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The bound-states in Quantum Field Theory : review of some analytic problems raised by the variational perturbation method

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Abstract

In the framework of the weakly-coupled $\mathcal{P}(\varphi)_2$ models we summarize some of the problems raised by a new method for finding bound states, called the *variational perturbation method.* To show first its interest, we present a result of this method, from which the existence of a bound state follows simply by solving a Schrödinger equation, and which allows to find time-zero eigenvectors at first perturbation orders. The main part of this paper is devoted to the review of the problems encountered by the restriction to zero-time vectors (existence of zero-time vectors in the domain of the Hamiltonian, asymptotic series of zero-time vectors approaching any vector, and particularly those of the one-particle subspace). Lastly we present a new quantum and almost-relativistic model for the two-particle system at low energy, deduced from the $\mathcal{P}(\varphi)_2$ models by these considerations.

Introduction

The first construction of Quantum Field Theory models, the weakly-coupled $\mathcal{P}(\varphi)_2$ models, by Glimm, Jaffe and Spencer [1] in 1973, was a capital step in the history of Physics, showing that the concepts of Relativity and Quantum Physics are not mathematically incompatible. Unfortunately the weakly-coupled $\mathcal{P}(\varphi)_2$ models describe a utopian world of massive, spinless, chargeless and weakly-coupled particles in a two-dimensional space-time. Since then many efforts have been made to find other models, more closely related to the observable world, in order to make comparisons with experimental measurements. Even though this goal has not yet been reached, the models already constructed [1] are encouraging and make this subject, called the *Constructive Quantum Field Theory*, an important branch of modern mathematical physics.

Meanwhile it would be an error to neglect the study of the $\mathcal{P}(\varphi)_2$ models only because they are too simple. Their construction is indeed not trivial, so we must take advantage of their existence. They can be used as a laboratory to test many ideas or questions about relativistic and quantum system behavior. Some important problems have already been solved, such as the existence of diffusion and bound states [1]. But many other questions have not been treated, at both mathematical and physical level.

Here we are interested in the two-particle phenomena at low energy. In that domain, among many others the following questions arise naturally :

Physical questions. What is the two-particle phenomenology at low energy ? Is it governed by the Schrödinger equation at first approximation ? If it is the case, what is the "effective" potential ? And what are the deviations to the Schrödinger previsions ? Are they supported by some underlying relativistic kinematical laws ?

Mathematical questions. We know that the representation of the state space by the fields acting on the vacuum state is not injective, which sometimes causes difficulties. Can we find a dense subspace with an injective parametrization, in which the dynamics of the two-particle system is easily described, in a natural way ?

More generally, these two questions can be summarized as follows : does there exist a simpler theory for the two-body system which gives the same predictions as the weakly-coupled $\mathcal{P}(\varphi)_2$ models at low energy with a precise error estimate ?

This approach, which emphasizes the role of the vectors of the state space, is generally neglected in Quantum Field Theory (Q.F.T.). We try to take a step in this direction and give a tentative of answer to some of the above questions. This paper summarizes other publications (except §3), to which we will refer for the detailed proofs. The presentation, sufficiently detailed for non-specialists, will follow the order of the problems listed just above. §1 gives a result for the two-body system, obtained by a new method for finding bound states, called the *variational perturbation method*, initially proposed by Glimm, Jaffe and Spencer. An equation is given, which connects two Rayleigh quotients, one of the Q.F.T. and the other of its non-relativistic limit. From this equation it is easy to deduce the existence of a bound state, by simply solving a Schrödinger equation. Moreover by comparison with the literature we conclude that we have obtained the eigenvectors at first perturbation order. In §2 we discuss the subspace of the state space in which the calculations have been done, the *zero-time subspace* $\mathscr{D}_o$. The restriction to this subspace is sufficient for our purpose, because all vectors can be approached by an asymptotic perturbation series in $\mathscr{D}_o$. Moreover $\mathscr{D}_o$ contains vectors in the domain of the Hamiltonian, and the orthogonal projection of any vector on the one-particle subspace can be also approached by an asymptotic perturbation series in $\mathscr{D}_o$. All these results need an analysis of the Schwinger functions in momentum space, that we deduce from a new programme, the *WTI Programme*. In §3 we present a new, simple, quantum and almost-relativistic model for a two-particle system at low energy, deduced from the above considerations, and for which the research of the bound states leads to the same equation as in §1.

Summary

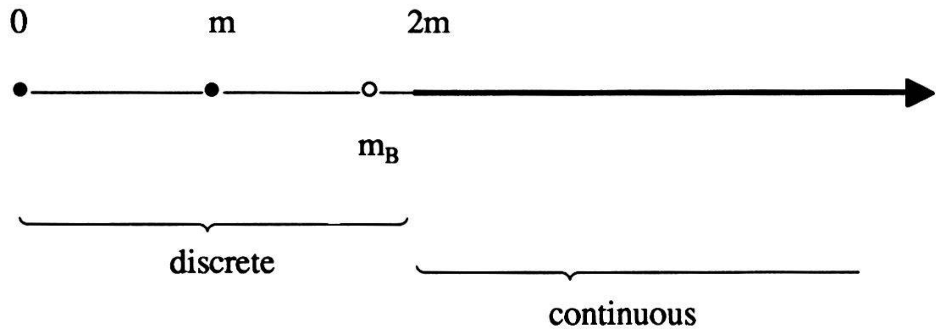
page

1. An equation for the two-particle system

A weakly-coupled $\mathcal{P}(\varphi)_2$ model is an example of the Wightman Quantum Field Theory in a two-dimensional space-time, describing massive, spinless, chargeless, identical particles with weak mutual interaction. Such a theory is given by four quantities ($\mathcal{H}$, $\mathcal{U}$, φ, Γ), that is a $\mathbb{C}$-Hilbert space $\mathcal{H}$ (the *state space*), a scalar, unitary and continuous representation $\mathcal{U}$ in $\mathcal{H}$ of the Poincaré group of the two-dimensional space-time, a dense subspace Γ in $\mathcal{H}$ and a field φ, mapping $\mathcal{S}(\mathbb{R}^2,\mathbb{R})$ in the self-adjoint operators with domain Γ and range in Γ, all these objects satisfying the Wightman Axioms [1].

We denote by H (Hamiltonian) and by P (momentum) the infinitesimal generators of the temporal and spatial translations respectively. The Mass operator M is given by $M = \sqrt{H^2-P^2}$.

The problem of the existence of bound states concerns the discrete part of the spectrum of M, which is expected, for such models, to be as follows :



where the eigenspace associated to 0, called the *vacuum subspace*, is one-dimensional and belongs to Γ, and the eigenspace corresponding to m>0 (the one-particle mass), called the *one-particle state subspace*, carries an irreducible representation of the Poincaré group. Above 2m, which is not an eigenvalue, the spectrum is continuous, and corresponds to the states with more than one particles.

The eigenvalue m_B, whenever it exists (depending of the model), is interpreted as the mass of a two-particle bound state. Its existence suffices to prove that the model describes really interacting particles.

The set of these models is parametrized by three quantities (m_o,$\mathcal{P}$, λ), where m_o>0 is the *one-particle free mass*, $\mathcal{P}$ a positive-valued $\mathbb{R}\rightarrow\mathbb{R}$ polynomial called the *interaction polynomial*, and

The set of these models is parametrized by three quantities (m_o,$\mathcal{P}$, λ), where $m_o>0$ is the *one-particle free mass*, $\mathcal{P}$ a positive-valued $\mathbb{R}\to\mathbb{R}$ polynomial called the *interaction polynomial*, and $\lambda\geq 0$ the *coupling constant*. The weak coupling refers to λ, which is taken small. $\lambda=0$ describes models without interaction (which we call *free models*). In the following we consider families of models which differ only by λ, that we note $\mathcal{F}=\{(\mathcal{H}_\lambda, \mathcal{U}_\lambda, \varphi_\lambda, \Gamma_\lambda), \lambda\in[0,\lambda_{max}]\}$ for some $\lambda_{max}>0$ (we put a λ-index everywhere). The set of these families can be parametrized by m_o and $\mathcal{P}$. In such a family there exists a vector Ω_λ in the vacuum subspace of $\mathcal{H}_\lambda$ for each λ such that the scalar products $(\Omega_\lambda\,;\,\varphi_\lambda(f)^n\,\Omega_\lambda)_{\mathcal{H}_\lambda}$ are C^∞ functions of λ on $[0,\lambda_{max}]$, for all $n\in\mathbb{N}^*$ and all $f\in\mathcal{S}(\mathbb{R}^2)$.

The complete information on the discrete part of the spectrum of M has been obtained by the Bethe-Salpeter method (see the references in [3] and [9]), which can be compared to the method of analysis of the resolvent-operator used in Quantum Mechanic (Q.M.), as much for the functional analysis technics involved as for the quality of its results. This method confirms the general structure of the spectrum pointed out above, where m_λ is a C^∞ function of λ on $[0,\lambda_{max}]$ with $m_{\lambda=0}=m_o$ for each family $\mathcal{F}$. Moreover this method gives necessary and sufficient conditions on the polynomial $\mathcal{P}$ for the existence of a bound state. If it exists, it is unique, and the corresponding eigenspace carries an irreducible representation of the Poincaré group. Its mass $m_{B,\lambda}$ is a C^∞ function of λ on $[0,\lambda_{max}]$ converging to $2m_o$ (which is not an eigenvalue) when $\lambda\to 0$.

An other method, called here the *variational perturbation method*, initially proposed (in a simpler version) by Glimm, Jaffe and Spencer [2, p. 175-7], has been explored. Let us suppose that we know all about the spectrum of M, except the existence of a bound state. Thus we are only interested to know if the spectrum of M is empty or not in the open interval (m_λ, $2m_\lambda$). This method is adapted to this question. It works in a given family $\mathcal{F}=\{(\mathcal{H}_\lambda, \ldots), \lambda\in[0,.]\}$ and combines three ideas.

First m_B is defined by the minimum of a Rayleigh quotient (we drop now the λ-index whenever it is not necessary for the understanding) :

$$m_B{}^2 \;=\; \inf_{0\neq\Psi\in\mathcal{V}} \frac{(\,\Psi\,;\,M^2\,\Psi\,)}{(\,\Psi\,;\,\Psi\,)}$$

where $\mathcal{V}$ is the intersection of D(M), the domain of M, and the subspace $(1-E_0-E_m)\mathcal{H}$, E_μ being the orthogonal projector associated to the eigenvalue μ of M (we take the Rayleigh quotient of M^2 rather then of M because it simplifies the calculations). Here (.;.) is the $\mathcal{H}$ scalar product. The interest of the Rayleigh quotient lies in its capacity of regularization : if Ψ_λ is an eigenvector for

$m_{B,\lambda}$ for all λ, then it must be singular when $\lambda \to 0$ (because the eigenvalue disappears), while the quotient $(\Psi_\lambda ; M^2 \Psi_\lambda)_{\mathscr{H}_\lambda} / (\Psi_\lambda ; \Psi_\lambda)_{\mathscr{H}_\lambda} = (m_{B,\lambda})^2$ is continuous in this limit (even C^∞), as we have just seen. Thus it allows a perturbation development in power of λ.

The second idea consists in a good choice of test vectors. We restrict ourselves to vectors of the following kind :

$$\sum_{i=0}^{N} \lambda^i \sum_{j=0}^{M} \Theta_\lambda^j(f_j^i)\, \Omega_\lambda \quad \in \mathscr{H}_\lambda$$

for all N, M$\in \mathbf{N}$ and suitable functions f_j^i, where $\Theta_\lambda^0(f) = 1$ and $\Theta_\lambda^j(f)$, $j \in \mathbf{N}^*$, are the *zero-time fields*, formally given by

$$\Theta_\lambda^j(f) = \int_{\mathbf{R}^j} d^j\vec{x} \;\; f(\vec{x}_1, \dots, \vec{x}_j) \;\; : \varphi_\lambda(0,\vec{x}_1) \cdots \varphi_\lambda(0,\vec{x}_j) : .$$

The double points :.: denotes the Wick polynomials (see §2, where the existence of such vectors is discussed). This choice of zero-time fields avoids the use of too many variables. The particular λ-dependance of the test vectors is not necessary, but it help the calculation. Note that λ plays here a double role: it indicates to which Hilbert space $\mathscr{H}_\lambda$ the vectors belong, and it is a small parameter which allows perturbation expansion. Because of the weak coupling, the solution of the minimization problem approaches that of the free model. Thus the term with f_2^0 is expected to play a dominant role, while the functions $f_j^0, j \neq 2$, will be suppressed.

The restriction to $\mathscr{D}_o$, the set of zero-time vectors, tends to approach the picture of the Q.M.. This can be understood as the choice for which the momentum operator P is diagonal, because it acts on $\mathscr{D}_o$ as follows :

$$P\, \Theta^i(f)\, \Omega = \Theta^i(Pf)\, \Omega ,$$

where Pf denotes the usual action of the momentum in Q.M.. While the operators H, M or L (Lorentz generator) are expected to act on $\mathscr{D}_o$ in a more complicated way (see §3).

The third idea consists in introducing the singularity at $\lambda=0$ in Ψ_λ before the calculation. We try to guess it, by the help of the following argument : the localization of the bound state (in the space of the relative variable) must vanish when the interaction disappears. We introduce a scaling for the relative variable of the function f_2^0 (which play the main role), replacing $\vec{x}_{rel}$ by $\delta\vec{x}_{rel}$, where δ is a function of λ which goes to 0 when $\lambda \to 0$.

We give here a result of this method, after having minimized the Rayleigh quotient in varying $\delta(\lambda)$ and the functions f_j^1, but not the function $f = f_2^0$ (that is, the functions f_j^1 are given in terms of f). To simplify, we give the result for some families of weakly-coupled $\mathcal{P}(\varphi)_2$ models, as in [3] (where the interaction polynomial $\mathcal{P}$ is even and has a non-zero fourth degree term ; these restrictions are removed in [4]). The analytic part uses some quotient of norms q(f) (quotient of some Sobolev norms), well defined for example if $f \in \mathcal{S}(\mathbb{R}^2)$ and $f \neq 0$.

Theorem. *For all* $f \in L^2(\mathbb{R}^2)$ *with well defined* q(f) *there exists a vector* Ψ_f *satisfying* :

i) $\Psi_f \in D(M) \cap (1-E_o-E_m)\mathcal{H}$,

ii) $P\,\Psi_f = \Psi_{Pf}$,

iii) $$\frac{(\Psi_f;M^2\Psi_f)}{(\Psi_f;\Psi_f)} = (2m)^2 + 4m^{-3}\lambda^2 \frac{\langle f;H^{NR}_{rel}(1)f\rangle}{\langle f;f\rangle} + \lambda^{5/2}\mathcal{R}(f,\lambda),$$

and there exist $K,K' \in (0,\infty)$ *such that, for all* $\lambda < [K'q(f)]^{-1}$:

$$|\mathcal{R}(f,\lambda)| < K \frac{q(f)^4}{1-\lambda K'q(f)} .$$

Here $\langle .;. \rangle$ is the $L^2(\mathbb{R}^2)$ scalar product and $H^{NR}_{rel}(\lambda/m_0^2)$ is the relative part of the Hamiltonian of the non-relativistic limit, obtained by Dimock [5].

The proof of the theorem, and the precise form of Ψ_f, H^{NR}_{rel} and q(f) are given in [3] and [4].

Note that in the theorem, the speed of light is a fixed constant. The effective non-relativistic limit comes from the scaling in the relatives variables. But the center of mass system is always treated as relativistic.

The problem of the part of the spectrum of M in (m, 2m) is now reduced to a problem of Q.M., i.e. if the spectrum of H^{NR}_{rel} has a negative part.

Corollary. *If there exist* $f \in L^2(\mathbb{R}^2)$ *with well defined* q(f) *and* $E>0$ *such that :*

$\langle f;H^{NR}_{rel}(1)\,f\rangle/\langle f;\, f\rangle = -E$, *then the spectrum of* M *is not empty in* $(m, 2m - m^{-4}\lambda^2E]$.

If we assume that the spectrum of M is purely discrete below 2m, as pointed out above, the corollary states that a bound state exists when some conditions on H^{NR}_{rel} are satisfied (this turns out to give conditions on the interaction polynomial $\mathcal{P}$). Moreover its mass m_B is bounded by : $m_B \leq 2m - m^{-4}\lambda^2E$.

The comparison with the precise results given by the Bethe-Salpeter method is surprising. The conditions on the interaction polynomial $\mathcal{P}$ for the existence of a bound state are the same, and the value for m_B is exactly the bound given above, at first perturbation orders. Thus we have actually reached the bottom of the spectrum of M on $(1-E_o-E_m)\mathcal{H}$, and $\{\Psi_f, f$ as in the corollary$\}$ can be seen as the eigenspace (given by zero-time vectors), at first perturbation orders.

This surprise allows a new view on the formula *iii)* of the Theorem. It gives not only a new connection between the Q.F.T. and its Q.M. non-relativistic limit, but also a new interpretation in terms of particles of some (zero-time) vectors of the state space in Q.F.T..

The theorem allows two interesting developments (passing over the adaptation to other models). We can calculate the next perturbation terms, minimizing over the functions f_j^2, f_j^3, etc... This could lead to the relativistic corrections to the Schrödinger equation (at zero-time), proposed by the weakly-coupled $\mathcal{P}(\varphi)_2$ models [6]. The second development leads to the creation of a new theory, by halting the calculation at an intermediate step, before doing the δ-expansion, that is before taking the non-relativistic limit. We obtain in that way a new model for the two-particle system, quantum and almost-relativistic (not exactly relativistic because of the perturbation approach), which has the property to have the same non-relativistic limit as the $\mathcal{P}(\varphi)_2$ models (see §3).

The proof of the theorem can be divided into two parts of different nature. The first part constructs a zero-time vector Ψ_f in $(1-E_o-E_m)\mathcal{H}$, which satisfies the formula *iii)* up to $O(\lambda^3)$ (see [3] or [4]). Here the analytic difficulties are neglected, and the expansions in powers of λ are taken as formal series. So this part is rather "algebraic". It can also be done, with suitable modifications, for other Q.F.T. models. The second part (the "analytic part"), concerns the control of the remainder $\mathcal{R}(f,\lambda)$. It is specific to models with rigorous mathematical construction. The method given here works for the weakly-coupled $\mathcal{P}(\varphi)_2$ models, but is certainly not strong enough for other models. To obtain the formula *iii)*, expansions in power of λ have been performed at three stages :

developments of the scalar products $(\Theta^i(f)\Omega; M^2\Theta^j(g)\Omega)$, $(\Theta^i(f)\Omega;(1-E_o)\Theta^j(g)\Omega)$
development of the Rayleigh quotient,
development in the scaling parameter δ (taken as a function of λ).

The control of the remainders of the last two expansions is easy but tedious. It is given in [3] and [4]. The control of the remainder of the first expansion is more difficult and we need to go back to the details of the construction of the models. It is obtained by combining [3], [4], [7], [8], [9]. In §2 we present an outline of these works.

2. Analysis of the zero-time subspace

We present here the analytic problems encountered in the use of zero-time vectors in the theorem of §1. More generally, let us go back to the presentation of the variational perturbation method. A subspace $\mathscr{V}$ of $\mathscr{H}$ is required, which, for the convenience of the calculations, should be :

1) contained in the domain of M, with scalar products $(.;M^2.)$ C^∞ in λ,
2) orthogonal to the vacuum and one-particle states,
3) large enough (dense in $(1-E_0-E_m)\mathscr{H}$),
4) small enough (parametrized injectively by a space of functions),
5) convenient (the operators P, H, M, L should act on it in a simple way).

