a framework in which to work. Our objection is that the axiom of weak modularity is absolutely essential to the spirit of such an approach, which can then not deal with, for example, separated quantum entities.

To be able to consider such an algebraic structure as a genuine logic one needs an implication and a semantics, that is notions of "deduction" and "truth". As our structure is algebraic we would like to define the implication as a binary connective, that is as a map

[2] It is important to note that a study of physics from this perspective leads naturally to more general logico-algebraic structures such as orthomodular posets [Pták and Pulmannová 1991] and orthoalgebras [Foulis *et al.* 1992]. In all such structures the notion of weak modularity is retained while other axioms defining the set of propositions are relaxed. More discussion of the differences between these approaches may be found in Piron [1993].

$\rightarrow: \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$. However Hardegree [1981a,b] has shown that any such connective that satisfies the three minimal deduction conditions entailment ($a \rightarrow b$ means that a implies b), modus ponens (a and $a \rightarrow b$ entails b) and modus tollens (b' and $a \rightarrow b$ entails a') leads to a weakly modular structure. Hence implication connectives are not available in the non-weakly modular case. We note that even in the weakly modular case the implication connectives do not satisfy all the classical conditions desired. Some authors have tried to remedy this problem by redefining the connectives $\wedge$ and $\vee$ [Finch 1969, Román and Rumbos 1988]. However the new connectives, defined in terms of the left adjoint of the implication, do not satisfy such conditions as commutativity. In our opinion this approach then causes more problems than it cures.

The lack of an implication connective is not necessarily fatal to the extension of quantum logical ideas however, as we can retreat to the idea of an implication relation. Such an approach has also been used in the weakly modular case [Pavičić 1987]. The motivation there was that the implication connective is not unique in non-distributive ortholattices, a fact first noticed by Weyl [1940]. Pavičić showed that the weakly modular ortholattices are precisely those where the possible implication connectives reduce to the implication relation.

As implication in the lattice of propositions *defines* the partial order $<$, it is clearly this relation that we must take in the general case. However for the partial order to be an implication relation in the usual sense it must satisfy the minimal condition of testability. Here testability has the usual intuitive meaning of allowing us the possibility to falsify a statement about which properties imply which others. This condition is minimal as the existence of a Kripkean (many-world) semantics requires the knowledge of the possible worlds in which an implication is true. We show that even this minimal requirement is not satisfied in the non-weakly modular case, making the extension of quantum logical ideas extremely difficult, if not impossible.

Now the implication $a < b$ is equivalent to the equation $a = a \wedge b$ and so $a < b$ if and only if $s(a) = s(a \wedge b)$ for all states s. However this does not provide genuine testability as it is a relation between two probabilities and so can, at best, be supported statistically. There is no way to falsify such a relation. For this stronger requirement we need a *definite equational characterisation* of the partial order, that is a two-place function $I : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ such that $I(a, b) = 1$ is equivalent to $a < b$. In this case $a < b$ if and only if $s(I(a, b)) = 1$ for all states s, a condition that does lead to testability in the usual intuitive sense.

We now show that such a definite equational characterisation exists if and only if the lattice of propositions is weakly modular. We first need the following lemma (see for example Kalmbach [1984]).

Lemma *Let $\mathcal{L}$ be an ortholattice. Then $\mathcal{L}$ is weakly modular if and only if $a < b$ and $a' \wedge b = 0$ implies $a = b$.*

We can now prove the main result of this section.

Theorem *Let $\mathcal{L}$ be an ortholattice. Then the partial order has a definite equational* *characterisation if and only if $\mathcal{L}$ is weakly modular.*

Proof: If the ortholattice $\mathcal{L}$ is weakly modular then it suffices to take $I(a,b) = a' \vee (a \wedge b)$. We now show the converse, namely that weakly modular lattices are the only ortholattices that allow such a characterisation. Let us write $\mathcal{F}_2$ for the set of two-place functions $I : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$. Imagine that the ortholattice $\mathcal{L}$ is not orthomodular. Then by the lemma there exist distinct elements x and y of $\mathcal{L}$ such that $x < y$ and $x' \wedge y = 0$. Now as $x < y$ we have that $y' < x'$ and so $x \wedge y' < x \wedge x' = 0$. Thus $x \wedge y' = 0$ and one can easily see that the subalgebra of $\mathcal{L}$ generated by x and y contains only the elements x, y, x', y', 0 and 1. We write $A_1 = \{x, y\}$, $A_2 = \{x', y'\}$, $A_3 = \{0\}$ and $A_4 = \{1\}$.

We proceed by showing that for any element $I(a,b)$ of $\mathcal{F}_2$ with $I(x,y) = 1$ we must necessarily have that $I(y,x) = 1$. Hence such an equation cannot represent the partial order as $x < y$ but $y \not< x$. Let p and q be arbitrary elements of $\{0, x, y, x', y', 1\}$. Then one can show that the subsets A_i to which $p \wedge q$, $p \vee q$ and p' belong depend only on the subsets to which p and q belong. Hence by recursion $I(x,y)$ and $I(y,x)$ must belong to the same A_i. In particular, we can have that $I(x,y) = 1$ if and only if $I(y,x) = 1$, completing the proof. ∎

Thus one cannot generalise the spirit of quantum logic, that the system is described by propositions that one can treat as a kind of logic, to non-weakly modular systems. Even in the weakly modular case we must take care, as the function I must necessarily invole the orthocomplementation. However this is only given implicitly once the entire order relation is known. Hence we can at most test whether the entire ascription of the partial order is correct.

Finally we note that the Kripkean semantics of the weakly modular case are not straightforward either. This is because the axiom of weak modularity cannot be expressed as a first order formula in the orthogonality relation. Hence the standard approach to the Kripkean semantics of a modal logic, namely the use of a frame (X, R) for some relation R on the set X, is not available. This argument is due to Goldblatt [1984].

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