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A New Asymptotic Condition for Absolutely Continuous Spectrum of the Sturm–Liouville Operator on the Half–Line

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Abstract. A new asymptotic condition $\lim_{N\to\infty} \int_0^N (f(x,\lambda))^2 dx / \int_0^N |f(x,\lambda)|^2 dx = 0$ is proposed to describe solutions $f(x,\lambda)$ at real spectral parameter λ of the Sturm–Liouville differential equation $-\frac{d^2 f(x,\lambda)}{dx^2} + V(x)f(x,\lambda) = \lambda f(x,\lambda)$ over the half–line $0 \leq x < \infty$. The asymptotic condition is shown to imply absolute continuity of the Schrödinger operator $T = -\frac{d^2}{dx^2} + V(x)$, and starting from this condition a wide variety of consequences for spectral theory and asymptotics of solutions is developed, including the precise determination of the spectral density function.
Applications will include a range of problems in classical and quantum theory, including the detailed analysis of continuous spectra in potential scattering.

1 Introduction

It has been commonly realised for many years by physicists and mathematicians that there is a close connection between the spectral properties of the Sturm–Liouville differential operator $T = -\frac{d^2}{dx^2} + V(x)$ acting on the half–line $0 \leq x < \infty$, with regular boundary condition at $x = 0$ and limit-point case at $x = \infty$, and the large x asymptotic behaviour of solutions $f(x,\lambda)$ of the corresponding real λ differential equation

$$-\frac{d^2f(x,\lambda)}{dx^2} + V(x)\, f(x,\lambda) = \lambda f(x,\lambda)\,.$$

Here $V(x)$ is a given real potential function, which we assume locally integrable, and $\lambda \in \mathbb{R}$ is the spectral parameter. Typically the differential operator T might be defined subject to Dirichlet or Neumann conditions at $x = 0$, but other more general boundary conditions are also used. The limit-point condition at infinity (see [1]), which will hold in particular if $V(x)$ is bounded at large distances but also much more generally, implies that no boundary condition is necessary at infinity in defining T.

Although the link between spectral properties and large x asymptotic behaviour of solutions is well known, it has not always been so clearly understood what precise mathematical form this link was to take. For example, it has been conjectured that the absolutely continuous spectrum corresponded precisely to such $\lambda \in \mathbb{R}$ for which solutions of the differential equation were bounded at large distances. Though this statement of the conjecture can be refuted by counterexamples, under certain restrictions the existence of bounded solutions can still play an important role.

The most clear-cut and decisive criterion able to distinguish between the various types of spectral behaviour for the general Sturm-Liouville operator depends on the notion of <u>**subordinacy**</u>. A solution $f(x,\lambda)$ is said to be subordinate, for given $\lambda \in \mathbb{R}$, if that solution is asymptotically smaller, in L^2 norm, than all other solutions $g(x,\lambda)$ for the same value of λ, apart from constant multiples of f. More precisely, f is said to be subordinate provided

$$\lim_{N\to\infty} \frac{\int_0^N (f(x,\lambda))^2 dx}{\int_0^N (g(x,\lambda))^2 dx} = 0$$ for any solution g which is not a constant multiple of f.

The use of subordinacy in spectral analysis leads to a characterisation of each component of the spectrum in terms of subsets of $\mathbb{R}$ on which subordinate solutions exist. For precise statements of these results, and proofs, see [2],[3],[4]; for two recent applications see [5],[6].

Applying the ideas to the continuous spectrum, it follows, for example, that the support of the absolutely continuous spectrum is located precisely on the <u>complement</u> of those λ for which there are subordinate solutions. For definitions and examples of absolutely continuous and other types of spectrum, see for example [4].

The purpose of the present paper will be to establish a criterion which is analogous to that of subordinacy, but will apply to λ in the absolutely continuous spectrum, and which moreover will allow a complete and detailed analysis of the nature of that spectrum, including the determination of the corresponding spectral density.

Spectral properties of the Sturm-Liouville operator are controlled by the boundary behaviour of the Weyl m-function m(z) as the complex variable z approaches the real axis from the upper half-plane. (See Section 2 for definition and properties). The differential operator T is unitarily equivalent to the multiplication operator $h(\lambda) \to \lambda h(\lambda)$ in the Hilbert space $L^2(\mathbb{R}, d\rho(\lambda))$, where the spectral function $\rho(\lambda)$ is monotonic non-decreasing and defined up to an additive constant by

$$\rho(\lambda_2) - \rho(\lambda_1) = \lim_{\epsilon \to 0+} \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \mathrm{Im}\, m(\lambda + i\epsilon)\, d\lambda ,$$

for any λ_1, λ_2 which are not discrete points of the measure. At points of absolute continuity, the spectral measure takes the form $d\rho = \frac{d\rho}{d\lambda} d\lambda$, where the spectral density $\frac{d\rho(\lambda)}{d\lambda}$ is given almost everywhere by

$$\frac{d\rho(\lambda)}{d\lambda} = \lim_{\epsilon \to 0+} \frac{1}{\pi} \mathrm{Im}\, m(\lambda + i\epsilon) .$$

In quantum mechanical applications to the Schrödinger equation, the spectral density may be thought of as a local probability density for the energy of the system. Mathematically, the spectral density provides a complete description of the absolutely continuous spectrum.