The aim of this paragraph is to show that some suitable subspace of $\mathscr{D}_o$, the zero-time subspace, could be an acceptable candidate for such a subspace.

About question 1): the zero-time vectors are given by some limits, so the question of their possible presence in the domain of M (which is an unbounded operator) makes it necessary to go back to the basic definitions. After having exposed the main points of the construction of the $\mathcal{P}(\varphi)_2$ models (§2.1) we can state the problem (§2.2). The difficulty consists in the control of the asymptotic decrease of the Fourier transform of the so-called Schwinger functions and of their derivatives with respect to λ. A new programme, the *WTI Programme*, is established to investigate this problem (§2.3). It allows us to prove (§2.4) that there exist zero-time vectors in the domain of M and M^2, and that for two such vectors ξ, ζ, the scalars product $(\xi, M^\nu \zeta)$, $\nu \in \{0,1,2,3,4\}$, are C^∞ in λ.

About question 3): we do not prove that $\mathscr{D}_o$ is dense in $\mathscr{H}$, but § 2.5 states that it is "almost dense", in the sense that all vectors of $\mathscr{H}$ can be approached by an asymptotic series (in power of λ) of vectors of $\mathscr{D}_o$. Then the restriction to $\mathscr{D}_o$ has no consequence for any investigation involving a perturbation calculation. The orthogonalization with respect to the vacuum and one-particle states, (question 2)), is studied in § 2.6. The projection $(1-E_0-E_m)\xi$ of any vector $\xi \in \mathscr{D}_o$ can be approached again by an asymptotic series (in power of λ) of vectors of $\mathscr{D}_o$.

About question 4): let us recall that we have chosen the zero-time vectors subspace especially for this property to hold. We do not need to prove it, because we do not encounter difficulties by having too many variables in working with $\mathscr{D}_o$.

About question 5): $\mathscr{D}_o$ is satisfying for the free models (§ 2.2). For the interaction case, we discuss this problem in § 3.

2.1 Construction of the weakly-coupled $\mathcal{P}(\varphi)_2$ models : the main points

The construction of the weakly-coupled $\mathcal{P}(\varphi)_2$ models by Glimm, Jaffe and Spencer [1] uses a large detour, passing by the so-called Euclidean Field Theory (i.e. with imaginary time). We expose first this theory (for more details see [7] and [8]). The basic tool is a probability space (Q,Σ,μ), where

$Q = \mathscr{S}'(\mathbb{R}^2,\mathbb{R})$,
Σ is the Borel σ-algebra of Q (given the weak topology),
μ is a probability (i.e.normed, positive) measure on Σ satisfying some conditions.

The conditions on μ are the *Probability Axioms for Quantum Field Theory*, stated below. By the Minlos theorem there is a one-to-one correspondence between the probability measures on Σ and the functions $\mathscr{C} : \mathscr{S}(\mathbb{R}^2,\mathbb{R}) \to \mathbb{C}$ satisfying :

i) Normalization : $\mathscr{C}(0) = 1$.
ii) Continuity : $\mathscr{C}$ is continuous.
iii) Positivity : for all $n \in \mathbb{N}^*$ and $f \in \mathscr{S}(\mathbb{R}^2)^n$, let $A_{i,j} = \mathscr{C}(f_i - f_j)$. Then A is a semi-positive definite matrix.

In this correspondence $\mathscr{C}$ is the *characteristic function of μ*, that is

$$\mathscr{C}(f) = \int_{q\in Q} e^{i\,\phi_f(q)}\, d\mu(q) \qquad \text{for all } f \in \mathscr{S}(\mathbb{R}^2,\mathbb{R})$$

where ϕ (the *Euclidean field*) is the map from $\mathscr{S}(\mathbb{R}^2)$ to the random variables on (Q,Σ) defined by $\phi_f(q) = q(f)$ for all $q \in Q$.

The Minlos theorem generalizes the Bochner theorem for infinite dimensional integration spaces. It gives an easy way to obtain probability measures on Σ. For example $f \mapsto F(b(f,f))$ satisfies *i), ii), iii)* if F is any continuous, positive-valued function on $\mathbb{R}$ satisfying $F(0) = 1$ and if $b(f,g)$ is any semi-positive definite bilinear form continuous on $\mathscr{S}(\mathbb{R}^2)$ (this statement follows from the proof of proposition I.1.2 of [6]).

Let $\mathscr{G}$ be the Euclidean group on $\mathbb{R}^2$ (rotations, translations and reflections), acting on $\mathscr{S}(\mathbb{R}^2)$ in the usual way. We single out a particular direction in $\mathbb{R}^2$ which we call *Euclidean time* ; a point in $\mathbb{R}^2$ will be written as $x = (\overset{\circ}{x}, \vec{x})$, where $\overset{\circ}{x}$ is an Euclidean time, and $\vec{x}$ is a space point. Let

$\{T(x),\ x\in \mathbb{R}^2\}\subset\mathcal{G}$ denote the subgroup of translations, and let $\theta\in\mathcal{G}$ denote the reflection in the $\overset{\circ}{x}=0$ hyperplane. We also define:

$$\mathcal{S}^+ = \left\{ f\in \mathcal{S}(\mathbb{R}^2),\ f(\overset{\circ}{x},\vec{x}) = 0 \text{ if } \overset{\circ}{x}<0 \right\}.$$

From now on, we write $\mathcal{S}$ instead of $\mathcal{S}(\mathbb{R}^2)$ or $\mathcal{S}(\mathbb{R}^2,\mathbb{R})$.

The *Probability Axioms for Q.F.T.* are the following conditions on μ, in term of $\mathscr{C}$:

i) Euclidean invariance : $\mathscr{C}(\gamma f) = \mathscr{C}(f)$ for all $\gamma\in\mathcal{G}$, $f\in\mathcal{S}$.

ii) Osterwalder-Schrader positivity : for all $n\in\mathbb{N}^*$ and $f\in(\mathcal{S}^+)^n$, let $B_{i,j} = \mathscr{C}(\theta f_i - f_j)$. Then B is a semi-positive definite matrix.

iii) Cluster property : $\lim\limits_{s\to\infty} \left[\mathscr{C}(f+T(sx)g) - \mathscr{C}(f)\mathscr{C}(g)\right] = 0$ for all $x\in\mathbb{R}^2-\{0\}$ and $f,g\in\mathcal{S}$.

iv) Regularity : for all $f\in\mathcal{S}$, $\alpha\mapsto\mathscr{C}(\alpha f)$ is of class C^∞ in an $\mathbb{R}$-neighborhood of, $\alpha=0$ and there exist a Schwartz space norm $|\ldots|$ and finite positive numbers a,b,c with

$$\left|\partial_\alpha^n \mathscr{C}(\alpha f)\big|_{\alpha=0}\right| \le a\, b^n\, (n!)^c\, |f|^n \quad \text{for all } n\in\mathbb{N}^*\ .$$

From the last axiom the moments of μ exist as tempered distributions :

$$S_n(f) = \int_{q\in Q} \phi_{f_1}(q)\cdots\phi_{f_n}(q)\ d\mu(q) \qquad \text{for all } n\in\mathbb{N}^* \text{ and } f\in\mathcal{S}^n.$$

As consequences of the above axioms, the distributions S_n satisfy the Osterwalder-Schrader axioms [7], so there exists a Wightman Q.F.T. model $(\mathcal{H}, \mathcal{U}, \varphi, \Gamma)$ such that the S_n are the analytic continuation of the distributions $(\Omega;\varphi(f_1)\cdots\varphi(f_n)\Omega)_{\mathcal{H}}$ to imaginary time [10]. The S_n are called the *Schwinger distributions*. Let us give the construction of the state space $\mathcal{H}$, the momentum operator, P the Hamiltonian H and the Lorentz generator L.

Let $\mathscr{E}$ (the *Euclidean Hilbert space*) be the closure in $L^2(Q,\mu)$ of the $\mathbb{C}$-span of the following set

$$\left\{1,\ (\phi_f)^n\ ;\ n\in\mathbb{N}^*,\ f\in\mathcal{S}\right\}.$$

Let $\mathscr{E}^+$ be the closed subspace of $\mathscr{E}$ obtained by restricting the functions f to be in $\mathcal{S}^+$:

$$\mathscr{E}^+ = \text{closure of the span of } \left\{1,\ (\phi_f)^n\ ;\ n\in\mathbb{N}^*,\ f\in\mathcal{S}^+\right\}.$$

We denote by E^+ the orthogonal projector on $\mathscr{E}^+$. Following Klein and Landau [11] we introduce the operator :

$$\mathscr{W} = E^+\,\theta\, E^+.$$

Note that $\mathscr{W}$ measures in some sense the "non-locality" of the measure μ. It is a bounded and positive operator by the axiom *i)* and *ii)*. The subspace $\mathscr{W}\mathscr{E}$ can be seen as a pre-Hilbert space with the scalar product $(\mathscr{W}\xi, \mathscr{W}\zeta) \mapsto (\mathscr{W}\xi\,;\zeta)_{\mathscr{E}}$ for all $\xi, \zeta \in \mathscr{E}$. Its completion is identified as the state space $\mathscr{H}$. We denote by $W : \mathscr{E} \to \mathscr{H}$ the canonical map generated by $\mathscr{W}$.

The Euclidean group $\mathscr{G}$ acts in a natural way in $\mathscr{E}$ (we write the same symbol for the group elements and for the operators of the representation). The translations of space $T((0,\vec{x}))$, $\vec{x} \in \mathbb{R}$, commute with $\mathscr{W}$, thus $\{WT((0,\vec{x}))W^{-1},\ \vec{x} \in \mathbb{R}\}$ is well defined on $\mathscr{H}$, and gives a continuous unitary group. By Stone's theorem there exists a self-adjoint operator P, (the momentum) such that

$$WT((0,\vec{x}))W^{-1} = exp\,(iP\vec{x}) \qquad \text{for all } \vec{x} \in \mathbb{R}\ .$$

The translation of a non-negative Euclidean time $T((t,0))$, $t \in \mathbb{R}_+ = \{s \geq 0\}$, maps $\mathscr{E}^+$ in $\mathscr{E}^+$ and $ker\,\mathscr{W}$ in $ker\,\mathscr{W}$. Thus $\{WT((t,0))W^{-1},\ t \in \mathbb{R}_+\}$ is well defined on $\mathscr{H}$, and gives a continuous self-adjoint semi-group. By the extension of Stone's theorem for semi-groups, there exists a self-adjoint operator H, (the Hamiltonian) such that

$$WT((t,0))W^{-1} = exp\,(-tH) \qquad \text{for all } t \in \mathbb{R}_+ .$$

Moreover $WT((t,0))W^{-1}$ has norm ≤ 1 for all $t \in \mathbb{R}_+$, so H is positive. The square of the mass operator $M^2 = H^2 - P^2$ satisfies the formula :

$$\begin{aligned}(W\xi\,; M^2 W\zeta)_{\mathscr{H}} &= \lim_{x \to 0,\ \overset{\circ}{x} \geq 0} \Delta_x\ (W\xi\,; WT(x)W^{-1}\ W\zeta)_{\mathscr{H}} \\ &= \lim_{x \to 0,\ \overset{\circ}{x} \geq 0} \Delta_x\ (\mathscr{W}\xi\,; T(x)\,\zeta)_{\mathscr{E}}\end{aligned}$$

for all $\xi, \zeta \in \mathscr{E}$ for which the limits exist. We have just found that the scalar products involved in the Rayleigh quotient of §1 can be obtained in the Euclidean framework.

Let $R(\alpha)$, $\alpha \in]-\pi,\pi]$, be the rotation of angle α in $\mathbb{R}^2$, respectively its representation in $\mathscr{E}$. For all $\alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ there exists a closed subspace $\mathscr{E}_\alpha$ of $\mathscr{E}^+$ such that $R(\alpha) : \mathscr{E}_\alpha \to \mathscr{E}^+$. Let W_α be the restriction of W to $\mathscr{E}_\alpha$. Then $\left\{W_{-\alpha}R(\alpha)W_\alpha^{-1},\ \alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right\}$ is a symmetric local semi-group ([11]), so by a generalization of the Stone theorem ([11]) there exists a self-adjoint operator L (Lorentz infinitesimal generator) such that

$$W_{-\alpha}R(\alpha)W_\alpha^{-1} = exp\,(-\alpha L) \qquad \text{for all } \alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

Thus all functions of the operators H, P and L can be studied in the Euclidean framework.

Note that $1 \in \mathcal{E}$ (more precisely, the random variable $Q \ni q \mapsto 1$), which is an invariant vector under the action of the group $\mathcal{G}$. The vacuum vector of $\mathcal{H}$ (which generates the vacuum subspace) is taken to be $\Omega = W1$. It satisfies automatically : $P\,\Omega = H\,\Omega = M\,\Omega = L\,\Omega = 0$.

We pass now to examples of such a theory, the weakly-coupled $\mathcal{P}(\varphi)_2$ models. We begin to expose the case where the coupling constant $\lambda = 0$.

Let $m_o>0$ be given. The free model, describing non-interacting particles of mass m_o, is constructed from the following characteristic function $\mathcal{C}_o$:

$$\mathcal{C}_o(f) = exp\ -\tfrac{1}{2}\langle f\,;C\,f\rangle$$

for all $f \in \mathcal{S}$, where $C = (-\Delta+m_o)^{-1}$. Here $\langle .;.\rangle$ is the $L^2(\mathbb{R}^2)$ scalar product and Δ is the Laplacian on $\mathbb{R}^2$. C is called the *covariance operator*. $\mathcal{C}_o$ satisfies the hypothesis of Minlos' theorem (see [7]), so there exists a probability measure μ_o on Σ with characteristic function $\mathcal{C}_o$. Moreover $\mathcal{C}_o$ satisfies the *Probability Axioms for Q.F.T.* (see [7]) so there exists a Wightman Q.F.T. model with state space $\mathcal{H}_o$, Hamiltonian H_o, momentum P_o, mass operator M_o and Lorentz generator L_o, constructed from $\mathcal{C}_o$ as mentioned above. We denote by W_o the canonical map $\mathcal{E}_o = L^2(Q,\mu_o) \to \mathcal{H}_o$, and by $\Omega_o = W_o\,1$ the vacuum vector.

The fact that these models describe free particles is due to the gaussian form of $\mathcal{C}_o(f)$. To obtain models with interaction we must perturb $\mathcal{C}_o(f)$ as much as to destroy the gaussian property. This turns out to be a difficult problem, whose solution involves very singular operations.

First we must control local products of distributions (the so-called problem of the *U.V. limit*). We introduce new random variables, the Wick polynomials of the fields, denoted by $:\phi_f^n:$ for $n \in \mathbb{N}^*$ and $f \in \mathcal{S}(\mathbb{R}^2)$, defined by the generating formula :

$$\sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!}\ :\phi_f^n:\ = \frac{e^{i\,\alpha\,\phi_f}}{\mathcal{C}_o(\alpha f)} \quad , \alpha \in \mathbb{R},$$

with $:\phi_f^0:=1$. We will use the algebraic notation $:A+B: = :A:+:B:$ and $:kA: = k:A:$ for k a constant. Let $\mathcal{P}$ be a $\mathbb{R} \to \mathbb{R}$ positive-valued polynomial (the *interaction polynomial*). We take a C^∞-function $g : \mathbb{R}^2 \to \mathbb{R}$ with compact support and satisfying $\int g=1$, and for all $n \in \mathbb{N}^*$ we define g_n by : $g_n(x) = n^2 g(nx)$ for all $x \in \mathbb{R}^2$.

Theorem (U.V. limit, Nelson). *The following limits exist in* $L^2(Q,\mu_o)$:

i) $$\prod_{i=1}^{N} :\phi^{m_i}:_{f_i} \;=\; \underset{n\to\infty}{s\text{-}lim} \prod_{i=1}^{N} \int_{\mathbb{R}^2} :\left(\phi_{T(x)g_n}\right)^{m_i}: f_i(x)\, d^2x$$

for all $N\in\mathbb{N}^*$, $m\in(\mathbb{N}^*)^N$, $f\in(\mathscr{S}(\mathbb{R}^2))^N$.

ii) $$e^{-\lambda V_\Lambda} \;=\; \underset{n\to\infty}{s\text{-}lim}\ exp\left(-\lambda \int_\Lambda :\mathcal{P}\left(\phi_{T(x)g_n}\right): d^2x\right)$$

for all $\lambda>0$ *and* Λ *compact set of* $\mathbb{R}^2$.

This theorem is due to Nelson [12] (see [7] for other details). The maps $:\phi^m:$ from $\mathscr{S}$ to $\mathscr{E}_o$ are called the *Euclidean Wick fields*. For each $\lambda\geq 0$ and Λ compact set of $\mathbb{R}^2$, the probability measure is well defined, as it follows from the theorem :

$$d\mu_{\lambda,\Lambda}(q) \;=\; \frac{1}{Z_{\lambda,\Lambda}}\, e^{-\lambda V_\Lambda(q)}\, d\mu_o(q)$$

for all $q\in Q$, where $Z_{\lambda,\Lambda}$ is the normalization factor, and its generalized moments (or *generalized Schwinger distributions*) are also well defined :

$$S^m_{\lambda,\Lambda}(f) \;=\; \int_Q \prod_{i=1}^{N} :\phi^{m_i}:_{f_i}\, d\mu_{\lambda,\Lambda}$$

for all $N\in\mathbb{N}^*$, $m\in(\mathbb{N}^*)^N$ and $f\in(\mathscr{S}(\mathbb{R}^2))^N$. The goal now is to perform the limit $\Lambda\to\mathbb{R}^2$. The key for this, and for the answer of our further analytic problems, is given by the following estimates for $S^m_{\lambda,\Lambda}$ which are uniform in λ and Λ.

Let us introduce some notations. $\mathcal{B}$ is the Banach space of Lebesgue-measurable functions f on $\mathbb{R}^2$ with the norm :

$$|||\, f\, ||| \;=\; \sum_{\Delta\in\mathcal{R}} \|\, \chi_\Delta f\, \|_{L^2}$$

where $\mathcal{R}$ is the lattice of $\mathbb{R}^2$: $\{(n,n+1)\times(m,m+1)\ ;\ n,m\in\mathbb{Z}\}$ and χ_Δ is the characteristic function of the compact set Δ. For $f = f_1\otimes\cdots\otimes f_n$ with all $f_i\in\mathcal{B}$ we also note $|||\,f\,||| = |||\,f_1\,|||\cdots|||\,f_n\,|||$. For two set in $\mathcal{B}$, $f=\{f_1,\cdots,f_n\}$ and $g=\{g_1,\cdots,g_m\}$, we denote by d(f,g) the smaller distance in $\mathbb{R}^2$ between $\cup\{\text{support } f_i\ ;\ i=1,\dots,n\}$ and $\cup\{\text{support } g_i\ ;\ i=1,\dots,m\}$. For $n,n'\in\mathbb{N}^*$ and $m\in(\mathbb{N}^*)^n$, $m'\in(\mathbb{N}^*)^{n'}$ we write $m+m' = (m_1,\dots,m_n,m'_1,\dots,m'_{n'})$ and $\ell_m = m_1+\cdots+m_n$.

Theorem (Uniform bounds, Glimm, Jaffe, Spencer). *There exist* K, $\underline{\lambda}$, $\underline{m} \in (0,\infty)$ *(depending only on* $\mathcal{P}$ *and* m_o*) such that, for all* $\lambda \in [0,\underline{\lambda}]$ *and* Λ *compact set of* $\mathbb{R}^2$:

i) $$\left|S^{m}_{\lambda,\Lambda}(f)\right| < K^n \ell_m! \;|||f|||,$$

ii) $$\left|S^{m+m'}_{\lambda,\Lambda}(f\otimes g) - S^{m}_{\lambda,\Lambda}(f)\, S^{m'}_{\lambda,\Lambda}(g)\right| < K^{n+n'} \ell_{m+m'}! \;|||f|||\;|||g|||\; e^{-\underline{m}d(f,g)},$$

for all $n,n' \in \mathbb{N}^*$, $m \in (\mathbb{N}^*)^n$, $m' \in (\mathbb{N}^*)^{n'}$, $f \in \mathcal{B}^n$ *and* $g \in \mathcal{B}^{n'}$.

The proof [1], which is very difficult, consists in controlling a series called the *cluster expansion.* This is the crucial step of the rigorous construction of any Q.F.T. model. Note that the theorem gives a continuous extension of the $S^m_{\lambda,\Lambda}$, always written with the same symbol, from $\mathcal{S}^n$ to $\mathcal{B}^n$.

It follows from the theorem that the $S^m_{\lambda,\Lambda}$ and the measure $\mu_{\lambda,\Lambda}$ converge when $\Lambda \to \mathbb{R}^2$ (the so-called *thermodynamic limit*). Let B_n be the ball in $\mathbb{R}^2$ of center 0 and radius n for each $n \in \mathbb{N}^*$.

Theorem (Thermodynamic limit) *Let* $\lambda \in [0,\underline{\lambda}]$.

a) *Convergence of the generalized Schwinger distributions : for all* $n \in \mathbb{N}^*$, $m \in (\mathbb{N}^*)^n$ *and* $f \in \mathcal{B}^n$, $S^m_{\lambda,B_n}(f)$ *converge when* $n \to \infty$, *and the limit* $S^m_\lambda(f)$ *satisfies* :

i) $$\left|S^{m}_{\lambda}(f)\right| < K^n \ell_m! \;|||f|||,$$

ii) $$\left|S^{m+m'}_{\lambda}(f\otimes g) - S^{m}_{\lambda}(f)\, S^{m'}_{\lambda}(g)\right| < K^{n+n'} \ell_{m+m'}! \;|||f|||\;|||g|||\; e^{-\underline{m}d(f,g)},$$

for all $n,n' \in \mathbb{N}^*$, $m \in (\mathbb{N}^*)^n$, $m' \in (\mathbb{N}^*)^{n'}$, $f \in \mathcal{B}^n$ *and* $g \in \mathcal{B}^{n'}$.

b) *Convergence of the measure : for all* $\sigma \in \Sigma$, $\mu_{\lambda,B_n}(\sigma)$ *converges when* $n \to \infty$, *and the limit* $\mu_\lambda(\sigma)$ *defines a probability measure* μ_λ *on* Σ *which satisfies the Probability Axioms for Quantum Field Theory.*

a) is proved in [1] and b) in [7]. Let $\mathcal{C}_\lambda$ be the characteristic function of μ_λ for $\lambda \in [0,\underline{\lambda}]$. It follows from the theorem that a Wightman Q.F.T. model $(\mathcal{H}_\lambda, \mathcal{U}_\lambda, \varphi_\lambda, \Gamma_\lambda)$ exists, with Hamiltonian H_λ, momentum P_λ and mass operator M_λ, constructed from $\mathcal{C}_\lambda$ as mentioned above. We denote by W_λ the canonical map $\mathcal{E}_\lambda = L^2(Q,\mu_\lambda) \to \mathcal{H}_\lambda$, and by $\Omega_\lambda = W_\lambda 1$ the vacuum vector. These models, called the *weakly-coupled* $\mathcal{P}(\varphi)_2$ *models*, describe a quantum and relativistic world of particles which actually interact ([13], [14]).

2.2 The zero-time vectors; the problem

Let $\lambda \in [0,\underline{\lambda}]$ be fixed. The combination of Euclidean fields ϕ^n (for $n \in \mathbb{N}^*$) given by

$$\phi^n(f) = :\phi_{f_1} \cdots \phi_{f_n}: \qquad \text{for all } f \in (\mathscr{S}(\mathbb{R}^2))^n$$

defines a continuous map $\phi^n : (\mathscr{S}(\mathbb{R}^2))^n \to \mathscr{E}_\lambda$. By the nuclear theorem it can be extended continuously to a map (also called ϕ^n) from $\mathscr{S}((\mathbb{R}^2)^n)$ to $\mathscr{E}_\lambda$.

The Fourier transform $\widetilde{\phi^n}$ of ϕ^n is defined by $\widetilde{\phi^n}(\tilde{f}) = \phi^n(f)$ for all $f \in \mathscr{S}((\mathbb{R}^2)^n)$ where $\tilde{f}$ is the ordinary Fourier transform of f. The map $\widetilde{\phi^n} : \mathscr{S}((\mathbb{R}^2)^n) \to \mathscr{E}_\lambda$ is also continuous.

To study the zero-time vectors we begin with the case of the free models (i.e. $\lambda=0$), where the situation is more clear. In that case the Euclidean scalar product of vectors $:\phi^n:_f$, $:\phi^m:_g$ for $n,m \in \mathbb{N}$ and $f,g \in \mathscr{S}(\mathbb{R}^2)$ is given by

$$(:\phi^n:_f ; :\phi^m:_g)_{\mathscr{E}_o} = \partial_\alpha^n \, \partial_\beta^m \frac{\mathscr{C}_o(\alpha f + \beta g)}{\mathscr{C}_o(\alpha f)\, \mathscr{C}_o(\beta g)} \Bigg|_{\alpha=\beta=0} = \delta_{n,m} \; n! \langle f ; C g \rangle^n .$$

We obtain after some calculations, with now $f \in \mathscr{S}((\mathbb{R}^2)^n)$ and $g \in \mathscr{S}((\mathbb{R}^2)^m)$:

$$(\phi^n(f) ; \phi^m(g))_{\mathscr{E}_o} = \delta_{n,m} \; n! \int_{\mathbb{R}^{2n}} d\xi^n(k) \quad \overline{\tilde{f}^{\,s}(k)}\, \tilde{g}(k)$$

where $d\xi^n(k) = \prod_{i=1}^{n} \dfrac{d^2k_i}{k_i^2 + m_o^2}$, and f^s is the following symmetrization of f :

$$f^s(x_1,\ldots,x_n) = \frac{1}{n!} \sum_{\pi \in \sigma_n} f(x_{\pi(1)},\ldots,x_{\pi(n)}) , \quad \text{for all } (x_1,\ldots,x_n) \in (\mathbb{R}^2)^n,$$

where σ_n is the set of all permutations of $\{1,\ldots,n\}$.

Let $\mathscr{A}$ be a $\mathbb{C}$-Hilbert space. For all $\in \mathbb{N}^*$ we denote by $\mathscr{A}_n = Sym\ \mathscr{A} \otimes \cdots \otimes \mathscr{A}$ (symmetrical n times tensorial product), which is a Hilbert space with the scalar product deduced from $(f \otimes \cdots \otimes f ; g \otimes \cdots \otimes g)_{\mathscr{A}_n} = \big((f;g)_{\mathscr{A}}\big)^n$ for all $f,g \in \mathscr{A}$. The Fock space over $\mathscr{A}$, denoted by $\mathscr{F}(\mathscr{A})$, is the Hilbert space $\mathscr{F}(\mathscr{A}) = \bigoplus_{0 \le n < \infty} \mathscr{A}_n$, where $\mathscr{A}_o = \mathbb{C}$, with the scalar product $((f_o,\ldots,f_n,\ldots); (g_o,\ldots,g_n,\ldots))_{\mathscr{F}(\mathscr{A})} = \Sigma_{0 \le n < \infty} (f_n;g_n)_{\mathscr{A}_n}$, where $f_n, g_n \in \mathscr{A}_n$ for all $n \in \mathbb{N}$.

We take $\mathscr{A} = L^2(\mathbb{R}^2,\xi^1)$; we have just seen that the map $j : \mathscr{F}(\mathscr{A}) \to \mathscr{E}_o$, given by

$$j((f_o,\dots,f_n,\dots)) = f_o + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \widetilde{\phi^n}(f_n{}^S)$$

is well defined and continuous (we have performed a continuous extension of $\widetilde{\phi^n}$ from $\mathcal{S}((\mathbb{R}^2)^n)$ to $\mathcal{A}_n$, that we still write $\widetilde{\phi^n}$). We can say more about j.

Proposition. *$j : \mathcal{F}(\mathcal{A}) \to \mathcal{E}_o$ is an Hilbert space isomorphism.*

Proof. Let $f = (f_o,\dots,f_n,\dots) \in \mathcal{F}(\mathcal{A})$. From the above scalar product it follows that $\| j(f) \|_{\mathcal{E}_o} = \| f \|_{\mathcal{F}(\mathcal{A})}$, so j is an isomorphism from $\mathcal{F}(\mathcal{A})$ to its range, *Ran j*. By construction of $\mathcal{E}_o$, $\{j(f)\ ;\ f \in \mathcal{F}(\mathcal{A})\}$ is a dense set of $\mathcal{E}_o$, thus $Ran\, j = \mathcal{E}_o$. ◊

Let $n \in \mathbb{N}^*$. The function $((\overset{o}{k}_1,\vec{k}_1),\dots,(\overset{o}{k}_n,\vec{k}_n)) \mapsto f((\overset{o}{k}_1,\vec{k}_1),\dots,(\overset{o}{k}_n,\vec{k}_n)) = g(\vec{k}_1,\dots,\vec{k}_n)$ (that is, constant in the variables $\overset{o}{k}_1,\dots,\overset{o}{k}_n$), belongs to $\mathcal{A}_n$ provided that g is Borel-measurable and bounded in the norm :

$$\int_{\mathbb{R}^{2n}} d\xi^n(k)\, |f^S(k)|^2 = (2\pi)^n \int_{\mathbb{R}^n} d\eta^n(\vec{k})\, |g^S(\vec{k})|^2$$

where $d\eta^n(\vec{k}) = \prod_{i=1}^{n} \frac{d\vec{k}_i}{2\omega(\vec{k}_i)}$ and ω is the function on $\mathbb{R}$: $\omega(p) = \sqrt{p^2 + m_o{}^2}$; g^S is now the complete symmetrization of g (in all variables). For such a function f we write : $\widetilde{\phi^n}(f) = \widetilde{\theta^n}(g)$. This defines a continuous map $\widetilde{\theta^n} : L^2(\mathbb{R}^n,\eta^n) \to \mathcal{E}_o$. Let $\mathcal{K}$ be the Hilbert space generated by the functions $g \in \mathcal{S}(\mathbb{R})$ with $\tilde{g} \in L^2(\mathbb{R},\eta^1)$ ($\mathcal{K}$ is a Sobolev space). The vectors $\theta^n(g) = \widetilde{\theta^n}(\tilde{g})$ for all $g \in \mathcal{K}_n$, are the *Euclidean vectors at zero-time*. They can be written formally as $\theta^n(g) = \phi^n(\delta^n \otimes g)$, where δ is the Dirac generalized function. By application of W_o we obtain the *zero-time vectors* of $\mathcal{H}_o$:

$$W_o\theta^n(g),\ g \in \mathcal{K}_n,\ n \in \mathbb{N}^*,$$

which, together with the vacuum vector Ω_o, generate the *zero-time subspace* $\mathcal{D}_o$. The zero-time vector "$\Theta^n_{\lambda=0}(f)\Omega_o$" introduced in §1 (by an imprecise definition) can now be identified with $W_o\theta^n(f)$.

The scalar product of $\mathcal{H}_o$ is the same as of $\mathcal{E}_o$, but with $\langle f;Cg \rangle$ replaced by $\langle \theta f;Cg \rangle$ for functions f,g with support property : $f((\overset{o}{x},\vec{x})) = g((\overset{o}{x},\vec{x})) = 0$ if $\overset{o}{x} < 0$. For such functions a simple calculation [7] gives :

$$\langle\, \theta f\, ; Cg\, \rangle = 2\pi \int_{\mathbb{R}} d\eta^1(\vec{k})\ \overline{\tilde{f}\,(i\omega(\vec{k}),\vec{k})}\ \tilde{g}(i\omega(\vec{k}),\vec{k})\ .$$

In particular $\langle \theta f;Cf \rangle = 0$ does not imply f=0. As a consequence the map $W_o \circ j : \mathcal{F}(\mathcal{A}) \to \mathcal{H}_o$ is not injective. To find a isomorphism between $\mathcal{H}_o$ and a Fock space of functions we must restrict ourselves to the zero-time vectors. We define the map $i : \mathcal{F}(\mathcal{K}) \to \mathcal{H}_o$ by

$$i((f_o,\dots,f_n,\dots)) = W_o f_o + W_o \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \theta^n(f_n{}^S)$$

which is well defined and continuous. We can say more.

Proposition. *$i : \mathcal{F}(\mathcal{K}) \to \mathcal{H}_o$ is an Hilbert space isomorphism.*

Proof. For all $f = (f_o,\dots,f_n,\dots) \in \mathcal{F}(\mathcal{K})$ we have $\| i(f) \|_{\mathcal{H}_o} = \| f \|_{\mathcal{F}(\mathcal{K})}$, so i is an isomorphism from $\mathcal{F}(\mathcal{K})$ to its range, *Ran i*. By construction of $\mathcal{H}_o$, the span of the set

$$\Big\{ W_o \circ j(f)\ ;\ f_o=1,\ f_n=g\otimes\cdots\otimes g \text{ (n times) for } 1\le n\le N,\ f_n=0 \text{ for } n>N, \text{ for all } N\in \mathbb{N}^* \\ \text{and } g\in \mathcal{S}(\mathbb{R}^2) \text{ with } g((\mathring{x},\vec{x})) = 0 \text{ if } \mathring{x} < 0 \Big\}$$

is a dense set of $\mathcal{H}_o$. By the above identity for $\langle \theta f;Cg \rangle$, this set is not distinguishable in $\mathcal{H}_o$ from the same set with now g is replaced by h, where $\tilde{h}((\mathring{k},\vec{k}) = \tilde{g}((i\omega(\vec{k}),\vec{k}))$. Thus the zero-time subspace $\mathcal{D}_o$ is dense, and $Ran\ i = \mathcal{H}_o$. ◊

From the Proposition and its proof the following results are easily deduced.

Proposition *For all* n, $\alpha\in \mathbb{N}^*$:

i) $P_o{}^\alpha W_o\theta^n(f) = W_o\theta^n(P^\alpha f)$, *provided* $P^\alpha f\in \mathcal{K}_n$,

ii) $H_o{}^\alpha W_o\theta^n(f) = W_o\theta^n(h^\alpha f)$, *provided* $h^\alpha f\in \mathcal{K}_n$,

iii) $L_o W_o\theta^n(f) = W_o\theta^n(L_o f)$, *provided* $L_o f\in \mathcal{K}_n$,

where for all $(\vec{k}_1,\dots,\vec{k}_n)\in (\mathbb{R}^2)^n$:

$$\widetilde{P^\alpha f}(\vec{k}_1,\dots,\vec{k}_n) = (\vec{k}_1+\dots+\vec{k}_n)^\alpha\, \tilde{f}(\vec{k}_1,\dots,\vec{k}_n),$$

$$\widetilde{h^\alpha f}(\vec{k}_1,\dots,\vec{k}_n) = \big(\omega(\vec{k}_1) + \dots + \omega(\vec{k}_n)\big)^\alpha\, \tilde{f}(\vec{k}_1,\dots,\vec{k}_n) \text{ and}$$

$$\widetilde{L_o f}(\vec{k}_1,\dots,\vec{k}_n) = -i\left(\omega(\vec{k}_1)\frac{\partial}{\partial \vec{k}_1} + \dots + \omega(\vec{k}_n)\frac{\partial}{\partial \vec{k}_n}\right)\tilde{f}(\vec{k}_1,\dots,\vec{k}_n).$$

From the last proposition the action of $M^2 = H^2 - P^2$ is easily calculated and we obtain :

$$M^2\,\Omega_o = 0,$$

$$M^2 W_o\theta^1(f) = m_o{}^2 W_o\theta^1(f) \quad \text{for all } f\in \mathcal{K}_1,$$

$$M^2 W_o\theta^n(f) = W_o\theta^n(\mathcal{M}^2 f) \text{ for } n\geq 2, \text{ where } \mathcal{M}^2 f = h^2 f - P^2 f, \text{ provided } \mathcal{M}^2 f\in \mathcal{K}_n.$$

The two first eigenspaces are the vacuum and the one-particle states subspaces respectively. Moreover $W_o\theta^n(f)$, $n\geq 2$, is in the domain of M if $\mathcal{M}^2 f\in \mathcal{K}_n$, which is realized if $h^2 f\in \mathcal{K}_n$.

Thus, to conclude, the following subspace of $\mathcal{D}_o$:

$$\left\{ W_o\theta^n(f),\ n\in \mathbf{N}^*,\ n\geq 2,\ h^2 f\in \mathcal{K}_n \right\}$$

is a good candidate for $\mathcal{V}$ satisfying all the requirements 1) to 5) of the beginning of this section.

For models with interaction (i.e. $\lambda\neq 0$) the situation is more complicated, because we have no explicit forms for the scalar products. Let us state the problem of the existence of zero-time vectors in the domain of M. Because the action of P on such vectors is trivial (as in the last proposition) we have only to look at the domain of H. A vector $WA = W\phi^n(f)$ (for some n,f) belong to the domain of H^α, and the scalar products $(WA; H^{2\alpha} WA)$ is a C^∞-function of λ, if

$$\partial_\lambda^\nu\, \partial_t^{2\alpha}\, \chi(t,\lambda) \Big|_{t=+0}$$

is well defined for all $\nu\in \mathbf{N}$, where

$$\chi(t,\lambda) \;=\; \left(WA\, ;\, e^{-tH}\, WA \right)_{\mathcal{H}} \;=\; \left(\mathcal{W}A\, ;\, T(t,0)\, A \right)_{\mathcal{E}}.$$

Without lost of generality we can take $\phi^n(f)\in \mathcal{E}^+$ (this imposes only a support property for f) so that $(\mathcal{W}A\, ;\, T(t,0)\, A)_{\mathcal{E}} = (\theta A\, ;\, T(t,0)\, A)_{\mathcal{E}}$. Let us write $\chi(t,\lambda)$ in the following form :

$$\chi(t,\lambda) \;=\; \int d^{2n}k\ d^{2n}p\ \ \overline{\tilde{f}\,(\theta k)}\ \tilde{f}(p)\ \tilde{s}_{n,n,\lambda}(k,p)\ e^{it(\sum_{j=1}^{n} \mathring{p}_j)}$$

where $s_{n,m,\lambda}$ is the pseudo-function generating the Schwinger distribution : $S_{n,m,\lambda}(f,g) = (\phi^n(f);\ \phi^m(g))_{\mathcal{E}}$ (for suitable m, g) and $\tilde{s}_{n,m,\lambda}$ is its Fourier transform (in the distribution sense). We speak about a zero-time vector if the function $\tilde{f}$ does not depend of the $\mathring{k}_i$ variables ; in that case, we separate these variables from the "momentum variables", which cut the problem in two steps :

$$\partial_\lambda^\nu\, \partial_t^{2\alpha}\, \chi(t,\lambda) \Big|_{t=+0} \;=\; \partial_\lambda^\nu\, \partial_t^{2\alpha} \int d^n\vec{k}\ d^n\vec{p}\ \ \overline{\tilde{f}\,(\vec{k})}\ \tilde{f}(\vec{p})\ \xi_\lambda(\vec{k},\vec{p}\,,t) \Big|_{t=+0}$$

where : $\xi_\lambda(\vec{k},\vec{p},t) = \displaystyle\int d^n\overset{\circ}{k}\; d^n\overset{\circ}{p}\; \tilde{s}_{n,n,\lambda}(k,p)\, e^{it(\sum_{j=1}^{n} \overset{\circ}{p}_j)}$.