The paper is organised as follows. The starting point, in Section 2, will be the asymptotic condition

$$\lim_{N \to \infty} \int_0^N (f(x,\lambda))^2 dx \Big/ \int_0^N |f(x,\lambda)|^2 dx = 0 \quad : \qquad \text{(A)},$$

to be satisfied at some real value of λ by a solution $f(x,\lambda)$ of the governing differential equation. Clearly, to satisfy condition (A), $f(x,\lambda)$ has to be complex. However, since λ is real, $f(x,\lambda)$ is a linear combination $u(x,\lambda) + M(\lambda)v(x,\lambda)$ of <u>real</u> solutions u,v, which we shall define subject to prescribed initial conditions at $x = 0$. In Section 2, we give a precise definition of the coefficient $M(\lambda)$ and provide an alternative formulation of Condition (A) in terms of the large x asymptotics of real solutions u,v, bringing out more clearly the relationship between Condition (A) and the definition of subordinate solutions.

In Section 3, we explore the consequences of Condition (A) for the large x asymptotics of solutions $u(x,z)$, $v(x,z)$ of the differential equation at <u>complex</u> spectral parameter z, for z close to a point λ of the real axis. Estimates are carried out in the

Hilbert space $L^2(0,N)$ in which N is allowed to vary in a controlled way with ϵ. The main results, summarised in Theorem 1, illustrate how very precise asymptotic estimates can be drawn from an apparently simple condition (A). We emphasise here that our original asymptotic condition need be satisfied at a single real value of λ only, and that no restrictions whatever have been imposed on the large x behaviour of the potential function V(x).

In Theorem 2 of Section 4, having determined the asymptotics of solutions at complex z, these are used to establish a link between $M(\lambda)$ and the boundary value of the m-function m(z). In particular, our asymptotic condition implies absolute continuity of the spectrum, and the spectral density is just $\frac{1}{\pi}\,\mathrm{Im}M(\lambda)$. Theorem 3 then provides necessary and sufficient conditions for Condition (A) to hold in terms of boundary behaviour of the m-function and asymptotics of solutions at complex z.

Finally, in Section 5, we exhibit the close connection between the asymptotic condition and local behaviour of the spectral function $\rho(\lambda)$.

From a single asymptotic condition (A), which describes the behaviour of solutions at just one, real, value of the spectral parameter, we are thus able to draw a variety of consequences relating to spectral properties, spectral functions and densities, complex z asymptotics and boundary behaviour of the m-function. We consider that the results justify the further investigation of absolutely continuous spectra through these methods, and we would anticipate their application to a wide range of problems in classical and quantum physics.

2 The asymptotic condition

We consider the family of differential operators

$$T_\alpha = -\frac{d^2}{dx^2} + V(x)\left(-\pi/2 < \alpha < \pi/2\right), \tag{1}$$

acting in $L^2(\mathbb{R}_+)$ and subject to the boundary condition

$$(\cos\alpha)f(0) + (\sin\alpha)f'(0) = 0,$$

where $V(\cdot)$ is real-valued and $V \in L_1(0,N)$ for any $N > 0$.

Associated with the differential expression $-\frac{d^2}{dx^2} + V(x)$ is the corresponding differential equation

$$-\frac{d^2f(x,z)}{dx^2} + V(x)\,f(x,z) = zf(x,z), \quad (0 < x < \infty;\ \mathrm{Im}z > 0) \tag{2}$$

with its real counterpart

$$-\frac{d^2}{dx^2}f(x,\lambda) + V(x)f(x,\lambda) = \lambda f(x,\lambda) \quad (0 < x < \infty \, ; \, \lambda \in \mathbb{R}) \tag{2)'$$

We denote by $u_\alpha(\cdot,z)$, $v_\alpha(\cdot,z)$ the solutions of eq. (2), and correspondingly solutions $u_\alpha(\cdot,\lambda)$, $v_\alpha(\cdot,\lambda)$ of eq. (2)′, subject to initial conditions

$$\left.\begin{array}{ll} u_\alpha(0,z) = \cos\alpha & v_\alpha(0,z) = -\sin\alpha \\ u_\alpha'(0,z) = \sin\alpha & v_\alpha'(0,z) = \cos\alpha \end{array}\right\} \tag{3}$$

The Weyl–Titchmarsh m–function $m_\alpha(z)$ [1],[7] is defined for $\mathrm{Im} z > 0$ by the condition that

$$u_\alpha(\cdot,z) + m_\alpha(z)v_\alpha(\cdot,z) \in L^2(0,\infty) .$$

We assume the limit–point case at infinity, so that the above condition defines $m_\alpha(z)$ uniquely. Then $m_\alpha(z)$ has the Herglotz representation ([7])

$$m_\alpha(z) = \cot\alpha + \int_{-\infty}^{\infty} \frac{d\rho_\alpha(t)}{t-z} \qquad (\alpha \neq 0 \, , \, \mathrm{Im} z > 0) \, , \tag{4}$$

with, for $\alpha = 0$, the modified representation

$$m_0(z) = c_1 + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) d\rho_0(t) \tag{4)'$$

The differential operator T_α is unitarily equivalent to a multiplication operator in the space $L^2(\mathbb{R},d\rho_\alpha)$. For notational convenience, we shall usually drop the suffix α , which will then be understood, and refer to the differential operator T , its corresponding m–function m(z) and spectral measure $\mu = d\rho$, and solutions u,v of eq. (2) subject to conditions (3).

Our principal concern in this paper is with the absolutely continuous spectrum of T. The absolutely continuous part μ_{ac} of the measure μ has density function given almost everywhere by $\frac{1}{\pi}$ Im $m_+(\lambda)$. Here $m_+(\lambda)$ is the boundary value of the m–function, defined for $\lambda \in \mathbb{R}$ by

$$m_+(\lambda) = \lim_{\epsilon \to 0+} m(\lambda + i\epsilon) . \tag{5}$$

The essential support of μ_{ac} consists of all $\lambda \in \mathbb{R}$ for which the limit in eq. (5) exists and has strictly positive imaginary part.

The underlying idea, as in [2], is to relate all spectral properties of the self–adjoint differential operator T to the asymptotics of <u>real λ</u> solutions $f(x,\lambda)$ of the corresponding differential equation (2)′. The starting point will be an asymptotic condition for $f(x,\lambda)$ which may be regarded as the analogue, for absolutely continuous spectra, of the subordinacy condition used to characterise spectral behaviour in [2],[3],[4].

Definition: We shall say that Condition (A) holds, for a given real value of λ, if a solution $f(x,\lambda)$ of eq. (2)′ exists, for which

$$\lim_{N\to\infty} \int_0^N (f(x,\lambda))^2 dx \Big/ \int_0^N |f(x.\lambda)|^2 dx = 0 \tag{6}$$

That Condition (A) is satisfied, for appropriate values of λ, for a wide range of standard types of potential, follows readily from the known asymptotic behaviour of solutions in these cases. If V is of short range, so that $\int_0^\infty |V(x)|dx < \infty$, eq. (6) will be satisfied for any $\lambda > 0$ by taking $f(x,\lambda)$ to have the asymptotic behaviour $f(x,\lambda) \sim e^{i\sqrt{\lambda}x}$, and a similar conclusion follows for a large class of long range potentials, by making a suitable modification to the asymptotic form in order to take account of the long range tail of the potential. For a periodic potential with period l, and λ in the interior of any band of the continuous spectrum, there is a solution $f(x,\lambda)$ and real valued function $\beta(\lambda)$ for which $f(x + nl,\lambda) = e^{in\beta(\lambda)}f(x,\lambda)$, $(n = 1,2,3,\cdots)$. Eq. (6) then follows from the fact that $\sum_1^n e^{2in\beta(\lambda)}$ is bounded for fixed λ provided $\beta(\lambda)$ is not a multiple of π, whereas $\sum_1^n |e^{in\beta(\lambda)}|^2 = n$. For the spectral theory of periodic differential equations see [8].

These examples, and others which could be adduced, demonstrate the relevance of Condition (A) to the analysis of one-dimensional Schrödinger operators. In the sequel, we shall assume eq. (6) but make no other detailed assumptions concerning the asymptotics either of the potential V or of solutions of the differential equation. We are then working within a very general framework, which will lead to far-reaching consequences for the asymptotics at both real λ and complex z, as well as a complete treatment of the absolutely continuous part of the measure.

We proceed now to an alternative formulation of Condition (A), which exhibits more clearly the link between this condition and the subordinacy criterion. Since, in eq. (6), any multiple of the solution $f(x,\lambda)$ will do just as well, without loss of generality we can write

$$f(x,\lambda) = u(x,\lambda) + M(\lambda)\, v(x,\lambda)\,, \tag{7}$$

where $u(\cdot,\lambda)$, $v(\cdot,\lambda)$ are the two solutions defined earlier.

Here the coefficient $M(\lambda)$ must satisfy $\operatorname{Im}M(\lambda) \neq 0$, since eq. (6) cannot hold for any real solution f . Moreover, in eq. (6), $f(x,\lambda)$ could be replaced by the complex conjugate $\bar{f}(x,\lambda)$, with $\overline{M}(\lambda)$ for $M(\lambda)$ in eq. (7). Hence without loss of generality we may define $M(\lambda)$ to have strictly positive imaginary part.

LEMMA 1

Let $\mathcal{M} \subseteq \mathbb{R}$ denote the set of λ for which Condition (A) holds. Then the complex valued function $M(\lambda)$ is uniquely defined for $\lambda \in \mathcal{M}$ by the properties $\operatorname{Im}M(\lambda) > 0$ and

$$\lim_{N\to\infty} \int_0^N (u(x,\lambda) + M(\lambda)v(x,\lambda))^2 dx \Big/ \int_0^N |u(x,\lambda) + M(\lambda)v(x,\lambda)|^2 dx = 0 . \tag{8}$$

Moreover, a necessary and sufficient condition for eq. (8) to hold, with $\operatorname{Im}M(\lambda) > 0$, is that

$$\lim_{N\to\infty} \int_0^N (u(x,\lambda) + M(\lambda)v(x,\lambda))^2 dx \Big/ \int_0^N (v(x,\lambda))^2 dx = 0 . \tag{9}$$

PROOF

For $\lambda \in \mathcal{M}$, we have already seen that $M(\lambda)$ exists, with $\operatorname{Im}M(\lambda) > 0$, such that eq. (8) is satisfied. Uniqueness will follow shortly. For $\lambda \in \mathcal{M}$, define $a_N(\lambda), b_N(\lambda)$ by

$$\left.\begin{aligned} a_N(\lambda) &= \int_0^N (u(x,\lambda))^2 dx \Big/ \int_0^N (v(x,\lambda))^2 dx \quad , \\ b_N(\lambda) &= \int_0^N u(x,\lambda)v(x,\lambda) dx \Big/ \int_0^N (v(x,\lambda))^2 dx \end{aligned}\right\}$$

Schwarz's inequality then implies $(b_N(\lambda))^2 \leq a_N(\lambda)$.