The problem leads to the study of the asymptotic decrease of the pseudo-function $\tilde{s}_{n,n,\lambda}$. If the decrease were sufficient, the t-derivatives would be given by the factor $(\Sigma\, \overset{\circ}{p}_j)^{2\alpha}$ inside the last integral. We will see that it is not so simple. In the free models for example, in the case n=m=1, the pseudo-function $\tilde{s}_{1,1,0}$ is given by $\tilde{s}(k,p) = \dfrac{\delta^{(2)}(k-p)}{p^2 + m_o{}^2}$, and the form of the denominator is not sufficient to guarantee the integrability of $\overset{\circ}{p}{}^{2\alpha}\tilde{s}(k,p)$ even for $\alpha=1/2$. But after integrating over $\overset{\circ}{k}$ and $\overset{\circ}{p}$ we obtain, in this example : $\xi(\vec{k},\vec{p},t) = \delta(\vec{k}-\vec{p})\,\dfrac{\pi}{\omega(\vec{p})}\, exp(-|t|\omega(\vec{p}))$, and then

$$\chi(t,0) = \int_{\mathbf{R}} \frac{d\vec{p}}{2\omega(\vec{p})}\; |\,\tilde{f}\,(\vec{p})|^2 \;\; exp(-|t|\omega(\vec{p}))$$

which, for all $f \in \mathscr{S}(\mathbf{R})$, is a C^∞-function of t for t>0, with a well defined limit for $t \to +0$; and also each derivative admits a well defined limit for $t \to +0$.

In the next paragraph we present a programme which decomposes each Schwinger distribution in a sum of products of two kinds of terms. The first one are distributions of the free models (for them we must "first integrate and then differentiate"), and the others have a Fourier transform with a good asymptotic decrease (for them we must "first differentiate and then integrate") and their derivatives with respect to λ have the same property. The consequences of these results on the zero-time vectors will be discussed in §2.4.

2.3 The WTI Programme

We present a new programme for the Schwinger distributions, in order to study the asymptotic decrease of their Fourier transforms. It is based on two analytic results, the inequalities *i)* and *ii)* of the theorem of the thermodynamic limit (last theorem of §2.1), and on three algebraic-like operations :

(W) : the Wick projection,

(T) : the truncation,

(I) : the integration by part formula.

First we expose these operations and then we discuss their consequences.

We will adopt the notation : $\langle \dots \rangle_\lambda$ for $\int_Q d\mu_\lambda \;\dots$

2.3.1 (W) : the Wick projection. We first give a definition. The *Wick-Schwinger distributions* $SW_{n,\lambda}$ with $\lambda\in[0,\underline{\lambda}]$ and $n\in\mathbb{N}^*$, are given by

$$SW_{n,\lambda}(f) = \langle :\phi_{f_1}\cdots\phi_{f_n}:\rangle_\lambda$$

for all $f\in\mathcal{S}(\mathbb{R}^2)^n$; for n=0 we take $SW_{0,\lambda} = 1$. Note that for $\lambda=0$, $SW_{n,0} = 0$ for all $n\in\mathbb{N}^*$. All Schwinger distributions can be decomposed linearly into Wick-Schwinger distributions, the coefficients of the decomposition being Schwinger distributions of the $\lambda=0$ case.

Lemma *Let* X *be a finite non-empty subset of* $\mathbb{N}^*$, p *a partition of* X *and* $f\in(\mathcal{S}(\mathbb{R}^2))^X$. *Then the following formula (W), called the "Wick projection", holds :*

$$\prod_{J\in p} :\prod_{j\in J}\phi_{f_j}: \;=\; \sum_{\emptyset\subseteq Y\subseteq X} \langle \prod_{J\in p_Y} :\prod_{j\in J}\phi_{f_j}:\rangle_0 \; :\prod_{j\in X-Y}\phi_{f_j}: \,.$$

Integrating over μ_λ *for all* $\lambda\in[0,\underline{\lambda}]$ *gives :*

$$\langle \prod_{J\in p} :\prod_{j\in J}\phi_{f_j}:\rangle_\lambda \;=\; \sum_{\emptyset\subseteq Y\subseteq X} \langle \prod_{J\in p_Y} :\prod_{j\in J}\phi_{f_j}:\rangle_0 \; SW_{|X-Y|,\lambda}(f_{X-Y})$$

where $f_{X-Y} = \otimes_{j\in X-Y} f_j$.

The proof is given for instance in [7]. p_Y is the restriction of p to Y.

2.3.2 (T) : the Truncation. Let $n\in\mathbb{N}^*$, $\mathcal{O}\subset\mathbb{R}^n$ a neighborhood of 0 and $\mathcal{G}\in C^n(\mathcal{O})$ satisfying $\mathcal{G}(0) = 1$ (the *generating function*). For $I\subset\{1,\dots,n\}$ (non empty) we consider the following numbers (with the notation $D_I = \prod_{i\in I}\partial_i$) :

M_I, a *moment of* $\mathcal{G}$, defined by : $M_I = (D_I\mathcal{G})(0)$ and
T_I, a *truncated moment of* $\mathcal{G}$, given by : $T_I = (D_I\, ln\,\mathcal{G})(0)$.

These quantities are related together as follows.

Lemma *Let* $\mathcal{G}$, $\{M_I, T_I, I\subset\{1,\dots,n\}\}$ *as before. For all* $I\subset\{1,\dots,n\}$ *(non empty) we have :*

i) $M_I = \sum_{p\in\mathcal{P}(I)} \prod_{J\in p} T_J$, *ii)* $T_I = \sum_{p\in\mathcal{P}(I)} (-1)^{|p|-1}(|p|-1)! \prod_{J\in p} M_J$.

Notation : $\mathcal{P}(I)$ is the set of all partitions of I. For a proof of this well known result, see [7].

As first example we take $\mathcal{G}(x) = \langle e^{ix\cdot A}\rangle_\lambda$, where $x\cdot A = x_1A_1+\ldots+x_nA_n$ and $\{A_j,\ 1\le j\le n\}$ are suitable random variables. The moments are $M_I = i^{|I|}\langle\Pi_{j\in I} A_j\rangle_\lambda$ and the truncated moments are noted by $T_I = i^{|I|}\langle\Pi_{j\in I}\,;A_j;\rangle^T_\lambda$.

An other example is given by : $\mathcal{G}(x) = \mathcal{C}_\lambda(x\cdot f)/\mathcal{C}_0(x\cdot f)$, where $x\cdot f = x_1f_1+\ldots+x_nf_n$. The moments are the Wick-Schwinger distributions $SW_{m,\lambda}(f_I)$, with $f_I=\otimes_{j\in I} f_j$. The truncated moments, called the *truncated Wick-Schwinger distributions*, are denoted by $SWT_{m,\lambda}(f_I)$. Because the logarithm of $\mathcal{G}$ is a sum, we have $SWT_{m,\lambda} = ST_{m,\lambda} - ST_{m,0}$. Note that a simple calculation gives $ST_{m,0} = \delta_{m,2} \langle . ; (1-\Delta)^{-1} . \rangle$.

2.3.3 (I) : the Integration by parts formula. This is a family of relations between Schwinger distributions obtained by an adaptation of the familiar "integration by parts formula" to the functional integral; see [1] or [7].

We denote by $\mathcal{P}^k$ the k-th derivative of the interaction polynomial $\mathcal{P}$ (if $k>deg\,\mathcal{P}$, $\mathcal{P}^k=0$). For all $\lambda\in[0,\underline{\lambda}]$, $n\in\mathbb{N}^*$ and $p\in\mathcal{P}_n$, the set of all partitions of $\{1,\ldots,n\}$, we consider the combinations of Schwinger distributions :

$$\mathcal{S}^p_\lambda(f) = \lim_{\Lambda\to\mathbb{R}^2} \langle \prod_{J\in p} :\mathcal{P}^{|J|}(\phi):_{f_J}\rangle_{\lambda,\Lambda},$$

$$\mathcal{ST}^p_\lambda(f) = \lim_{\Lambda\to\mathbb{R}^2} \langle \prod_{J\in p} ;\ :\mathcal{P}^{|J|}(\phi):_{f_J};\rangle^T_{\lambda,\Lambda}$$

for $f\in(\mathcal{S}(\mathbb{R}^2))^p$, with the notation : $\langle \ldots \rangle_{\lambda,\Lambda}$ for $\int_Q d\mu_{\lambda,\Lambda}$ .... The limit $\Lambda\to\mathbb{R}^2$ has to be taken in the sense of the thermodynamic limit theorem, §2.1.

Lemma *For all* $n\in\mathbb{N}^*$, $f\in(\mathcal{S}(\mathbb{R}^2))^n$ *and* $\lambda\in[0,\underline{\lambda}]$, *the following "integration by parts formula" (I) holds :*

i) $$SW_{n,\lambda}(f) = \sum_{p\in\mathcal{P}_n} (-\lambda)^{|p|}\ \mathcal{S}^p_\lambda(f_p),$$

ii) $$SWT_{n,\lambda}(f) = \sum_{p\in\mathcal{P}_n} (-\lambda)^{|p|}\ \mathcal{ST}^p_\lambda(f_p),$$

where $f_p\in(\mathcal{S}(\mathbb{R}^2))^{|p|}$ *is given by* $f_p = \otimes_{J\in p}(\Pi_{j\in J} Cf_j)$.

In the lemma, C is now the operator $(1-\Delta)^{-1}$. Note that the r.h.s. involve the functions Cf_j instead of f_j, $1\le j\le n$. This is the key for finding the local regularity of the Schwinger distributions.

2.3.4. Local regularity. The integration by parts formula (I), *i)* (previous lemma), together with the inequality a), *i)* of the thermodynamic limit theorem (last theorem of §2.1), have the following consequence ([7]).

> **Lemma.** *For all* $\lambda \in [0, \underline{\lambda}]$ *and* $n \in \mathbb{N}^*$, *the Wick-Schwinger distribution* $SW_{n,\lambda}$ *is generated by a continuous and bounded function* $sw_{n,\lambda}$, *called a Wick-Schwinger function ; moreover* $\| sw_{n,\lambda} \|_{L^\infty}$ *is also a bounded function of* λ *on* $[0, \underline{\lambda}]$.

This lemma has three important consequences. First, by (W) all Schwinger distributions $S_{n,\lambda}$ are generated by functions $s_{n,\lambda}$ which have the same singularities as the functions $s_{n,\lambda=0}$. But $s_{n,0}$ can be studied in great detail (see §2.2 or [7]) ; they are C^∞ functions on the *non-coincident points* NC(n) defined by

$$NC(n) = \left\{ (x_1, \ldots, x_n) \in (\mathbb{R}^2)^n \, , \, x_i \neq x_j \text{ for all } i \neq j \right\}$$

and have only logarithmic singularities. The functions $s_{n,\lambda}$ are then continuous on NC(n) and locally integrable at any power.

The second consequence is that the first limits of the U.V. limit theorem (§2.1), which lead to the Wick-fields, can also be obtained when $\lambda \neq 0$.

> **Proposition.** *For all* $\lambda \in [0, \underline{\lambda}]$ *the following limits exist in* $L^2(Q, \mu_\lambda)$ *(with the notation of the U.V. limit theorem* §2.1):
>
> $$\prod_{i=1}^{N} :\phi_\lambda^{m_i}:_{f_i} \quad = \quad \underset{n \to \infty}{s\text{-}lim} \prod_{i=1}^{N} \int_{\mathbb{R}^2} :(\phi_{T(x)g_n})^{m_i}: f_i(x) \, d^2x$$
>
> *and they satisfy* : $$\langle \prod_{i=1}^{N} :\phi_\lambda^{m_i}:_{f_i} \rangle_\lambda \quad = \quad S_\lambda^m(f)$$
>
> *for all* $N \in \mathbb{N}^*$, $m \in (\mathbb{N}^*)^N$, $f \in (\mathscr{S}(\mathbb{R}^2))^N$.

The $:\phi_\lambda^m:$ are the *Euclidean Wick fields of the interaction model.* The proof of the convergence (see [7]) consists in writing the corresponding $L^2(Q, \mu_\lambda)$-norms as sums of products, by (W), of distributions with $\lambda = 0$ (which converge by the U.V. limit theorem) times distributions $SW_{m,\lambda}$ (which converge because they are generated by continuous and bounded functions). The problem is now one of convergence of a product of convergent distributions. It can be solved by performing the limit in two steps in a standard way.

The third consequence is an extension of the formula (W).

Lemma *For all* $n\in\mathbb{N}^*$, $m\in\mathbb{N}^n$ *and* $\lambda\in[0,\underline{\lambda}]$ *the distribution* S^m_λ *is generated by a function* s^m_λ *continuous on* NC(n) *and locally integrable at any power satisfying the extended formula* (W) *for all* $x\in$ NC(n) :

$$s^m_\lambda(x_1,\ldots,x_n) \;=\; \sum_{0\leq v\leq m} K^m_v \;\; s^n_0(x_1,\ldots,x_n) \;\; sw_{m-v,\lambda}(x_1,\ldots,x_1,\ldots,x_n)\,.$$

In the lemma m and v are multi-indexes, $K^m_v = \prod_{i=1}^{n} \binom{m_i}{v_i}$ (binomial coefficients) and $m-v = \sum_{i=1}^{n} (m_i-v_i)$. In $sw_{m-v,\lambda}(x_1,\ldots,x_1,\ldots,x_n)$ the variable x_1 appears m_1-v_1 times, etc.... If an index v_i is 0, then the x_i-dependence in $s^v_0(x_1,\ldots,x_n)$ is dropped.

The proof of the formula (see [8]) consists in showing again the convergence of a product of convergent distributions (because the Euclidean Wick fields are given by limits, and the formula (W) introduces products).

2.3.5. Asymptotic decrease. The truncated Schwinger distributions ST^m_λ are defined as the truncated moments (times i^{-n}) of the generating function

$$\mathbb{R}^n \ni \alpha \mapsto \mathscr{G}(\alpha) \;=\; \int_Q d\mu_\lambda \, exp\left(i\sum_{i=1}^{n} \alpha_i \, {:}\phi^{m_i}_\lambda{:}\,f_i \right).$$

They are connected with the distributions S^m_λ by the formula in lemma 2.3.2. Thus they are also generated by functions st^m_λ continuous on NC(n) and locally integrable at any power. Moreover they have an asymptotic decrease property, which follows now from the second inequality a), *ii)* of the thermodynamic limit theorem (last theorem of §2.1). Note that by the translation invariance, $st^m_\lambda(x_1,\ldots,x_n)$, where $(x_1,\ldots,x_n)\in(\mathbb{R}^2)^n$, do not depend on the $\mathbb{R}^2$-variable $x_1+\ldots+x_n$. Thus st^m_λ is a function of the *relative variables* $(x_1-x_n,\ldots,x_{n-1}-x_n)\in(\mathbb{R}^2)^{n-1}$ alone. We need one more definition. For $n\in\mathbb{N}$, $n\geq2$, for $(x_1,\ldots,x_n)\in(\mathbb{R}^2)^n$ and for $\{I,J\}$ a partition of $\{1,\ldots,n\}$ in two parts, let us denote by $d_{I,J}$ the smallest distance between the convex envelopes in $\mathbb{R}^2$ of $\{x_i, i\in I\}$ and of $\{x_j, j\in J\}$. Then we define : $\sigma(x_1,\ldots,x_n) = max\ \{\ d_{I,J}\ , \{I,J\}\in\mathscr{P}_n\}$ (see figure 1).

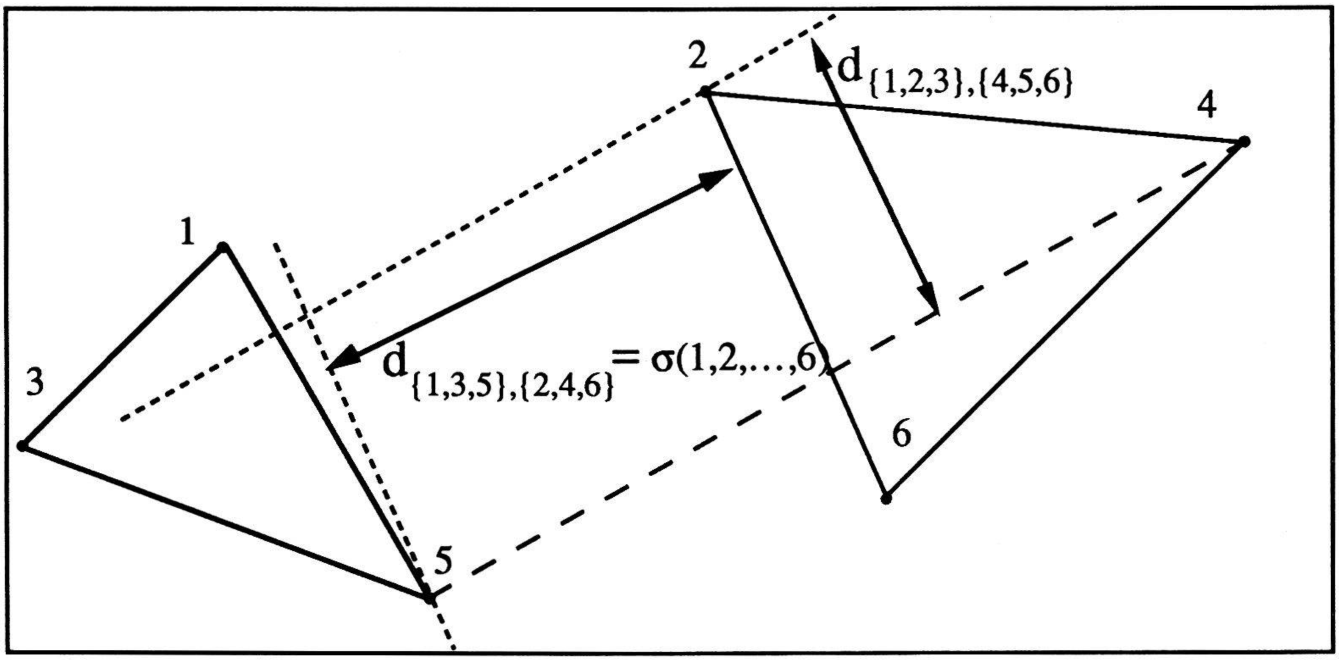


Figure 1

Lemma *For all* $n\in\mathbb{N}$, $n\geq 2$ *and* $m\in(\mathbb{N}^*)^n$ *there exist* $K\in(0,\infty)$ *and* $\nu\in\mathbb{N}^*$ *such that*

$$\left|st_\lambda^m(x_1,\ldots,x_n)\right| < K\left(\prod_{1\leq i<j\leq n} L(x_i-x_j)\right)^\nu \mathit{exp}\left(-\,\underline{m}\,\sigma(x_1,\ldots,x_n)\right)$$

for all $x\in NC(n)$ *and* $\lambda\in[0,\underline{\lambda}]$.

Here $L(x) = \ell(m_o\|x\|)$, where $\ell(r) = 1 + |\mathit{ln}\, r|$ if $0<r\leq 1$ and $\ell(r) = 1$ if $r\geq 1$.

The proof (see [8]) is complicated. It does not follow directly from the inequality a), *ii)* of the last theorem of § 2.1 (which concerns only smeared distributions), but uses a consequence of this inequality on the spectrum of the Hamiltonian, which has a gap of length $\underline{m}$. The exponential in the lemma comes from this gap. Thus the lemma is a consequence not only of the inequality mentioned above, but also from the existence of the Hamiltonian, that is from all the Axioms.