Expanding integrands and dividing numerator and denominator by $\int_0^N (v(x,\lambda))^2 dx$, eq. (8) becomes

$$\lim_{N\to\infty} \frac{a_N(\lambda) + 2M(\lambda)b_N(\lambda) + (M(\lambda))^2}{a_N(\lambda) + (M(\lambda) + \overline{M}(\lambda))b_N(\lambda) + |M(\lambda)|^2}$$

$$= \lim_{N\to\infty} \frac{a_N(\lambda) + 2A(\lambda)b_N(\lambda) + (A(\lambda))^2 - (B(\lambda))^2 + 2i(B(\lambda)b_N(\lambda) + A(\lambda)B(\lambda))}{a_N(\lambda) + 2A(\lambda)b_N(\lambda) + (A(\lambda))^2 + (B(\lambda))^2}$$

$$= 0 , \tag{10}$$

where we have defined $A(\lambda)$, $B(\lambda)$ by

$$\left.\begin{aligned} A(\lambda) &= \operatorname{Re}M(\lambda) \\ B(\lambda) &= \operatorname{Im}M(\lambda) \end{aligned}\right\} \tag{11}$$

Suppose a sequence $\{N_j\}$ can be found, $j = 1,2,3,\cdots$ such that $N_j \to \infty$ and $|b_{N_j}| \to \infty$. With $a_{N_j} \geq (b_{N_j})^2$, this would imply $b_{N_j}/a_{N_j} \to 0$. Taking the second ratio in (10), with $N = N_j$, and dividing numerator and denominator by a_{N_j}, this ratio would then converge, for this sequence, to unity. Hence, in fact, the postulated sequence $\{N_j\}$ cannot exist; from this it follows that b_N is bounded. Moreover a_N must also be bounded, since again any sequence $\{N_j\}$ with $a_{N_j} \to \infty$ implies $b_{N_j}/a_{N_j} \to 0$, and leads to a contradiction. Hence the denominators in eq. (10) are bounded in the limit $N \to \infty$, for fixed λ. Multiplying throughout by the denominator and taking the limit now yields

$$\left.\begin{aligned} &\lim_{N\to\infty} a_N(\lambda) + 2A(\lambda)b_N(\lambda) + (A(\lambda))^2 - (B(\lambda))^2 = 0 \\ &\lim_{N\to\infty} 2B(\lambda)\,(b_N(\lambda) + A(\lambda)) = 0 \end{aligned}\right\} \tag{12}$$

Eqs. (12) are just the real and imaginary parts, respectively, of eq. (9). Conversely, starting from eq. (9), and observing that the denominator in (10) approaches $2(\mathrm{Im}M(\lambda))^2$, eq. (8) may be verified (provided $\mathrm{Im}M(\lambda) > 0$).

We can also use eqs. (12) to deduce the limits $b_N(\lambda) \to -A(\lambda)$, $a_N(\lambda) \to (A(\lambda))^2 + (B(\lambda))^2$.

Recalling, from eq. (11), the definitions of $A(\lambda)$ and $B(\lambda)$ we have, then,

$$\left.\begin{aligned} &\lim_{N\to\infty} \int_0^N u(x,\lambda)v(x,\lambda)dx \Big/ \int_0^N (v(x,\lambda))^2 dx = -A(\lambda) = -\mathrm{Re}M(\lambda) \\ &\lim_{N\to\infty} \int_0^N (u(x,\lambda))^2 dx \Big/ \int_0^N (v(x,\lambda))^2 dx = (A(\lambda))^2 + (B(\lambda))^2 = |M(\lambda)|^2 \end{aligned}\right\} \tag{13}$$

Note that eqs. (13) determine $A(\lambda)$ and $(B(\lambda))^2$ for every $\lambda \in \mathcal{M}$. With the positive square root for the imaginary part, the uniqueness of $M(\lambda)$ is assured, completing the proof of the Lemma.

<u>Definition</u>: The function $M(\lambda)$, defined for $\lambda \in \mathcal{M}$ by eq. (10), with $\mathrm{Im}M(\lambda) > 0$, is called the M-function for the differential operator
$T = -\frac{d^2}{dx^2} + V(x)$ in $L^2(0,\infty)$.

Note that an equation identical to eq. (9), with $m(\lambda)$ instead of $M(\lambda)$, is the defining equation for the notion of subordinacy, where $m(\lambda) = \infty$ whenever $v(\cdot,\lambda)$ is subordinate. Two important differences between the m-function $m(\lambda)$ and the M-function are (i) that $m(\lambda)$ is real whereas $M(\lambda)$ is complex, and (ii) that, with $m(\lambda)$ for $M(\lambda)$, the limit in eq. (8) is 1 not 0. These two cases lead to entirely different asymptotics and spectral behaviour, as we shall find in the following sections.

3 Asymptotics of solutions

The key to most of the results of this paper will be a detailed asymptotic analysis of the large x behaviour of solutions of the differential equations (2) and (2)′. In particular, we shall be interested in a comparison between asymptotics of the solutions $u(x,\lambda)$, $v(x,\lambda)$ and $u(x,z)$, $v(x,z)$ respectively, defined subject to initial conditions (3). Note first of all, by a straightforward application of the variation of constants formula, that $u(x,z)$ is given in terms of $u(x,\lambda)$, $v(x,\lambda)$, with $z = \lambda + i\epsilon$, $\epsilon > 0$, by the solution of the integral equation

$$u(x,z) = u(x,\lambda) + i\epsilon\, u(x,\lambda)\int_0^x v(t,\lambda)u(t,z)dt - i\epsilon\, v(x,\lambda)\int_0^x u(t,\lambda)u(t,z)dt\,. \tag{14}$$

We shall write this equation in the form

$$u(\cdot,z) = u(\cdot,\lambda) + L_\epsilon u(\cdot,z)\,, \tag{15}$$

where L_ϵ is the linear operator defined by

$$(L_\epsilon f)(x) = i\epsilon\, u(x,\lambda)\int_0^x v(t,\lambda)f(t)dt - i\epsilon\, v(x,\lambda)\int_0^x u(t,\lambda)f(t)dt \tag{16}$$

The basic idea will be to iterate eq. (15) to obtain

$$u(\cdot,z) = (1 + L_\epsilon + L_\epsilon^2 + \cdots)\, u(\cdot,\lambda)$$

in the Hilbert space $L^2(0,N)$, where N is chosen with reference to the value of the small parameter ϵ. It will by no means be true that $\|L_\epsilon\| < 1$. The following Lemma will provide appropriate norm estimates for powers of L_ϵ.

<u>LEMMA 2</u>

Let L be a linear operator in $L^2(0,N)$, given by

$$(Lf)(x) = \int_0^x k(x,t)f(t)dt\,,$$

and suppose that

$$|k(x,t)| \le \varphi(x)\varphi(t)\,(t \le x)\,,$$

where $\varphi(\cdot) \in L^2(0,N)$.

Let $\|\cdot\|_N$ denote norm in $L^2(0,N)$, i.e. $\|h\|_N = \sqrt{\int_0^N |h(x)|^2 dx}$,

and define two operator norms $\|\cdot\|_N$ and $\|\cdot\|_N$ by

$$\|L\|_N = \sup_{h\neq 0} \|Lh\|_N/\|h\|_N ,$$

$$\|L\|_N = \sup_{h\neq 0} \frac{\operatorname{ess.sup}_{0\le x\le N} |(Lh)(x)|}{\|h\|} .$$

(Thus $\|\cdot\|_N$ regards L as an operator from $L^2(O,N)$ to $L^\infty(O,N)$; we shall assume φ has finite L^∞ norm).

Then the following norm estimates hold for powers of L

$$\|L^n\|_N \le (\|\varphi\|_N)^{2n}/n\,! \tag{17}$$

$$\|L^n\|_N \le (\|\varphi\|_N)^{2n-1} \operatorname{ess.sup}_{0\le x\le N}|\varphi(x)|/\sqrt{n!(n-1)!} \tag{17$'$}$$

PROOF

To avoid complications of notation, we consider the case $n = 3$; the argument for general n follows similarly.

The kernel $K(x,t)$ of L^3 is given for $t \le x$ by $K(x,t) = \int\int k(x,t_1)k(t_1,t_2)k(t_2,t)dt_1dt_2$, where the integration is over the region $x\ge t_1\ge t_2\ge t$. Applying the Schwarz inequality, together with the given bound on the kernel $k(x,t)$, we have

$$|K(x,t)|^2 \le \left(\iint_{x\ge t_1\ge t_2} \varphi^2(x)\varphi^2(t_1)\varphi^2(t_2)dt_1dt_2\right)\left(\iint_{t_1\ge t_2\ge t} \varphi^2(t_1)\varphi^2(t_2)\varphi^2(t)dt_1dt_2\right)$$

The Hilbert-Schmidt norm estimate [9] for integral operators now gives

$$(\|L^3\|_N)^2 \le \left(\iiint_{x\ge t_1\ge t_2} \varphi^2(x)\varphi^2(t_1)\varphi^2(t_2)dxdt_1dt_2\right)\left(\iiint_{t_1\ge t_2\ge t} \varphi^2(t_1)\varphi^2(t_2)\varphi^2(t)dt_1dt_2dt\right) .$$

By a symmetry argument, each of the integrals on the right-hand side gives $\|\varphi\|_N^6/3!$ Hence (17) follows in the case $n = 3$, and the extension to general n is straightforward.

The estimate (17)$'$ comes from the bound

$$(\|\cdot\|_N)^2 \le \operatorname{ess.sup}_{0\le x\le N} \int|K(x,t)|^2dt .$$

We are now ready to estimate the norm of the operator L_ϵ defined by eq. (16). Here we use the fact that the kernel is bounded in absolute value, through Schwarz's inequality, by

$$\epsilon\sqrt{(u(x,\lambda)^2 + (v(x,\lambda))^2}\,\sqrt{(u(t,\lambda))^2 + (v(t,\lambda))^2} \qquad (t \le x) .$$

Thus (17) gives in this case

$$\|(L_\epsilon)^n\|_N \le \epsilon^n[(\|u(\cdot,\lambda)\|_N)^2 + (\|v(\cdot,\lambda)\|_N)^2]^n/n!$$

However, inspection of eq. (14) shows that the kernel of L_ϵ is unaltered if $u(\cdot,\lambda)\sqrt{\|v(\cdot,\lambda)\|_N/\|u(\cdot,\lambda)\|_N}$ and $v(\cdot,\lambda)\sqrt{\|u(\cdot,\lambda)\|_N/\|v(\cdot,\lambda)\|_N}$ are substituted respectively for $u(\cdot,\lambda)$ and $v(\cdot,\lambda)$. Making these substitutions into our bound for $\|(L_\epsilon)^n\|_N$ results in the improved estimate

$$\|(L_\epsilon)^n\|_N \le (2\epsilon\|u(\cdot,\lambda)\|_N\|v(\cdot,\lambda)\|_N)^n/n! \tag{18}$$