We will use two consequences of the lemma. The first one is that the truncated functions are integrable in the relative variables.

Proposition *For all* $n\in\mathbb{N}$, $n\geq 2$, $m\in(\mathbb{N}^*)^n$, $\lambda\in[0,\underline{\lambda}]$ *and* $1\leq p<\infty$ *the function* :

$(x_1,\ldots,x_{n-1},0)\mapsto st_\lambda^m(x_1,\ldots,x_{n-1},0)$ *is in* $L^p((\mathbb{R}^2)^{n-1})$, *and its* L^p*-norm is bounded in* λ.

To prove the proposition (see [8]) we bound the involving integral using the previous lemma and we introduce a lattice on $(\mathbb{R}^2)^{n-1}$ to separate the local and the asymptotic difficulties.

The second consequence of the lemma concerns the derivative of the Schwinger functions with respect to λ, which have to be well controlled before doing perturbation calculations. The following theorem concerns the derivatives of the Schwinger distributions. The interaction polynomial $\mathcal{P}$ is written as $\mathcal{P}(x) = \sum_{1\le i\le N} a_i\, x^i$, where N is even and $a_N>0$. In what follows we take $a_i=0$ for i=0 or i>N.

Theorem *For all* $\lambda\in[0,\underline{\lambda}]$, $n\in\mathbb{N}^*$, $m\in(\mathbb{N}^*)^n$ *and* $f\in\mathcal{S}((\mathbb{R}^2)^n)$, *the function* $\lambda\mapsto ST_\lambda^m(f)$ *is in* $C^\infty([0,\underline{\lambda}])$ *and the following identity holds for all* $\nu\in\mathbb{N}^*$:

$$\partial_\lambda^\nu\, ST_\lambda^m(f) = (-1)^\nu \sum_{i\in\mathbb{N}^n} a_i \int_{\mathbb{R}^{2(n+\nu)}} d^{2n}x\; d^{2\nu}y\;\; f(x)\;\; st_\lambda^{m+i}(x,y)\ .$$

In the Theorem, $i\in\mathbb{N}^n$ is a multi-index, $i=\{i_1,\ldots,i_\nu\}$, $a_i = \prod_{1\le j\le\nu} a_{i_j}$, and $m+i = (m_1,\ldots,m_n, i_1,\ldots,i_\nu)$. Note that the l.h.s. of the formula is well defined, because of the integrability properties of the truncated functions. The proof is due to Dimock [15].

Let us pass to the differentiability of the Schwinger functions ([8]).

Proposition *For all* $\lambda\in[0,\underline{\lambda}]$, $n\in\mathbb{N}^*$, $m\in(\mathbb{N}^*)^n$ *and* $x\in NC(n)$, *the function* $\lambda\mapsto st_\lambda^m(x)$ *is in* $C^\infty([0,\underline{\lambda}])$ *and the following identity holds for all* $\nu\in\mathbb{N}^*$

$$\partial_\lambda^\nu\, st_\lambda^m(x) = (-1)^\nu \sum_{i\in\mathbb{N}^n} a_i \int_{\mathbb{R}^{2\nu}} d^{2\nu}y\;\; st_\lambda^{m+i}(x,y)\ .$$

In particular, $x\mapsto\partial_\lambda^\nu\, st_\lambda^m(x)$ *is continuous on* NC(n) *and locally integrable at any power*.

This formula, often used in the literature but never completely proved, is a consequence of the theorem and the fact that the r.h.s. is continuous on NC(n). This last result follows from the integrability property and from the local regularity seen in §2.3.4.

We summarize the consequences of the asymptotic decrease by a statement on the Fourier transform of the truncated functions. By the integrability property, the Fourier transform in the relative variables gives well defined, continuous and bounded functions. By derivation with respect to λ they become a sum of Fourier transform of other truncated functions. Thus their properties of continuity and boundedness are conserved by derivation with respect to λ.

2.3.6. The Wick-Schwinger truncated functions. The Wick-Schwinger truncated distributions $SWT_{n,\lambda}$ introduced in §2.3.2 are also generated by functions $swt_{n,\lambda}$, the *Wick-Schwinger truncated functions*. They are the most interesting functions occurring in the weakly-coupled $\mathcal{P}(\varphi)_2$ models, because they have the local regularity of the Wick-Schwinger functions (bounded and continuous) and the integrability property in the relative variables of all truncated functions. Moreover they generate all the objets of the models in the sense that by successive applications of (W) and (T), all Schwinger functions are sums of products of only

- Schwinger functions of the free models (i.e for $\lambda=0$),
- Wick-Schwinger functions,

and only the last ones depend on λ. The formula of integration by parts (I), *ii)* (lemma 2.3.3) has the following consequences.

Proposition *For all* $\lambda\in[0,\underline{\lambda}]$ *and* $n\in\mathbf{N}$, $n\geq 2$:

i) $swt_{n,\lambda}$ *is of class* $C^1((\mathbb{R}^2)^n)$ *and belongs to* $L^p((\mathbb{R}^2)^{n-1})$ *(w.r.t. relative variables) for all* $1\leq p\leq\infty$, *the* L^p*-norm being bounded in* λ ;

ii) for all $x\in(\mathbb{R}^2)^n$ *the function* $\lambda\mapsto swt_{n,\lambda}(x)$ *is of class* $C^\infty([0,\underline{\lambda}])$ *and its derivatives* $\partial_\lambda^\nu\, swt_{n,\lambda}$ *for all* $\nu\in\mathbf{N}$ *satisfy all the properties stated in i)* ;

iii) the derivatives of the Fourier transforms $\partial_\lambda^\nu\widetilde{swt}_{n,\lambda}$ *satisfies, for all* $\nu\in N$ *and* $p\in(\mathbb{R}^2)^n$

$$\partial_\lambda^\nu\, \widetilde{swt}_{n,\lambda}(p) = \frac{\delta^{(2)}\left(\sum_{i=1}^{n} p_i\right)}{\prod_{i=1}^{n}\left(p_i^2+m_o^2\right)}\ \partial_\lambda^\nu\, \Sigma_\lambda^n(p_1,\ldots,p_{n-1})$$

where $\partial_\lambda^\nu\,\Sigma_\lambda^n$ *is a continuous function on* $(\mathbb{R}^2)^{n-1}$ *such that there exists* $K\in(0,\infty)$, *independant of* λ *and of* $p_1,\ldots,p_{n-1}$ *with*

$$\left|\partial_\lambda^\nu\, \Sigma_\lambda^n(p_1,\ldots,p_{n-1})\right| < K.$$

The proof consists in establishing *iii)*. The formula is simply (I), *ii)*, (in lemma 2.3.2) written after having performed a Fourier transformation in all variables. (To have symmetric notation we often take the Fourier transform in all variables rather than in the relative variables; this simply introduces an extra δ pseudo-function). Σ_λ^n is a sum of Fourier transforms (here in the relative variables) of truncated functions :

$$\Sigma_\lambda^n(p_1,\ldots,p_{n-1}) = \sum_{p=\{I_1,\ldots,I_r\}\in\mathscr{P}_n} (-\lambda)^r\, \mathscr{A}_{\lambda,p}\left(\sum_{i\in I_1} p_i\,,\ \ldots,\ \sum_{i\in I_r} p_i\right)$$

where $p_n=-(p_1,\ldots,p_{n-1})$, $\mathscr{A}_{\lambda,\{1,\ldots,n\}} = (2\pi)^{n+2}\langle :\mathcal{P}^n(\delta): \rangle_\lambda$ (finite constant bounded in λ, by the first lemma of 2.3.4) and, for $p=\{I_1,\ldots,I_r\}\in\mathscr{P}_n$, $r\geq 2$, $\mathscr{A}_{\lambda,p}(q_1,\ldots,q_{r-1},-(q_1+\ldots+q_{r-1}))$ is the Fourier transform (up to 2π-factors) of $(x_1,\ldots,x_{r-1}) \mapsto \langle \Pi_{1\leq i\leq r} ;:\mathcal{P}^{|I_i|}(\phi(x_i)):; \rangle^T_\lambda$ with $x_r = 0$ (where $\phi(x) = \phi(T(x)\delta)$). The differentiability with respect to λ and the continuity and boundedness of $\partial_\lambda^v\Sigma_\lambda^n$ are those of all Fourier transforms of truncated functions.

Note that Σ_λ^n contains a constant ($\mathscr{A}_{\lambda,\{1,\ldots,n\}}$), thus have no asymptotical decrease.

2.3.7. A formula for the truncated functions. The last proposition allows us to write the functions $\widetilde{swt}_{n,\lambda}$ in term of functions Σ_λ^n, which are made of Fourier transforms of truncated functions st_λ^m. For iterating our programme, we have to apply it to these functions. But the Wick projection (W) gives expressions where the truncated properties are no more evident. So we need for the functions st_λ^m a formula which generalizes the previous proposition.

We introduce some notation. For $n\in\mathbb{N}^*$, and $m\in(\mathbb{N}^*)^n$ let $\mathscr{G}_m$ be the set of all connected graphs linking n vertices of $m_1,\ldots,m_n$ segments respectively. To each such graph G we associate a $n\times\ell$ matrix $\varepsilon(G)$, (the incidence matrix), where ℓ is the number of lines of the graph (that is $\ell = \frac{1}{2}\Sigma_{1\leq i\leq n} m_i$), as follows : we choose a direction to each line (arbitrarily), and we put

$$\varepsilon(G)_{i,j} = \begin{cases} +1 & \text{if the line j get out of the vertex i} \\ -1 & \text{if the line j goes in the vertex i} \\ 0 & \text{if the line j do not reach the vertex i} \end{cases}$$

A tedious calculation gives the formula (called Wick theorem) ([1]) :

$$\widetilde{st}_0^m(p_1,\ldots,p_n) = (2\pi)^{n-2\ell} \sum_{G\in\mathscr{G}_m} \int d\xi^\ell(k) \prod_{i=1}^n \delta^{(2)}\left(p_i - \sum_{j=1}^{\ell} \varepsilon(G)_{i,j}\, k_j\right)$$

where $d\xi^\ell(k) = \prod_{j=1}^{\ell} \frac{d^2k_j}{k_j^2 + m_o^2}$. The WTI Programme gives a similar expression for $\widetilde{st}_\lambda^m$ for all λ, involving graphs with more lines (but less than $\Sigma_{1\leq i\leq n} m_i$) and functions remaining under the integrals. These functions are bounded and continuous.

Proposition. *For all* $\lambda \in [0, \underline{\lambda}]$, $n \in \mathbb{N}^*$ *and* $m \in (\mathbb{N}^*)^n$ *the function* $\widetilde{st}_\lambda^m$ *satisfies, for all* $p \in (\mathbb{R}^2)^n$

$$\widetilde{st}_\lambda^m(p_1,\ldots,p_n) = \sum_{G \in \mathscr{E}_m} \int d\xi^{\ell(G)}(k) \prod_{i=1}^n \delta^{(2)}\left(p_i - \sum_{j=1}^{\ell(G)} \varepsilon(G)_{i,j}\, k_j\right) \mathscr{F}_\lambda^G(k_1,\ldots,k_{\ell(G)})$$

where $\mathscr{E}_m$ *is a set of connected graphs of* n *vertices and less than* $\Sigma_{1 \le i \le n}\, m_i$ *lines,*

$\ell(G)$, *for* $G \in \mathscr{E}_m$, *is the number of lines of* G,

$\mathscr{F}_\lambda^G$, *for* $G \in \mathscr{E}_m$, *is a continuous and bounded function such that, for fixed* $k \in (\mathbb{R}^2)^{\ell(G)}$, $\lambda \mapsto \mathscr{F}_\lambda^G(k)$ *is of class* $C^\infty([0,\underline{\lambda}])$, *and which satisfy : for all* $\nu \in \mathbb{N}$ *there exists* $K \in (0,\infty)$ *such that :*

$\left|\partial_\lambda^\nu \mathscr{F}_\lambda^G(k_1,\ldots,k_{\ell(G)})\right| < K$ *for all* $k \in (\mathbb{R}^2)^{\ell(G)}$ *and* $\lambda \in [0,\underline{\lambda}]$.

The formula of the proposition can be compared with the formula for the swt functions (proposition of § 2.3.6), the functions $\mathscr{F}$ playing the role of the functions Σ. But it remains to integrate on variables associated to the loops of some graphs with less than $\Sigma_{1 \le i \le n}\, m_i$ lines. The proof (see [8]) is based on a formula which generalizes the extended (W) formula (last lemma of § 2.3.4), in which only truncated functions appear :

Lemma. *For all* $\lambda \in [0,\underline{\lambda}]$, $n \in \mathbb{N}^*$ *and* $m \in (\mathbb{N}^*)^n$ *the following formula holds for all* $x \in NC(n)$:

$$st_\lambda^m(x_1,\ldots,x_n) = \sum_{p \in \mathscr{P}_{p_o}} \prod_{J \in p} \left(swt_{\lambda,|J|}(x_1,\ldots,x_1,\ldots,x_n) - \sum_{i \neq j = 1}^{n} K^J_{i,j}\, c(x_i - x_j) \right)$$

where p_o *is a partition* $\{I_1,\ldots,I_n\} \in \mathscr{P}_r$ *with* $|I_i| = m_i$ *for all* i *and* $r = \Sigma_{1 \le i \le n}\, m_i$; $K^J_{i,j} = \delta_{2,|J|}\, \delta_{1,|I_i \cap J|}\, \delta_{1,|I_j \cap J|}$; *in* $swt_{\lambda,|J|}(x_1,\ldots,x_1,\ldots,x_n)$, x_1 *appears* $|J \cap I_1|$ *times, etc ...*

Notation : $\mathscr{P}_{p_o}$ is the set of all partitions of $\{1,\ldots,r\}$ mutually connected with p_o , that is those partitions $p \in \mathscr{P}_r$ for which all unions of $J \in p$ differs from all unions of I_i's (except for the total union). The restriction of those partitions is necessary to avoid the factorization of the variables $x_1,\ldots,x_n$. That is, each term of the sum cannot be a product $f(y)g(z)$ with y and z disjoint subsets of $\{x_1,\ldots,x_n\}$.

Performing a Fourier transformation and using the proposition 2.3.6 leads to

$$\widetilde{st}_\lambda^m(p_1,\ldots,p_n) = \sum_{p \in \mathscr{P}_{p_o}} \int d\xi^r(k) \left(\prod_{i=1}^n \delta^{(2)}\left(p_i - \sum_{j \in I_i} k_j\right)\right) \prod_{J \in p} \delta^{(2)}\left(\sum_{j \in J} k_j\right) \Sigma'^J_\lambda(\{k_j, j \in J\})$$

where $\Sigma'^{J}_{\lambda}(\{k_j, j\in J\}) \;=\; \Sigma^{|J|}_{\lambda}(\{k_j, j\in J\}) - \sum_{i\neq j=1}^{n} K^{J}_{i,j}\,(k_i^2 - m_o^2).$

Note that the two products of δ pseudo-functions cause no difficulties, because the partitions p_o and p are mutually connected (so there is no δ-square!). We separate the integration variables k_j in two sets. We define the partition *inf*(p_o,p) by collecting the sets $I\cap J$ for all $I\in p_o$ and $J\in p$. Now in each element of p we choose an element of *inf*(p_o,p), and we call K their union. We introduce the function :

$$\mathscr{F}^{G}_{\lambda}(q_1,\dots,q_s) \;=\; \int\left(\prod_{j\in K}\frac{d^2k_j}{k_j^2+m_o^2}\right)\prod_{J\in p}\delta^{(2)}\left(\sum_{j\in J}k_j\right)\Sigma'^{J}_{\lambda}(\{k_j, j\in J\})$$

which is a function of the k_i , $i\in L = \{1,\dots,r\} - K$ (which we have called $q_1,\dots,q_s$). Note that, due to the good properties of Σ', these functions satisfy all the analytic properties stated in the proposition. We insert these functions in the above formula for $\widetilde{st}^{m}_{\lambda}$, which leads to the announced formula (the graphs G depend on p and on the choice of K).

2.4 Zero-time vectors in the domain of the Hamiltonian

We are now able to answer the first question of the beginning of this § 2, about the smoothness of the zero-time vectors. There exist indeed zero-time vectors in the domain of M and even in the domain of M^2 (this will be usefull in §2.6). Moreover for two such vectors ξ, ζ, the scalar products $(\xi, M^{\nu}\,\zeta)$, $\nu\in\{0,1,2,3,4\}$, are C^{∞} in λ. Because the operator P acts trivially on the zero-time vectors we have only to study the case where M is replaced by H. We follow essentially the way of the free model case §2.2, the WTI Programme bringing enough information about the scalar products of the models with interaction.

Due to the previous analysis, the scalar products of Euclidean fields can be defined for functions in a larger space than $\mathscr{S}$, which admit functions with δ pseudo-functions of the Euclidean time. Moreover, these scalar products, smeared now only in the space variables, are differentiable with respect to λ and to an extra Euclidean time variable obtained by translating one of the two vectors (§ 2.4.1)

§ 2.4.2 deduces the existence of zero-time Euclidean fields and § 2.4.3 translates these results in the Minkowskian Hilbert space, constructs the zero-time vectors and states their properties. § 2.4.4 presents some generalizations.

2.4.1 Regularity of some Euclidean scalar products. Let us take $n,m \in \mathbb{N}^*$, $f \in \mathcal{S}((\mathbb{R}^2)^n)$ and $g \in \mathcal{S}((\mathbb{R}^2)^m)$. We consider the function $\chi_{f,g}$ on $\mathbb{R}^2 \times [0,\underline{\lambda}]$ given by

$$\chi_{f,g}(s,\lambda) = (\phi^n(f)\,;T(s)\,\phi^m(g))_{\mathscr{E}} - (\phi^n(f)\,;1)_{\mathscr{E}}\,(1\,;\phi^m(g))_{\mathscr{E}}$$

$$= \int d^{2n}x\; d^{2m}y\;\; f(x)\; g(y-s)\; s_{n,m,\lambda}(x,y)$$

with $y - s = (y_1 - s,\ldots,y_m - s)$. Here $s_{n,m,\lambda}$ is the function generated by the distribution $S_{n,m,\lambda}(f,g) = \langle\, \phi^n(f)\,;\phi^m(g)\,\rangle_\lambda^T$ (Note the differences with the notation of §2.2 : $s \in \mathbb{R}^2$, the truncation property in $s_{n,m,\lambda}$ and the absence of the operator $\mathscr{W}$).

We will see that $\chi_{f,g}(s,\lambda)$ is well defined and differentiable with respect to $\vec{s}$ and λ even if f and g are of the following kind

$$(\mathbb{R}^2)^j \ni y \mapsto h(\vec{y})\; \delta^j(\mathring{y} - \mathring{x}) \text{ for some } \mathring{x} \in \mathbb{R}^j$$

for j=n or =m (with the notation : $\delta^j(\mathring{y} - \mathring{x}) = \prod_{1\le i\le j} \delta(\mathring{y}_i - \mathring{x}_i)$), provided the functions of the space variables h belong to some function spaces. Moreover, even in this case, it can be differentiable with respect to $\mathring{s}$ if the Euclidean times $\mathring{s}$ and $\mathring{x}_i$ have a right sign.

$\chi_{f,g}(s,\lambda)$ involves only the orthogonalized part of the smeared fields $\phi^n(f)$ with respect to the constants (recall that the constant random variables $Q \ni q \mapsto c \in \mathbb{C}$ belongs to $\mathscr{E}_\lambda$). We will also use the projection on the constant, so we introduce

$$\chi_f^0(\lambda) = (1\,;\phi^n(f))_{\mathscr{E}} = \int d^{2n}x\; f(x)\; sw_{n,\lambda}(x)\;.$$

The function spaces we need are build on the semi-norms $\mathfrak{b}_n^0$ and norms $\mathfrak{b}_{n,\alpha}$ (with $\alpha \in \mathbb{N}$), defined for all suitable functions f as follows

$$\mathfrak{b}_n^0(f)^2 = \int_{\mathbb{R}^n} d\eta^n(\vec{k})\; |\tilde{f}(\vec{k})|^2 \sum_{p\in\mathscr{P}_n} \prod_{J\in p} \delta\Big(\sum_{j\in J} \vec{k}_j\Big)$$

$$\mathfrak{b}_{n,\alpha}(f)^2 = \sum_{i=0}^{n-1} \int_{\mathbb{R}^n} d\eta^n(\vec{k})\; |\tilde{f^S}(\vec{k})|^2 \Big(\sum_{j=i+1}^{n} \omega(\vec{k}_j)\Big)^{\alpha} \sum_{p\in\mathscr{P}_i} \prod_{J\in p} \delta\Big(\sum_{j\in J} \vec{k}_j\Big)$$

(Recall that : $d\eta^n(\vec{k}) \doteq \prod_{i=1}^{n} \dfrac{d\vec{k}_i}{2\omega(\vec{k}_i)}$; ω is the function on $\mathbb{R}$: $\omega(p) = \sqrt{p^2 + m_o^2}$; f^S is the complete symmetrization of f and $\mathscr{P}_i$ is the set of partitions of $\{1,\ldots,i\}$. If i=0 or if n=1 the sum over $\mathscr{P}_i$ must be omited.)

Mathematically the norm $\mathfrak{b}_{n,\alpha}$ is the L^2-norm of a Borel-measure on $\mathbb{R}^n$.

Examples. For suitable functions f :

$$\mathfrak{b}_1^0(f) = m_o^{-1/2}\,|\tilde{f}(0)|\ ,$$

$$\mathfrak{b}_2^0(f)^2 = \int_{\mathbb{R}} \frac{d\vec{k}}{4\omega(\vec{k})^2}\,|\tilde{f}(\vec{k},-\vec{k})|^2 + m_o^{-2}\,|\tilde{f}(0,0)|^2,$$

$$\mathfrak{b}_{1,\alpha}(f)^2 = \int_{\mathbb{R}} \frac{d\vec{k}}{2\omega(\vec{k})}\,|\tilde{f}(\vec{k})|^2\,\omega(\vec{k})^\alpha,$$

$$\mathfrak{b}_{2,\alpha}(f)^2 = \int_{\mathbb{R}^2} \frac{d\vec{k}_1}{2\omega(\vec{k}_1)}\,\frac{d\vec{k}_2}{2\omega(\vec{k}_2)}\,|\tilde{f}(\vec{k}_1,\vec{k}_2)|^2\left(\omega(\vec{k}_1)+\omega(\vec{k}_2)\right)^\alpha +$$

$$+\ 2\,m_o^{-1}\int_{\mathbb{R}} \frac{d\vec{k}}{2\omega(\vec{k})}\,|\tilde{f}^S(\vec{k},0)|^2\,\omega(\vec{k})^\alpha\ .$$

Let us introduce the following function spaces :

$\mathfrak{B}_n^0$ (respectively $\mathfrak{B}_{n,\alpha}$) is the space of functions $f\in L^1_{loc}(\mathbb{R}^n)$, with at most polynomial growth, having continuous Fourier transforms (in the distribution sense) and well defined $\mathfrak{b}_n^0(f)$ (respectively $\mathfrak{b}_{n,\alpha}(f)$).

These spaces are given the topology induced by their norms or semi-norms. We do not close them, because we do not need completeness.

We state the differentiability properties of $\chi_{f,g}(s,\lambda)$ and $\chi_f^0(\lambda)$ for functions f and g as described above. We will use the notation $P^\beta f$ introduced in the last proposition of § 2.2, and $\mathbb{R}_+ = \{x\in\mathbb{R},\ 0\le x<\infty\}$.

Proposition. *For all* $\lambda\in[0,\underline{\lambda}]$, $s\in\mathbb{R}^2$, $n,m\in\mathbb{N}^*$, $\mathring{x}\in\mathbb{R}^n$ *and* $\mathring{y}\in\mathbb{R}^m$:

1) for all $f\in\mathfrak{B}_n^0$ *let us write* $F(z) = f(\vec{z})\,\delta^n(\mathring{z}-\mathring{x})$ *for all* $z\in(\mathbb{R}^2)^n$;
then $\chi_F^0(\lambda)$ *is a well defined* C^∞ *function of* λ *satisfying for all* $\nu\in\mathbb{N}$:
there exist $K\in(0,\infty)$ *independent of* f, $\mathring{x}$ *and* λ, *with* $\left|\partial_\lambda^\nu\,\chi_F^0(\lambda)\right| < K\,\mathfrak{b}_n^0(f)$.

2) for all $\beta,\beta_1\,\beta_2\in\mathbb{N}$ *with* $\beta_1+\beta_2=\beta$, *for all* $P^{\beta_1}f\in\mathfrak{B}_{n,0}$ *and* $P^{\beta_2}g\in\mathfrak{B}_{m,0}$ *let us write*
$F(z) = f(\vec{z})\,\delta^n(\mathring{z}-\mathring{x})$ *for all* $z\in(\mathbb{R}^2)^n$ *and* $G(z) = g(\vec{z})\,\delta^m(\mathring{z}-\mathring{y})$ *for all* $z\in(\mathbb{R}^2)^m$;

then for s *fixed,* $\lambda \mapsto \chi_{F,G}(s,\lambda) \in C^{\infty}[0,\underline{\lambda}]$,
and for all $\nu \in \mathbb{N}$, $\vec{s} \mapsto \partial_\lambda^\nu \chi_{F,G}(s,\lambda) \in C^\beta(\mathbb{R})$ *for fixed* $\mathring{s}$ *and* λ ;

the derivations commute : $\partial_{\vec{s}}^\beta \partial_\lambda^\nu \chi_{F,G}(s,\lambda) = \partial_\lambda^\nu \partial_{\vec{s}}^\beta \chi_{F,G}(s,\lambda)$;

moreover there exists $K \in (0,\infty)$ *independent of* f, g, $\mathring{x}$, $\mathring{y}$ *and* λ, *with*

$$\left| \partial_{\vec{s}}^\beta \partial_\lambda^\nu \chi_{F,G}(s,\lambda) \right| < K\, \mathfrak{b}_{n,0}(P^{\beta_1}f)\, \mathfrak{b}_{m,0}(P^{\beta_2}g) .$$

3) *for all* $\alpha \in \{1,2,3\}$, *with the same hypothesis and notation as in 2)*
but $P^{\beta_1}f \in \mathcal{B}_{n,\alpha}$, $P^{\beta_2}g \in \mathcal{B}_{m,\alpha}$ *and* $\mathring{x} \in (\mathbb{R}_+)^n$ *and* $-\mathring{y} \in (\mathbb{R}_+)^m$,
$\mathring{s} \mapsto \partial_{\vec{s}}^\beta \partial_\lambda^\nu \chi_{F,G}(s,\lambda) \in C^\alpha(\mathbb{R}_+)$ *for fixed* $\vec{s}$ *and* λ ;

all the derivations with respect to $\mathring{s}$, $\vec{s}$ *and* λ *commute* ;

moreover there exists $K \in (0,\infty)$ *independent of* f, g, $\mathring{x}$, $\mathring{y}$ *and* λ, *with*

$$\left| \partial_{\mathring{s}}^\alpha \partial_{\vec{s}}^\beta \partial_\lambda^\nu \chi_{F,G}(s,\lambda) \right| < K\, \mathfrak{b}_{n,\alpha}(P^{\beta_1}f)\, \mathfrak{b}_{m,\alpha}(P^{\beta_2}g) .$$

4) *The statement 3) is also true for* $\alpha = 4$ *if* $\mathring{x}_i = s$ *for all* $1 \le i \le n$ *and* $\mathring{y}_j = t$ *for all* $1 \le j \le m$, *for some* $s, t \in \mathbb{R}_+$.

The proof (see [8]) begins with functions F, G in $\mathscr{S}$ and establishes the proposition with, in the r.h.s. of the estimations, the semi-norms $N_n^0(F)$ (resp. norms $N_{n,\alpha}(F)$), defined as $\mathfrak{b}_n^0(f)$ (resp. $\mathfrak{b}_{n,\alpha}(f)$), but with $\mathbb{R}^n$, f and $d\eta^n$ replaced by $(\mathbb{R}^2)^n$, F and $d\xi^n$. Let us denote by $\mathscr{F}_n^0$ (resp. $\mathscr{F}_{n,\alpha}$) the space of functions $F \in L^1_{loc}((\mathbb{R}^2)^n)$, with at most polynomial growth, having continuous Fourier transforms (in the distribution sense) and with well defined $N_n^0(F)$ (resp. $N_{n,\alpha}(F)$). Because $\mathscr{S}((\mathbb{R}^2)^n) \hookrightarrow \mathscr{F}_n^0$ and $\mathscr{S}((\mathbb{R}^2)^n) \hookrightarrow \mathscr{F}_{n,\alpha}$ (continuous injections) we have obtained the extension of χ_F^0 (resp. $\chi_{F,G}$) from $\mathscr{S}$ to $\mathscr{F}_n^0$ (resp. $\mathscr{F}_{n,\alpha}$). We call them again χ_F^0 and $\chi_{F,G}$.

Functions like F and G in the proposition belong to $\mathscr{F}_n^0$ or $\mathscr{F}_{n,\alpha}$, and the semi-norm $\mathfrak{b}_n^0$ (resp. norms $\mathfrak{b}_{n,\alpha}$) are simply the restriction of N_n^0 (resp. $N_{n,\alpha}$) to this set of functions.

Let us present the others steps of the proof. The functions $s_{n,m,\lambda}$ and $sw_{n,\lambda}$ are decomposed according to the WTI Programme. The permutation of differentiation and integration : $\partial_s \int h(s,t)\, dt = \int \partial_s h(s,t)\, dt$, is allowed when the functions h(s,t) and $\partial_s h(s,t)$ are Lebesgue-integrable in s for all t and continuous in t for all s. This permits us to perform the derivatives for 1) and 2). Using that $\partial_\lambda^\nu \Sigma_\lambda^n(k)$ is bounded in k and λ we obtain the existence of $\partial_\lambda^\nu \chi_F^0(\lambda)$ provided that

$$\int d\xi^n(k)\ |\widetilde{F}(k)|\ \sum_{p\in\mathscr{P}_n}\prod_{J\in p}\delta^{(2)}\Big(\sum_{j\in J}k_j\Big)$$

is well defined. But this expression exists for functions F like in 1), and becomes then, after integration over the variables $\mathring{k}_i$:

$$\pi^{n-1}\int d\eta^n(\vec{k})\ |\tilde{f}(\vec{k})|\ \Big(\sum_{j\in J}\omega(\vec{k}_j)\Big)^{-1}\ \sum_{p\in\mathscr{P}_n}\prod_{J\in p}\delta\Big(\sum_{j\in J}\vec{k}_j\Big)$$

which can be bound by a constant times $\mathfrak{b}_n^0(f)$, using the Cauchy-Schwartz inequality. 2) is proven with similar technics.

To show 3) and 4) we start from 2), that is from $\partial^\beta_{\vec{s}}\,\partial^\nu_\lambda\,\chi_{F,G}(s,\lambda)$ with F and G like in 2). The WTI Programme decomposes $s_{n,m,\lambda}$ in products of functions of the free models and functions $swt_{j,\lambda}$. According to the Leibniz rule we have only to look at the differentiability of each term separately. The functions of the free model are differentiable only if $\mathring{s}$, $\mathring{x}_i$ and $\mathring{y}_j$ have the right signs (see § 2.2), which is realized for $\mathring{s}\in\mathbb{R}_+$, $\mathring{x}\in(\mathbb{R}_+)^n$ and $-\mathring{y}\in(\mathbb{R}_+)^m$. Moreover each differentiation gives a factor $\sum\omega(\vec{k}_j)$ which for integrability reasons requires that f is sufficiently regular at infinity (in particular if $f\in\mathscr{S}$, these terms are $C^\infty(\mathbb{R}_+)$). Let us look at the differentiation of the factors $swt_{j,\lambda}$. By the formula of the proposition § 2.3.6 we have to control a product of factors of the following type :

$$\lim_{t\to+0}\partial^\alpha_t\int\Big(\prod_{j\in I}\frac{d\mathring{k}_j}{\mathring{k}_j^2+\omega(\vec{k}_j)^2}\Big)\,\delta\Big(\sum_{j\in I}\mathring{k}_j\Big)\,e^{it(\sum_{j\in J}\mathring{k}_j)}\,e^{i(\sum_{j\in K}\mathring{x}_j\mathring{k}_j)}\,e^{-i(\sum_{j\in L}\mathring{y}_j\mathring{k}_j)}\,\partial^\nu_\lambda\Sigma^{|I|}_\lambda(\{k_j,j\in I\})$$

(multiplied by $\delta(\Sigma_{j\in I}\vec{k}_j)$) for some sets $I\subset\{1,\dots,n+m\}$, J, K, $L\subset I$ with $L = I - K$. Let us suppose that $\alpha < 3$. Because the derivatives of Σ^n_λ are bounded, the above expression is bounded by a constant times

$$\int\Big(\prod_{j\in I}\frac{d\mathring{k}_j}{\mathring{k}_j^2+\omega(\vec{k}_j)^2}\Big)\,\delta\Big(\sum_{j\in I}\mathring{k}_j\Big)\,\Big|\sum_{j\in J}\mathring{k}_j\Big|^\alpha$$

which gives the announced result. For $\alpha\geq 3$ we need more information on Σ^n_λ. Recall that it is a sum of functions $\mathscr{S}_{\lambda,p}$, which are truncated Schwinger functions evaluated at some sums of variables k_j (let us call them p_i). We iterate the WTI Programme using now the proposition 2.3.7 in order to find the decrease in those variables p_i. In the other variables we have only to study the following limit of derivatives of one-dimensional Feynmann integrals :

$$\lim_{t\to+0} \partial_t^\alpha \int \left(\prod_{j\in I} \frac{d\mathring{k}_j}{\mathring{k}_j^2 + \omega(\vec{k}_j)^2} \right) \delta\left(\mathring{p}_i - \sum_{j\in I} \mathring{k}_j\right) e^{it(\sum_{j\in J} \mathring{k}_j)} e^{i(\sum_{j\in K} \mathring{x}_j \mathring{k}_j)} e^{-i(\sum_{j\in L} \mathring{y}_j \mathring{k}_j)} .$$

The conditions on $\mathring{x}_i$ and on $\mathring{y}_j$ in 4) simplify only this last problem.

Remark. Continuing iterating the WTI Programme it is perhaps possible to prove the proposition for all $\alpha\in \mathbb{N}$, but with stronger conditions on f and g.

2.4.2 Euclidean vectors with precise Euclidean time. The operations of extension and restriction of the scalar products can also be performed for the Euclidean fields. As before we must distinguish the subset of $\mathscr{E}_\lambda$ generated by the constant random variables. Let P_o be its orthogonal projector.

Let $\lambda\in [0,\underline{\lambda}]$, $n,m\in \mathbb{N}^*$, $\mathring{x}\in \mathbb{R}^n$ and $\mathring{y}\in \mathbb{R}^m$ be fixed.

The field : $P_o{\cdot}\phi^n$ can be extended continuously from $\mathscr{S}((\mathbb{R}^2)^n)$ to $\mathscr{F}_n^0$ (this gives $F\mapsto\chi_F^0(\lambda)$ itself) and then restricted to the functions $(\mathbb{R}^2)^n\ni z\mapsto F(z) = f(\vec{z})\,\delta^n(\mathring{z} - \mathring{x})$ with $f\in \mathcal{B}_n^0$. This gives a map $\Phi_{n,\lambda}^0 : \mathcal{B}_n^0\times\mathbb{R}^n \to P_o\mathscr{E}_\lambda$ that we call the *Euclidean field with precise Euclidean time along the constants*. The scalar product involving $\Phi_{n,\lambda}^0$ is given by the formula :

$$(1 ; \Phi_{n,\lambda}^0(f,\mathring{x}))_{\mathscr{E}_\lambda} = \chi_F^0(\lambda).$$

The field : $(1-P_o){\cdot}\phi^n$ can be extended continuously from $\mathscr{S}((\mathbb{R}^2)^n)$ to $\mathscr{F}_{n,0}$ (by a standard analysis argument) and then restricted to the functions $(\mathbb{R}^2)^n\ni z\mapsto F(z) = f(\vec{z})\,\delta^n(\mathring{z} - \mathring{x})$ with $f\in \mathcal{B}_{n,o}$. This gives a map $\Phi_{n,\lambda} : \mathcal{B}_{n,o}\times \mathbb{R}^n \to (1-P_o)\mathscr{E}_\lambda$ that we call the *Euclidean field with precise Euclidean time*. The scalar product involving $\Phi_{n,\lambda}$ is given by the formula :

$$(\Phi_{n,\lambda}(f,\mathring{x}) ; \Phi_{m,\lambda}(g, \mathring{y}))_{\mathscr{E}_\lambda} = \chi_{F,G}(0,\lambda)$$

for all $(\mathbb{R}^2)^m\ni z\mapsto G(z) = g(\vec{z})\,\delta^m(\mathring{z} - \mathring{y})$ with $g\in \mathcal{B}_{m,o}$.

If $\mathring{x} = 0$, $\Phi_{n,\lambda}^0(f,0)$ and $\Phi_{n,\lambda}(f,0)$ are the *Euclidean zero-time vectors*. If $\lambda = 0$ they coincide with the corresponding vectors of §2.2 :

$$\Phi_{n,\lambda}^0(f,0) = P_o\theta^n(f) \text{ and } \Phi_{n,\lambda}(f,0) = (1-P_o)\,\theta^n(f).$$

For more details, see [8].

2.4.3 Zero-time vectors in the domain of the Hamiltonian. We translate now the results obtained in the Euclidean theory to the Minkowskian Hilbert space $\mathscr{H}_\lambda$.

We choose an Euclidean-time direction in $\mathbb{R}^2$ and construct an operator W_λ which passes from the Euclidean Hilbert space $\mathscr{E}_\lambda$ to the Minkowskian Hilbert space $\mathscr{H}_\lambda$, as explained in §2.1. The map

$$\Psi_{n,\lambda} = W_\lambda \circ \Phi_{n,\lambda} : \mathcal{B}_{n,o} \times (\mathbb{R}_+)^n \to (1-E_o)\mathscr{H}_\lambda$$

is called the *field with precise Euclidean time* (note the sign of the Euclidean time). Its restriction to the zero Euclidean time $\Psi_{n,\lambda}(.,0)$ is the *zero-time field*, and its values $\Psi_{n,\lambda}(f,0)$, $f \in \mathcal{B}_{n,o}$ are the *zero-time vectors*. Formally they should not be distinct from the zero-time Wightman fields applied on the vacuum state :

$$\Psi_{n,\lambda}(f_1 \otimes \cdots \otimes f_n, 0) = (1-E_o)\, \varphi_\lambda(f_1 \otimes \delta) \cdots \varphi_\lambda(f_n \otimes \delta)\, \Omega_\lambda = (1-E_o)\, \Theta\, {}^{n}_{\lambda}(f_1 \otimes \cdots \otimes f_n)\, \Omega_\lambda$$

where the last notation is that one of the zero-time vectors used in §1.

Let us denote by $D_\lambda(A_\lambda) \subset \mathscr{H}_\lambda$ the domain of any self-adjoint operator A_λ.