A similar argument, from (17)′, leads to the result

$$\|(L_\epsilon)^n\|_N \le \frac{\epsilon^n(2\|u(\cdot,\lambda)_N\| \ \|v(\cdot,\lambda)\|_N)^{n-1} \operatorname*{ess.sup}_{0\le x\le N} \sqrt{u^2(x,\lambda)\|v(\cdot,\lambda)\|^2 + v^2(x,\lambda)\|u(\cdot,\lambda)\|^2}}{\sqrt{n!(n-1)!}} . \tag{18′}$$

We now have, by iteration of eq. (15),

$$u(\cdot,z) = (1-L_\epsilon)^{-1}u(\cdot,\lambda) , \tag{19}$$

where $(1-L_\epsilon)^{-1}$ is bounded in $L^2(0,N)$ and

$$\|(1-L_\epsilon)^{-1}\|_N \le \exp\{2\epsilon\|u(\cdot,\lambda)\|_N \ \|v(\cdot,\lambda)\|_N\} \tag{20}$$

Given any $\epsilon > 0$, and a fixed positive constant c, the relationship $N = N(\epsilon)$ between ϵ and N will be defined by the condition

$$\epsilon(\|v(\cdot,\lambda)\|_N)^2 = c . \tag{21}$$

Our aim will be to determine the asymptotics of $u(\cdot,z)$, $v(\cdot,z)$ $(z = \lambda + i\epsilon)$ in the Hilbert space $L^2(0,N)$ as $\epsilon \to 0+$ and $N \to \infty$, with ϵ,N related by eq. (21). In the sequel, the relationship $N = N(\epsilon)$ will be understood even when not explicitly stated. From eqs. (13), we have $\|u(\cdot,\lambda)\|_N \le \text{const}\ \|v(\cdot,\lambda)\|_N$ for large N, so that eq. (21) guarantees, in (20), that $\|(1-L_\epsilon)^{-1}\|_N$ is uniformly bounded in the limit $\epsilon \to 0+$.

By the notation $Y_\epsilon = X_\epsilon + o_N(\psi)$ we shall mean that

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0+}} \|Y_\epsilon - X_\epsilon\|_N/\|\psi\|_N = 0 ;$$ in other words, in the limit $N \to \infty$,

$Y_\epsilon = X_\epsilon +$ a correction term which is asymptotically much smaller than ψ in norm.

The following Theorem summarises the asymptotic behaviour of $u(\cdot,z)$, $v(\cdot,z)$ which is a consequence of the Condition (A) stated in Section 2. As in section 2, we write $M(\lambda) = A(\lambda) + iB(\lambda)$, where $M(\lambda)$ is the M–function; estimates are not intended to be uniform in λ.

THEOREM 1

Suppose Condition (A) is satisfied, for some $\lambda \in \mathbb{R}$, and let $z = \lambda + i\epsilon$ for $\epsilon > 0$. With $N = N(\epsilon)$ defined by eq. (21), u(x,z), v(x,z) satisfy the following asymptotic formulae, in the limit $\epsilon \to 0+$,

$$u(x,z) = u(x,\lambda) \cosh \{\epsilon B(\lambda)\int_0^x (v(t,\lambda))^2 dt\}$$
$$-i((A(\lambda)/B(\lambda))u(x,\lambda) + ((A^2(\lambda) + B^2(\lambda))/B(\lambda))v(x,\lambda)) \sinh \{\epsilon B(\lambda)\int_0^x (v(t,\lambda))^2 dt\} + o_N(v(\cdot,\lambda)) \tag{22}$$

$$v(x,z) = v(x,\lambda) \cosh\{\epsilon B(\lambda)\int_0^x (v(t,\lambda))^2 dt\}$$
$$+ i((A(\lambda)/B(\lambda))v(x,\lambda) + ({}^1/B(\lambda))\, u(x,\lambda)) \sinh\{\epsilon B(\lambda)\int_0^x (v(t,\lambda))^2 dt\} + o_N(v(\cdot,\lambda)) . \tag{22$'$}$$

PROOF

We operate with L_ϵ on the right hand side of (22). Operating on the first term gives

$$i\, \epsilon\, u(x) \int_0^x v(t)u(t) \cosh(\epsilon B \int_0^t v^2(s)ds)\, dt$$
$$- i\, \epsilon\, v(x) \int_0^x u^2(t) \cosh (\epsilon B \int_0^t v^2(s)ds)\, dt , \tag{23}$$

where we have simplified the notation by dropping explicit reference to the parameter λ.

The first of the two contributions on the right hand side of (23) may be integrated by parts to give

$$Y_\epsilon(x) \equiv i\, \epsilon\, u(x) \int_0^x v(t)\, u(t)\, dt \cosh(\epsilon B\int_0^x v^2(s)\, ds)$$
$$- i\, \epsilon\, u(x) \int_0^x \epsilon B\, v^2(t)\, (\int_0^t v(s)\, u(s)\, ds) \sinh(\epsilon B \int_0^t v^2(s)\, ds)\, dt . \tag{23$'$}$$

Given any $\delta > 0$, eqs. (13) show that we can find $C > 0$ such that, for $x > C$, $(-A-\delta) \int_0^x v^2(t)dt \le \int_0^x v(t)\, u(t)\, dt \le (-A+\delta) \int_0^x v^2(t)\, dt$.

Note that $|Y_\epsilon(x)| \le \text{const}\ \epsilon\, |u(x)|$ for $0 \le x \le C$, so that $\int_0^C |Y_\epsilon(x)|^2 dx/\|v(\cdot)\|_N^2 \to 0$ as $\epsilon \to 0$. Moreover, the contribution to $Y_\epsilon(x)$ coming from the t integration over the interval $0 \le t \le C$ is again bounded by const. $\epsilon|u(x)|$. Hence, if we are to neglect anything which is vanishingly small in norm compared with $v(\cdot)$ in the limit $\epsilon \to 0$, we need consider $Y_\epsilon(x)$ only for $x \ge C$, and with $t \ge C$ in the integrand of (23)′. We can therefore write

$$\int_0^x v(t)u(t)dt = -A \int_0^x v^2(t)dt + 0(\delta\|v(\cdot)\|_N^2) \text{, and}$$

$$\int_0^t v(s)u(s)ds = -A \int_0^t v^2(s)ds + 0(\delta\|v(\cdot)\|_N^2)$$ in the integrand. Noting that $\epsilon\|v(\cdot)\|_N^2$ is bounded, this leads to the result

$$Y_\epsilon(x) = -i\epsilon Au(x) \int_0^x v^2(t)dt \cosh(\epsilon B \int_0^x v^2(s)ds)$$
$$+ i\epsilon Au(x) \int_0^x \epsilon Bv^2(t)(\int_0^t v^2(s)ds) \sinh(\epsilon B \int_0^t v^2(s)ds)dt$$
$$+ 0\,(\delta|u(x)|)\,.$$

Since $\delta > 0$ may be chosen arbitrarily small, and $\|u(\cdot)\|_N \leq \text{const}\ \|v(\cdot)\|_N$, the error term in this asymptotic expression for $Y_\epsilon(x)$ is vanishingly small compared with $v(\cdot)$ in the limit. On integrating by parts, we can go further and carry out the integration explicitly, to give

$$Y_\epsilon(x) = -i\epsilon Au(x) \int_0^x v^2(t) \cosh(\epsilon B\int_0^t v^2(s)ds)dt + o_N(v)$$
$$= -i(A/B)u(x) \sinh(\epsilon B \int_0^x v^2(t)dt) + o_N(v). \tag{24}$$

This is the first of the two contributions to the right hand side of (23). The second contribution to (23) may be evaluated in the same manner, to give

$$-i\epsilon(A^2+B^2)v(x) \int_0^x v^2(t)\cosh(\epsilon B \int_0^t v^2(s)ds)dt + o_N(v)$$
$$= -i((A^2+B^2)/B)\ v(x) \sinh(\epsilon B \int_0^x v^2(t)dt) + o_N(v)\,.$$

Writing eq. (22) in the form $u(x,z) = U_\epsilon(x,\lambda) + o_N\ (v(\cdot,\lambda))$, we can proceed as above to evaluate the remaining terms of $L_\epsilon U_\epsilon$; the basic idea is that, in each term, in the asymptotic limit to order v, one can justify in each term the replacement in the integrand of $u^2(t)$ by $(A^2+B^2)v^2(t)$, and of $u(t)v(t)$ by $-Av^2(t)$, after which integrations may be carried out explicitly. This leads, after simplification, to

$$L_\epsilon U_\epsilon = u(x)\cosh(\epsilon B \int_0^x v^2(t))dt) - i\{(A/B)u(x) + (A^2+B^2)/Bv(x)\}$$
$$\sinh\,(\epsilon B \int_0^x v^2(t)dt) - u(x) + o_N(v)\,.$$

Comparing with the original expression, in (22), for U_ϵ, we have $L_\epsilon U_\epsilon = U_\epsilon - u(x) + o_N(v)$.

Using eq. (15), where now we refer explicitly to the arguments z and λ, we have

$$(1 - L_\epsilon)(u(\cdot,z) - U_\epsilon(\cdot,\lambda)) = u(\cdot,\lambda) - (1 - L_\epsilon)U_\epsilon(\cdot,\lambda) = o_N(v(\cdot,\lambda))\,.$$

However, $\|(1-L_\epsilon)^{-1}\|$ is bounded in the limit $\epsilon \to 0+$, and it follows that $u(\cdot,z) - U_\epsilon(\cdot,\lambda) = o_N(v(\cdot,\lambda))$. This is exactly eq. (22), and eq. (22)′ is verified by a similar argument. This completes the proof of Theorem 1.

Although eqs. (22) and (22)′ present the most convenient expressions of the asymptotic behaviour of $u(x,z)$ and $v(x,z)$, an alternative approach using (17)′ and the norm $\|\cdot\|_N$ instead of $\|\cdot\|_N$, is to derive pointwise estimates. In that case one arrives at identical asymptotic formulae to (22) and (22)′, with the error term replaced by a correction $z_\epsilon(x,\lambda)$ which is dominated by u and v in the sense that

$$\lim_{\epsilon\to 0+} z_\epsilon(x,\lambda)/\sqrt{(u(x,\lambda))^2 + (v(x,\lambda))^2} = 0 .$$

(The limit is uniform in x for $0 \leq x \leq N(\epsilon)$). This formulation of asymptotics is a natural one particularly if a priori bounds on $u(x,\lambda)$ and $v(x,\lambda)$ are known.

The following Corollary is a simple consequence of eqs. (22) and (22)′ on taking the appropriate linear combination, and will be the starting point in Section 4 to investigating the boundary value of the Weyl–Titchmarsh m–function.