Theorem. *For all* $\lambda \in [0, \underline{\lambda}]$, $n, m \in \mathbb{N}^*$, $\mathring{x} \in (\mathbb{R}_+)^n$ *and* $\mathring{y} \in (\mathbb{R}_+)^m$:

Existence of vectors with precise Euclidean time in $D_\lambda(P_\lambda)$ *and* $D_\lambda(H_\lambda)$:

for all $\beta \in \mathbb{N}$: $\Psi_{n,\lambda}(f,\mathring{x}) \in D_\lambda(P_\lambda^\beta)$ *if* $P^\beta f \in \mathcal{B}_{n,o}$;

$\Psi_{n,\lambda}(f,\mathring{x}) \in D_\lambda(P_\lambda^\beta) \cap D_\lambda(H_\lambda)$ *if* $P^\beta f \in \mathcal{B}_{n,2}$;

if $\mathring{x}_i = s \geq 0$ *for all* $1 \leq i \leq n$: $\Psi_{n,\lambda}(f,\mathring{x}) \in D_\lambda(P_\lambda^\beta) \cap D_\lambda(H_\lambda^2)$ *if* $P^\beta f \in \mathcal{B}_{n,4}$.

Scalar products of these vectors :

for all β, β_1 $\beta_2 \in \mathbb{N}$ *with* $\beta_1 + \beta_2 = \beta$, $\alpha \in \{0,1,2,3\}$, $P^{\beta_1} f \in \mathcal{B}_{n,\alpha}$ *and* $P^{\beta_2} g \in \mathcal{B}_{m,\alpha}$,*the scalar product* $\left(\Psi_{n,\lambda}(g, -\mathring{y}) ; P^\beta H^\alpha \Psi_{n,\lambda}(f,\mathring{x}) \right)_{\mathscr{H}_\lambda}$ *is a* C^∞ *function of* λ ;

it and its derivatives are given by the formula, for all $\nu \in \mathbb{N}$:

$$\partial_\lambda^\nu \left(\Psi_{n,\lambda}(g, -\mathring{y}) ; P^\beta H^\alpha \Psi_{n,\lambda}(f,\mathring{x}) \right)_{\mathscr{H}_\lambda} = \partial_{\mathring{s}}^\alpha\, \partial_{\vec{s}}^\beta\, \partial_\lambda^\nu\, \chi_{F,G}(s,\lambda) \Big|_{\vec{s}=0,\ \mathring{s}=+0}$$

where $F : (\mathbb{R}^2)^n \ni z \mapsto f(\vec{z})\, \delta^n(\mathring{z} - \mathring{x})$ *and* $G : (\mathbb{R}^2)^m \ni z \mapsto g(\vec{z})\, \delta^m(\mathring{z} - \mathring{y})$

and there exists $K\in(0,\infty)$, *independent of* λ, $\mathring{x}$, $\mathring{y}$, f *and* g, *such that* :

$$\left|\partial_\lambda^\nu \left(\Psi_{n,\lambda}(g,-\mathring{y}) \,;\, P^\beta H^\alpha \Psi_{n,\lambda}(f,\mathring{x}) \right)_{\mathcal{H}_\lambda}\right| < K\, \mathfrak{b}_{n,\alpha}(P^{\beta_1}f)\, \mathfrak{b}_{m,\alpha}(P^{\beta_2}g) .$$

The above statement is also true for $\alpha = 4$ *if* $\mathring{x}_i=s$ *for all* $1\le i\le n$ *and* $\mathring{y}_j=t$ *for all* $1\le j\le m$, *for some* $s, t\in \mathbb{R}_+$.

The theorem also concerns the zero-time vectors : it suffices to take $\mathring{x}=0$ and $\mathring{y}=0$. Note that the projection of the zero-time vectors on the vacuum state is neglected. It can be treated as in the Euclidean case.

The proof of the theorem (see [8]) follows from the Proposition of §2.4.1 by standard analytic arguments.

Remark. By successive applications of the WTI Programme it is perhaps possible to verify the following conjecture : $\Psi_{n,\lambda}(f,\mathring{x}) \in D_\lambda(H_\lambda^\alpha)$ for all $\alpha\in\mathbb{N}$ if $\mathring{x}\in(\mathbb{R}_+)^n$ and $f\in\mathcal{S}(\mathbb{R}^n)$.

2.4.4 Beside the norms $\mathfrak{b}_{n,\alpha}$. Does the $\mathfrak{b}_{n,\alpha}$ be the optimal norms for which the theorem holds ? Let us look at the possible improvements in the definition of $\mathfrak{b}_{n,\alpha}$, §2.4.1. We have seen that the combination of δ pseudo-functions arises from the invariance of the truncated functions under the space translations, so they can not be avoided. The $(\Sigma\omega_i)^\alpha$-factors are also necessary, because they are those which enter in the case where $\lambda=0$ (see §2.2). Nevertheless we will see two possible improvements.

The first one is concerned with the special case where the interaction polynomial is even. In that case the Wick-Schwinger truncated functions $swt_{n,\lambda}$ with odd n vanish all identically. Then we can replace $\mathfrak{b}_{n,\alpha}$ by the same expressions, but involving only even partitions, that is partitions $\{I_1,\dots,I_k\}$ where all $|I_j|$, $1\le j\le k$ are even numbers.

The second improvement can be useful when the test functions f depend on λ in a singular way, as we have seen in §1. The δ pseudo-functions in $\mathfrak{b}_{n,\alpha}$ come all from a factor $swt_{n,\lambda}$. Due to the nice properties of these functions and because they vanish when $\lambda\to0$ we can put a λ-factor in front of each one without destroying any properties necessary for the theorem. Then we can replace the $\mathfrak{b}_{n,\alpha}$ by the same expressions, but with a λ-factor in front of all δ pseudo-functions.

2.5 "Almost density" of the zero-time vectors

We try now to answer the third question of the beginning of this § 2, about the density of the zero-time vectors. We will only see the weaker property that all vectors can be approached by an asymptotic series of zero-time vectors.

To be more precise, let $\mathscr{D}_\lambda$ be the span of $\left\{\Omega_\lambda, \Psi_{n,\lambda}(f,\mathring{x}), n\in \mathbb{N}^*, f\in \mathcal{B}_{n,o}, \mathring{x}\in (\mathbb{R}_+)^n\right\}$, the set of vectors with *precise Euclidean times*. $\mathscr{D}_\lambda$ is clearly a dense subspace of $\mathscr{H}_\lambda$. The span of $\left\{\Omega_\lambda, \Psi_{n,\lambda}(f,0), n\in \mathbb{N}^*, f\in \mathcal{B}_{n,o}\right\}$, denoted by $\mathscr{D}_{0,\lambda}$, is the set of *zero-time vectors*. Then the following statement holds.

Theorem. *For all* $\lambda\in [0,\Lambda]$ *and* $\vartheta = \Psi_{n,\lambda}(f,\mathring{x}) \in \mathscr{D}_\lambda$ *there exists a sequence of* $\mathscr{D}_{0,\lambda}$: $\{\zeta_i, i\in \mathbb{N}\}$ *such that, for all* $N\in \mathbb{N}$:

$$\left\|\vartheta - \sum_{i=0}^{N} \lambda^i \zeta_i\right\|_{\mathscr{H}_\lambda} < \lambda^{N+1} K\, \mathfrak{b}_{n,o}(f)$$

for some $K\in (0,\infty)$ *independent of* λ, $f\in \mathcal{B}_{n,o}$ *and* $\mathring{x}\in (\mathbb{R}_+)^n$.

Before presenting the proof of the theorem (which constructs the vectors ζ_i) let us discuss its result. Nothing is said about the growth of K when N increases. Thus nothing is known about the possible convergence or resummability of the series. So the theorem does not imply the density of $\mathscr{D}_{0,\lambda}$ but only to the weaker statement, that for all perturbation calculations, $\mathscr{D}_\lambda$ and $\mathscr{D}_{0,\lambda}$ are undistinguishable. We call this property the "almost-density of $\mathscr{D}_{0,\lambda}$".

In the theorem we must suppose that f and $\mathring{x}$ do not depend too wildly on λ. Recall that $\mathscr{H}_\lambda$ itself depends on λ (via the construction of the measure μ_λ) and $\vartheta, \zeta_i \in \mathscr{H}_\lambda$. So λ plays a double role : it indicates which Hilbert space the vectors belong to, and it is a small parameter which allows perturbation expansion. To get a better understanding of the situation, consider the fibre bundle with base $[0,\Lambda]$ and fibre $\mathscr{H}_\lambda$ (see figure 2), and let $\mathscr{S}$ be the set of the cross sections. An element $f\in \mathscr{S}$ is given by a family of vectors $\{f_\lambda\in \mathscr{H}_\lambda, \lambda\in [0,\Lambda]\}$. The zero-time vectors generate a subspace $\mathscr{S}_0 \subset \mathscr{S}$ which satisfies according to the theorem :

for all $f\in \mathscr{S}$ and $N\in \mathbb{N}$ there exist $g\in \mathscr{S}_0$ and $K\in (0,\infty)$ such that $\|f_\lambda - g_\lambda\|_{\mathscr{H}_\lambda} < K\lambda^N$.

It would be dense in $\mathscr{S}$ provided the topology would be generated by the following set of neighborhoods of any $f\in \mathscr{S}$:

$$\mathscr{V}_N = \left\{ g \in \mathscr{T},\ \| f_\lambda - g_\lambda \|_{\mathscr{H}_\lambda} < K\,\lambda^N \text{ for some } K \in (0,\infty) \right\}$$

for all $N \in \mathbf{N}$, which is without interest for us because this topology is not fine enough to allow calculations by any resummation procedure.

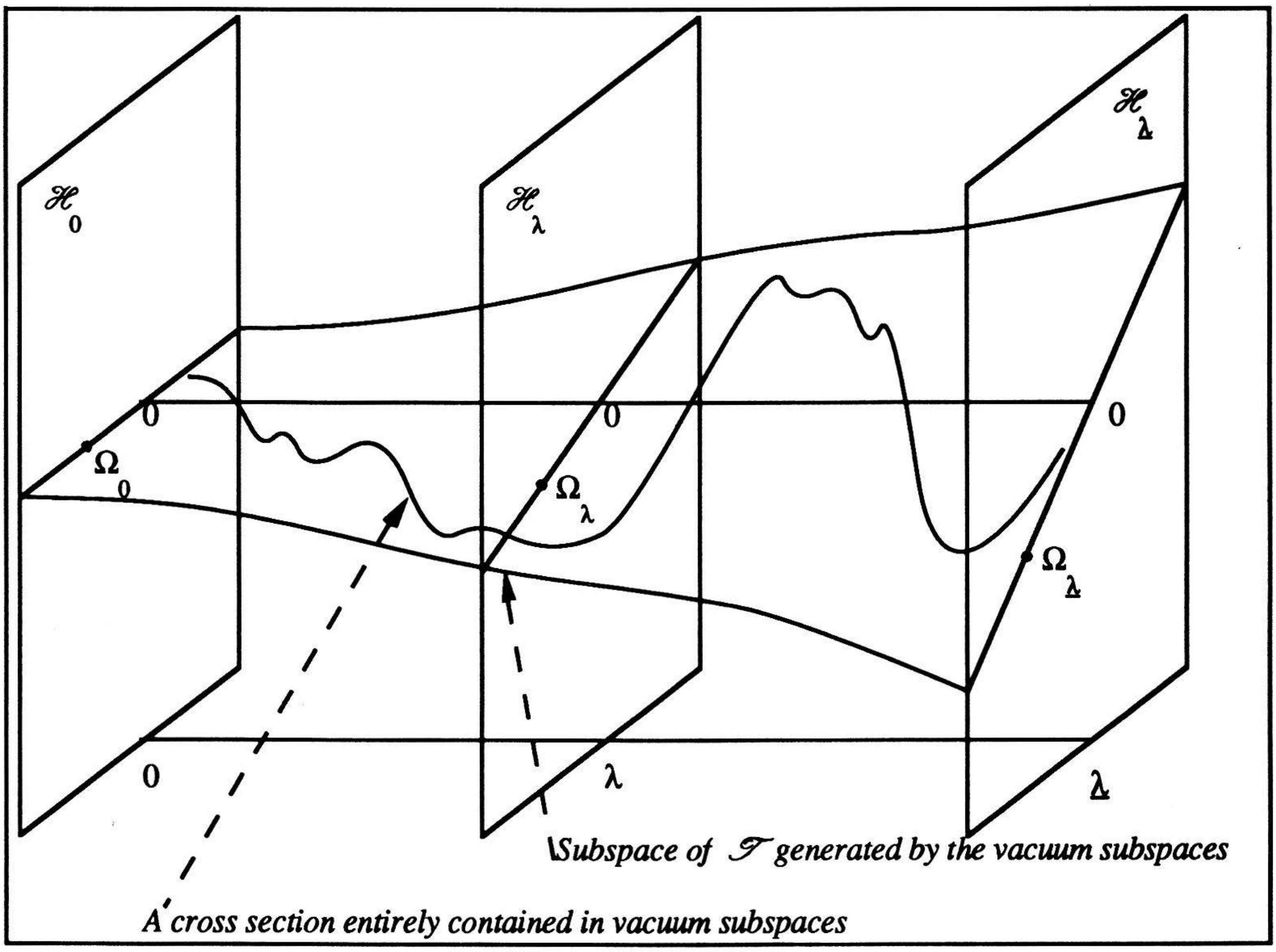


Figure 2

The proof of the theorem (see [9]) starts by supposing that there exists a perturbation expansion :

$$\vartheta = \sum_{i \geq 0} \lambda^i \zeta_i$$

for all $\vartheta \in \mathscr{D}_\lambda$ and with each $\zeta_i \in \mathscr{D}_{0,\lambda}$. Then for all $\xi \in \mathscr{D}_{0,\lambda}$:

$$0 = (\xi ; \vartheta - \Sigma \lambda^i \zeta_i) = \sum_{k \geq 0} \lambda^k \left[(\xi ; \vartheta)_k - \sum_{i=0}^{k} (\xi ; \zeta_i)_{k-i} \right]$$

where we have expanded the scalar products, with the notation $(A,B) = \Sigma \lambda^\ell (A,B)_\ell$. Note that for fixed k, i takes only a finite number of values. Thus for all $k \in \mathbf{N}$ we must have :

$$(\xi ; \zeta_k)_0 = (\xi ; \vartheta)_k - \sum_{i=0}^{k-1} (\xi ; \zeta_i)_{k-i} \ .$$

Now we take $\vartheta=\Psi_{n,\lambda}(f,\mathring{x})$ and write ζ_i as $\Sigma_{j\geq 1}\Psi_{j,\lambda}(A^{n,i,j}_{f,\mathring{x}},0)$, for some unknown functions $A^{n,i,j}_{f,\mathring{x}}$. If we take $\xi=\Psi_{r,\lambda}(g,0)$, the l.h.s. of the previous formula is easily calculated (estimation from the free theory), and is $(\xi;\Psi_{r,\lambda}(A^{n,k,r}_{f,\mathring{x}},0))|_{\lambda=0}$. So the above formula gives $A^{n,k,r}$ in terms of the sets $\{A^{n,i,j}, j\in \mathbb{N}\}$ with i<k. This leads to the following formal definition of $A^{n,i,j}_{f,\mathring{x}}$.

Definition. For all $n\in \mathbb{N}^*$, $f\in \mathcal{B}_{n,o}$ and $\mathring{x}\in (R_+)^n$ let $A^{n,i,j}_{f,\mathring{x}}$ be the functions given, for all $i\in \mathbb{N}$, $j\in \mathbb{N}^*$ and $\vec{p}\in \mathbb{R}^j$, by :

$$\tilde{A}^{n,i,j}_{f,\mathring{x}}(\vec{p}) = \frac{2^j}{j!}\tilde{F}(\vec{p}) \prod_{k=1}^{j} \omega(\vec{p}_k) \quad \text{where F is given, for all } \vec{y}\in \mathbb{R}^j, \text{ by :}$$

$$F(\vec{y}) = \partial^i_\lambda \ \frac{1}{i!} \int d^n\vec{x} \ f(\vec{x}) \ sp_{j,n,\lambda}((\vec{y},0);x)\Big|_{\lambda=0} \ -$$

$$- \sum_{k=0}^{i-1} \partial_\lambda^{i-k} \frac{1}{(i-k)!} \sum_{r\geq 1} \int d^r\vec{x} \ A^{n,k,r}_{f,\mathring{x}}(\vec{x}) \ sp_{j,r,\lambda}((\vec{y},0) \ ; \ (\vec{x},0))\Big|_{\lambda=0}$$

(for i=0, the term containing the sum over k does not appear).

The definition uses the functions $sp_{j,r,\lambda}$ instead of the $s_{j,r,\lambda}$, introduced in §2.4.1, which would appear naturally according to the previous considerations. $sp_{j,r,\lambda}$ are called the *partially connected Schwinger functions* and are defined as follows. We write $s_{j,r,\lambda}$ as a sum of products over the partitions of $\{1,\ldots,j+r\}$ of some truncated functions (which are defined by this operation, by the lemma of §2.3.2). Let us denote by $J=\{1,\ldots,j\}$ and $R=\{j+1,\ldots,j+r\}$. $sp_{j,r,\lambda}$ is obtained by summing only on the partitions of $J\cup R$ which connect all element of J to R. This connectedness property is necessary to avoid δ pseudo-functions in $\tilde{A}^{n,i,j}_{f,\mathring{x}}(\vec{p})$, and for the following results.

Lemma. *For all* n, $i\in \mathbb{N}$ *and* $j\in \mathbb{N}^*$, $A^{n,i,j}: \mathcal{B}_{n,o}\times(\mathbb{R}_+)^n\rightarrow\mathcal{B}_{j,o}$. *Moreover,* $b_{j,o}(A^{n,i,j}_{f,\mathring{x}}) < K\, b_{n,o}(f)$ *for some* $K\in (0,\infty)$ *independent of* $f\in \mathcal{B}_{n,o}$ *and* $\mathring{x}\in (\mathbb{R}_+)^n$.

This result is not trivial : it says that some Schwinger distributions can be evaluated at the functions $A^{n,i,j}_{f,\mathring{x}}$, which are themselves combinations of Schwinger functions partially evaluated at f.

To complete the proof of the theorem only operations on the terms of the perturbation series are needed. A stronger version of the theorem (unpublished) states that for $\vartheta\in D_\lambda(H_\lambda)$ (that is $f\in \mathcal{B}_{n,2}$) the functions $A^{n,i,j}_{f,\mathring{x}}$ belong to $\mathcal{B}_{j,2}$, that is all the ζ_i are in $D_\lambda(H_\lambda)$. So the intersection of the subspaces $\mathcal{D}_{0,\lambda}\cap D_\lambda(H_\lambda)$, which we really needed in §1, is also "almost-dense".

2.6 Zero-time one-particle states projection

We try now to answer to the second question of the beginning of this § 2 concerning the orthogonalization with respect to the vacuum and one-particle state subspaces. We have seen in §2.2 that in the free models the span of $\{W_0\theta^n(g),\ n\geq 2,\ g\in \mathcal{K}_n\}$ have this property. This is no more true in the interacting models. The orthogonalization with respect to the vacuum state causes no problem because this subspace is generated by the unique vector Ω_λ. The difficulty in $(1-E_0-E_m)$ comes from the projector E_m. As in §2.5, we will not really obtain it, but only approach $E_m\xi$, for any vector $\xi\in \mathscr{D}_{0,\lambda}$, by an asymptotic series of zero-time vectors.

To get a better approach of the spectrum of the mass operator we begin to estimate the action of the resolvent operator of M on the zero-time vectors, by approaching $(M^2\text{-}z)^{-1}\xi$ for $\xi\in\mathscr{D}_{0,\lambda}$ and suitable $z\in\mathbb{C}$ by an asymptotic series in $\mathscr{D}_{0,\lambda}$.

Let us introduce some notation: for a self-adjoint operator A, d(A,z) is the smallest distance in $\mathbb{C}$ between z and the spectrum of A. We introduce on $\mathfrak{B}_{n,m+1}$ with $n,m\in\mathbb{N}^*$, the norms $f\mapsto \mathfrak{d}_z^{n,m}(f)$ given by

$$\mathfrak{d}_z^{n,m}(f) = \left(1 + \left(\frac{R}{d(M_0^2,z)}\right)^{m+1}\right)\mathfrak{b}_{n,o}(f) + \left(\frac{1}{d(M_0^2,z)}\right)^{m+1} \mathfrak{b}_{n,o}(P^{2m+2}f)$$

where R is some fixed arbitrary number such that $R>>m_o$. We denote by $\mathfrak{O}$ the following open subset of $\mathbb{C}$:

$$\mathfrak{O} = \left\{ z\in\mathbb{C},\ |z|<R,\ d(M_\lambda^2,z)>0,\ d(M_0^2,z)>0\right\}.$$

Theorem. *For all* $\lambda\in[0,\Lambda]$, $N\in\mathbb{N}$, $z\in\mathfrak{O}$ *and* $\vartheta=\Psi_{n,\lambda}(f,0)\in\mathscr{D}_{0,\lambda}$ *with* $\mathfrak{d}_z^{n,N+1}(f)<\infty$ *there exists a finite sequence of* $\mathscr{D}_{0,\lambda}$: $\{\zeta_{z,i},\ 0\leq i\leq N\}$ *such that :*

$$\left\|\frac{1}{M_\lambda^2-z}\vartheta - \sum_{i=0}^{N}\lambda^i\,\zeta_{z,i}\right\|_{\mathscr{H}_\lambda} < \lambda^{N+1} K\,\frac{\mathfrak{d}_z^{n,N+1}(f)}{d(M_\lambda^2,z)}$$

for some $K\in(0,\infty)$ *independent of* λ, f *and* z.

Note that the conditions on f depend on the order N ; but this concerns only the dependence of $f(\vec{x}_1,\cdots,\vec{x}_n)$ in the variable $\vec{x}_1+\cdots+\vec{x}_n$.

To construct the vectors $\zeta_{z,i}$ for $\vartheta=\Psi_{n,\lambda}(f,0)\in\mathscr{D}_{0,\lambda}$, $\vartheta\neq 0$, and suitable $z\in\mathbb{C}$, let us suppose that there exists a perturbation expansion such that :

$$\vartheta = (M_\lambda^2-z)\sum_{i\geq 0}\lambda^i\,\zeta_{z,i}$$

where all $\zeta_{z,i} \in \mathscr{D}_{0,\lambda}$. (Note that $\zeta_{z,i}$ must be in the domain of M^2). Then we write $\zeta_{z,i}$ as $\Sigma_{j\geq 1}\Psi_{j,\lambda}(B^{n,i,j}_{f,z},0)$ for some functions $B^{n,i,j}_{f,z}$. We follow now the same way as in § 2.5, and obtain analogous results (see [9]). We find that $B^{n,i,j} : \mathbf{B}_{n,j+1}\times\mathcal{O}\to\mathbf{B}_{i,4}$ and that $(z,\vec{p}) \mapsto \tilde{B}^{n,i,j}_{f,z}(\vec{p})$ is continuous on $\mathcal{O}\times\mathbb{R}^i$.

We use now the information on the mass operator M given in the beginning of this paper, §1. For λ sufficiently small, a circle $\mathscr{C}$ in $\mathbb{C}$ can be drawn, with center m^2, such that $d(M_0^2,z)>\frac{1}{4}m_0^2$ and $d(M_\lambda^2,z)>\frac{1}{4}m_0^2$ for all $z\in\mathscr{C}$(see the figure 3). Let us call $\underline{\underline{\lambda}}$ the maximal value in $(0,\underline{\lambda}]$ for which these conditions hold for all $\lambda\in[0,\underline{\underline{\lambda}}]$. We fix now $\lambda\in[0,\underline{\underline{\lambda}}]$. The projector E_m can now be written as

$$E_m = -\frac{1}{2\pi i}\int_{\mathscr{C}} dz \; \frac{1}{M_\lambda^2 - z} \; .$$

Because of the nice properties of the functions $B^{n,i,j}_{f,z}$ stated above, the new functions $D^{n,i,j}_f$ given by

$$\tilde{D}^{n,i,j}_f(\vec{p}) = -\frac{1}{2\pi i}\int_{\mathscr{C}} dz \; \tilde{B}^{n,i,j}_{f,z}(\vec{p})$$

for all $\vec{p}\in\mathbb{R}^j$, belong to $\mathbf{B}_{i,4}$, and then the vectors $\zeta_i = \Sigma_{j\geq 1}\Psi_{j,\lambda}(D^{n,i,j}_f,0)$ lie in the domain of M^2 and have the property stated in the following theorem.

Theorem. *For all* $\lambda\in[0,\underline{\underline{\lambda}}]$, $N\in\mathbb{N}$ *and* $\vartheta = \Psi_{n,\lambda}(f,0) \in \mathscr{D}_{0,\lambda}$ *with* $\mathbf{b}_{n,o}(f)$ *and* $\mathbf{b}_{n,o}(P^{2N+4}f) < \infty$ *there exists a finite sequence of* $\mathscr{D}_{0,\lambda}$: $\{\zeta_i \,,\, 0\leq i\leq N\}$ *such that :*

$$\left\| E_m\,\vartheta - \sum_{i=0}^{N}\lambda^i\,\zeta_i \right\|_{\mathscr{H}_\lambda} < \lambda^{N+1}\left(K\,\mathbf{b}_{n,o}(f) + K'\,\mathbf{b}_{n,o}(P^{2N+4}f)\right)$$

for some K, K'$\in(0,\infty)$ *independent of* λ *and* f.

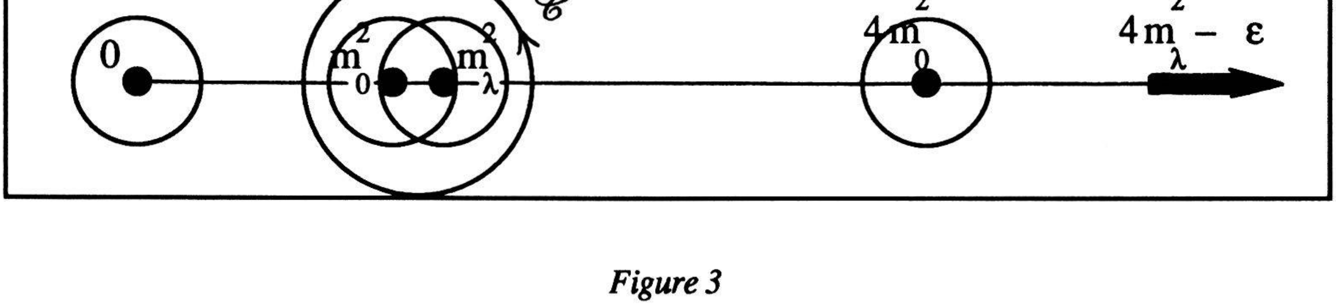


Figure 3

3. Zero–time representation of the Poincaré group

Here we try to construct an intermediate theory between the Q.F.T. and the Q.M. adapted to the two–particle system at low energy. More precisely we go back to the theorem of §1 and to the details of its proof, and we stop the calculation at an intermediate level, before doing the expansion in the δ parameter (in order to avoid the non-relativistic limit). Let us denote by $\Psi(f)$ the vector obtained by minimizing the Rayleigh quotient in varying the functions f_j^1, $j \in \mathbb{N}$, but not $f = f_2^0$. We change the representation of the relevant Hilbert space, introducing for each f a new function F (on $\mathbb{R}^2$) such that :

$$\| \Psi(f) \|_{\mathcal{H}}^2 = \int_{\mathbb{R}^2} d\eta^2(\vec{k}) \, | F(\vec{k}) |^2$$

where $d\eta^2(\vec{k}) = \prod_{i \in \{1,2\}} \dfrac{d\vec{k}_i}{2\omega(\vec{k}_i)}$, and ω is now the function on $\mathbb{R}$: $p \mapsto (p^2 + m^2)^{1/2}$. Note that here m is the one–particle mass of the model with interaction (it depends on λ).

We can compute the first perturbation orders of $(\Psi(f), \Theta\Psi(f))_{\mathcal{H}}$ for $\Theta = H^\alpha$, or P^β, or L, which give bilinear forms for the F's. It is then possible to define operators which generate these forms. We obtain in this way a new quantum and relativistic model, which is simple but not completely satisfactory because it comes from the first orders of a perturbation calculation, so that the Lorentz invariance is realized only at $O(\lambda^2)$. Nevertheless this model is interesting for many reasons. It is simpler than the $\mathcal{P}(\varphi)_2$ models and well adapted to the two-particle problem; moreover it has the same mass spectrum (in the neighborhood of 2m and at first perturbation orders) and it admits the same non-relativistic limit.

A Quantum Relativistic model is a representation in a Hilbert space of the Poincaré group or of its Lie algebra, which for a two-dimensional space-time is generated by three operators : P (momentum), H (Hamiltonian) and L (Lorentz generator), satisfying the commutation rules :

$$\begin{aligned} [P, H] &= 0 \\ [P, L] &= iH \\ [H, L] &= iP \end{aligned}$$

The last proposition of §2.2 gives an example of such representation, the "representation for n free particles", which has the interesting property that it works at zero-time. We want now to introduce interaction terms in this representation, deduced from the $\mathcal{P}(\varphi)_2$ model in the way explained just before. For the two-particle system at zero-time we obtain the following representation.

$\mathscr{H} = L^2(\mathbb{R}^2,\eta^2)$. On $\mathscr{H}$ we define (formally) the operators P, H, L by

$$P\,f(\vec{k}) = (\vec{k}_1+\vec{k}_2)\,f(\vec{k})$$

$$H\,f(\vec{k}) = \Omega(\vec{k})\,f(\vec{k}) + \lambda\frac{4!\,a_4}{2\pi}\int d\eta^2(\vec{k}')\,f(\vec{k}')\,\delta(\vec{k}'_1+\vec{k}'_2-\vec{k}_1-\vec{k}_2)$$

$$L\,f(\vec{k}) = L_o\,f(\vec{k}) + \lambda\frac{4!\,a_4}{2\pi}\,L_o\int d\eta^2(\vec{k}')\,f(\vec{k}')\,\frac{\delta(\vec{k}'_1+\vec{k}'_2-\vec{k}_1-\vec{k}_2)}{\Omega(\vec{k})+\Omega(\vec{k}')} + $$
$$+ \lambda\frac{4!\,a_4}{2\pi}\int d\eta^2(\vec{k}')\,(L_o f)(\vec{k}')\,\frac{\delta(\vec{k}'_1+\vec{k}'_2-\vec{k}_1-\vec{k}_2)}{\Omega(\vec{k})+\Omega(\vec{k}')}$$

where $\Omega(\vec{k}) = \omega(\vec{k}_1) + \omega(\vec{k}_2)$ and L_o is as in the last proposition of §2.2, with now m_o replaced by m. For $\lambda=0$ we obtain the representation for 2 free particles. These operators are well defined and symmetric on the domain $\mathscr{S}(\mathbb{R}^2)$. Their self–adjointness will be shown in another paper [16].

We can now reverse the situation, forget the $\mathcal{P}(\varphi)_2$ models and consider the above expressions for $\mathscr{H}$, H, P, L as definitions (we also forget the restriction to weak relative energy). To simplify the notation we introduce the operator $\mathcal{O}$ on $\mathscr{H}$ defined by

$$\mathcal{O}f(\vec{k}) = \frac{4!\,a_4}{2\pi}\int d\eta^2(\vec{k}')\,f(\vec{k}')\,\frac{\delta(\vec{k}'_1+\vec{k}'_2-\vec{k}_1-\vec{k}_2)}{\Omega(\vec{k})+\Omega(\vec{k}')}$$

$\mathcal{O}$ is a bounded operator [16]. We also denote by Ω the multiplication operator by the function $\Omega(\vec{k})$. Then H and L can be written as

$$H = \Omega + \lambda\,(\Omega\,\mathcal{O} + \mathcal{O}\,\Omega) = \Omega + \lambda\,\{\Omega\,,\mathcal{O}\}$$
$$L = L_o + \lambda\,(L_o\,\mathcal{O} + \mathcal{O}\,L_o) = L_o + \lambda\,\{L_o\,,\mathcal{O}\}$$

where $\{A,B\} = AB + BA$. Then P, H, L satisfy the following commutation relations :

Proposition. On $\mathscr{S}(\mathbb{R}^2)$ the operator H, P and L satisfy :

$$[P,H] = 0$$
$$[P,L] = iH$$
$$[H,L] = iP + \lambda^2\,[\,\{\Omega,\mathcal{O}\},\{L_o,\mathcal{O}\}\,]$$

The proposition shows that H, P and L give "almost" a presentation of the Poincaré group. The third relation states that the term proportional to λ in [H, L] vanishes identically. If we want that the term proportional to λ^2 vanishes also, we must add to H and L new appropriate terms multiplied by λ^2 (see [16]).

The proof of the proposition will give the answer to another question : how can we modify the interaction terms in H and L so that the commutation relations of the proposition hold again (see the remark after the proof).

Proof. Because of the δ pseudo-function in the definition of $\mathcal{O}$ we have $[\mathcal{O}, P] = 0$. Thus $[P, H] = [P, \Omega] + \lambda\{[P, \Omega], \mathcal{O}\}$. Because two multiplication operators commute we have $[P, \Omega] = 0$, and then $[P, H] = 0$. We also have : $[P, L] = [P, L_o] + \lambda\{[P, L_o], \mathcal{O}\}$. But $[P, L_o] = i\,\Omega$, so: $[P, L] = i\,\Omega + \lambda\{\Omega, \mathcal{O}\} = i\,H$. Now :

$$[H, L] = [\Omega, L_o] + \lambda([\,\Omega, \{L_o, \mathcal{O}\}] + [\{\,\Omega, \mathcal{O}\}, L_o]) + \lambda^2[\{\Omega, \mathcal{O}\},\{L_o, \mathcal{O}\}].$$

Using $[\Omega, L_o] = i\,P$, we obtain :

$$[H, L] = i\,P + 2\,\lambda\,(i\,P\mathcal{O} + \Omega\mathcal{O}L_o - L_o\mathcal{O}\Omega) + \lambda^2[\{\Omega, \mathcal{O}\},\{L_o, \mathcal{O}\}].$$

Until now we have not used the explicit form of the Kernel of $\mathcal{O}$ (except from the presence of the δ pseudo-function). The proposition is proved if for all $f \in \mathcal{S}(\mathbb{R}^2)$

$$\begin{aligned}\frac{2\pi}{4!\,a_4}\,(i\,P\mathcal{O} + \Omega\mathcal{O}L_o - L_o\mathcal{O}\Omega)\,f(\vec{k}) \;=\;& i\left(\vec{k}_1+\vec{k}_2\right)\int d\eta^2(\vec{k}')\,f(\vec{k}')\,\frac{\delta\left(\vec{k}'_1+\vec{k}'_2-\vec{k}_1-\vec{k}_2\right)}{\Omega(\vec{k}) + \Omega(\vec{k}')}\\ &+\;\Omega(\vec{k})\int d\eta^2(\vec{k}')\,(L_o f)(\vec{k}')\,\frac{\delta\left(\vec{k}'_1+\vec{k}'_2-\vec{k}_1-\vec{k}_2\right)}{\Omega(\vec{k}) + \Omega(\vec{k}')}\\ &-\;L_o\int d\eta^2(\vec{k}')\,\Omega(\vec{k}')\,f(\vec{k}')\,\frac{\delta\left(\vec{k}'_1+\vec{k}'_2-\vec{k}_1-\vec{k}_2\right)}{\Omega(\vec{k}) + \Omega(\vec{k}')}\end{aligned}$$

vanishes. Let us denote by (∗) the r.h.s. of this expression.

We make the change of variables $(\vec{k}_1, \vec{k}_2) \to (\alpha, \chi)$ given by :

$$\begin{aligned}\vec{k}_1 &= m\left(ch\,\chi\;sh\,\alpha + sh\,\chi\;ch\,\alpha\right)\\ \vec{k}_2 &= m\left(ch\,\chi\;sh\,\alpha - sh\,\chi\;ch\,\alpha\right)\end{aligned}$$

(note that $L_o(\vec{k}) = -i\,\partial_\alpha$), and the change : $(\vec{k}'_1, \vec{k}'_2) \to (\alpha', \chi')$ given by the same formulas. Note that $d\eta^2(\vec{k}') = \frac{1}{2}\,d\alpha' d\chi'$ and

$$\delta\left(\vec{k}'_1+\vec{k}'_2-\vec{k}_1-\vec{k}_2\right) = \frac{\delta(\alpha' - \bar{\alpha})}{\Omega'}$$

where $\Omega' = \Omega(\vec{k}')$ and $\bar{\alpha} = arg\ sh \dfrac{ch\,\chi\ sh\,\alpha}{ch\,\chi'}$ (see [3, Appendix I] for other formulas). Then (*) becomes :

$$(*) = i\left(\vec{k}_1+\vec{k}_2\right) \int \tfrac{1}{2} d\chi' \frac{F(\bar{\alpha},\chi')}{\Omega'(\Omega+\Omega')} - i\Omega \int \tfrac{1}{2} d\chi' \frac{\partial_{\bar{\alpha}} F(\bar{\alpha},\chi')}{\Omega'(\Omega+\Omega')}$$

$$+ i\partial_\alpha \int \tfrac{1}{2} d\chi' \frac{F(\bar{\alpha},\chi')}{\Omega+\Omega'}$$

where $F(\alpha,\chi) = f(\vec{k})$ and $\Omega = \Omega(\vec{k})$. With : $\partial_\alpha = \dfrac{\partial_{\bar{\alpha}}}{\partial_\alpha}\ \partial_{\bar{\alpha}} = \dfrac{\Omega}{\Omega'}\ \partial_{\bar{\alpha}}$ we obtain :

$$(*) = \frac{i}{2} \int d\chi'\ F(\bar{\alpha},\chi') \left(\partial_\alpha \frac{1}{\Omega+\Omega'} + \frac{\vec{k}_1+\vec{k}_2}{\Omega'(\Omega+\Omega')}\right)$$

But $\partial_\alpha \dfrac{1}{\Omega+\Omega'} = -\dfrac{\vec{k}_1+\vec{k}_2}{\Omega'(\Omega+\Omega')}$, so $(*) = 0$. ◊

Remark Let us replace in the kernel of $\mathcal{O}$ the factor $\frac{1}{\Omega+\Omega'}$ by a function $\xi(\alpha,\chi,\chi')$. From the proof it follows that the commutation relations of the proposition hold again provided that :

$$\partial_\alpha \xi(\alpha,\chi,\chi') = -\frac{\vec{k}_1+\vec{k}_2}{\Omega'}\ \xi(\alpha,\chi,\chi')$$

This differential equation is easily solved and gives : $\xi(\alpha,\chi,\chi') = \dfrac{C(\chi,\chi')}{\Omega+\Omega'}$ for an arbitrary function C. Then the proposition is true even if we multiply the kernel of $\mathcal{O}$ by an arbitrary function of χ and χ'.

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