<u>Corollary to Theorem 1</u>

The linear combination $u + Mv$ has the asymptotic behaviour

$$\begin{aligned} &u(x,z) + M(\lambda)v(x,z) \\ &\quad = (u(x,\lambda) + M(\lambda)v(x,\lambda)) \exp\{-\epsilon B(\lambda) \textstyle\int_0^x (v(t,\lambda))^2 dt\} \\ &\qquad + o_N(v(\cdot,\lambda)) \end{aligned} \tag{25}$$

We conclude this Section with a number of asymptotic formulae which follow from eqs. (22) and (22)′ on integrating products. In these formulae we shall say that an error term $z_\epsilon(x,\lambda)$ satisfies $z_\epsilon(x,\lambda) = o(^1/\epsilon)$ provided $\lim_{\epsilon\to 0+} \epsilon z_\epsilon(x,\lambda) = 0$ and the limit is uniform in x for $0 \leq x \leq N(\epsilon)$.

We have, then,

$$\left.\begin{aligned} &\int_0^x (u(t,z))^2 dt = \int_0^x (u(t,\lambda))^2 dt + o(^1/\epsilon) && \text{(i)} \\ &\int_0^x (v(t,z))^2 dt = \int_0^x (v(t,\lambda))^2 dt + o(^1/\epsilon) && \text{(ii)} \\ &\int_0^x u(t,z)v(t,z)\,dt = \int_0^x u(t,\lambda)v(t,\lambda)\,dt + o(^1/\epsilon) && \text{(iii)} \\ &\int_0^x |v(t,z)|^2 dt = (^1/2\epsilon B(\lambda))\sinh\{2\epsilon B(\lambda)\textstyle\int_0^x (v(t,\lambda))^2 dt\} + o(^1/\epsilon) && \text{(iv)} \end{aligned}\right\} \tag{26}$$

The proof of eqs. (26(i)–(iv)) follows the same lines as were taken in evaluating expressions such as (23) in the asymptotic limit.

4 Boundary values of the m–function

It is well known [7],[10], that spectral properties of the differential operator $T = -\frac{d^2}{dx^2} + V(x)$ in $L^2(0,\infty)$ are governed by the behaviour of the Weyl–Titchmarsh function $m(z)$ as z approaches the real axis. The following Theorem establishes the link between this and the function $M(\lambda)$ defined in Section 2.

<u>THEOREM 2</u>

Let $\mathcal{M} \in \mathbb{R}$ be the set of $\lambda \in \mathbb{R}$ for which Condition (A) holds. Then the m–function $m(z)$ and the M–function $M(\lambda)$ are related, for all $\lambda \in \mathcal{M}$, by

$$M(\lambda) = \lim_{\epsilon\to 0+} m(\lambda+i\epsilon) \tag{27}$$

Moreover the spectral measure μ, restricted to the set $\mathcal{M}$, is purely absolutely continuous, with density function $\frac{1}{\pi}\operatorname{Im}M(\lambda)$.

<u>PROOF</u>

The second part of the Theorem follows the first part from the standard characterisation of μ_{ac} in terms of boundary values of $m(z)$. Hence we have only to prove eq. (27). To do so, we use the Weyl limit point/limit circle theory; see [1],[4].

For any $\epsilon, N > 0$, with $z = \lambda + i\epsilon$, the Weyl theory defines a circle $C_N(z)$ in the upper half plane such that in the limit $N \to \infty$, and with z held fixed, the family of circles C_N converges to the single point $m(z)$. The radius R_N of the Weyl circle C_N is given by

$$R_N = \left(2\epsilon \int_0^N |v(t,\lambda)|^2 dt\right)^{-1}.$$

Here we shall <u>not</u> hold z fixed, but instead, as in Section 3, hold λ fixed and define by eq. (21) the relation $N = N(\epsilon)$ between N and ϵ. In that case, R_N will no longer converge to zero in the limit $N \to \infty$, since we also have $\epsilon \to 0$. We can, however, use (iv) of eqs. (26) to evaluate the limit, giving

$$\lim_{\substack{N\to\infty \\ \epsilon\to 0+}} R_N(\lambda+i\epsilon) = 2B(\lambda)/\sinh(2CB(\lambda)) . \tag{28}$$

A further consequence of the asymptotic formulae derived in the last section is that, for N sufficiently large, $M(\lambda)$ is in the <u>interior</u> of the circle C_N. To verify this, we have to check that $M(\lambda)$ satisfies the defining inequality for the interior of C_N, namely

$$\epsilon \int_0^N |u(t,z) + M(\lambda)v(t,z)|^2 dt < \operatorname{Im} M(\lambda) . \tag{29}$$

Using eq. (25), the left hand side of (29) is given asymptotically for large N and small ϵ by

$$\epsilon \int_0^N |u(t,\lambda) + M(\lambda)v(t,\lambda)|^2 \exp\{-2\epsilon B(\lambda)\int_0^t (v(s,\lambda))^2 ds\} dt .$$

Using the same methods as in the proof of eqs. (26), we can take the limit $\epsilon \to 0$, $N \to \infty$ with $N = N(\epsilon)$, to give

$$\lim_{\substack{N\to\infty\\ \epsilon\to 0+}} 2\epsilon(B(\lambda))^2 \int_0^N (v(t,\lambda))^2 \exp\{-2\epsilon B(\lambda) \int_0^t (v(s,\lambda))^2 ds\} dt$$

$$= B(\lambda)\,(1 - \exp\{-2CB(\lambda)\}\,) < B(\lambda) . \tag{25}$$

Since $B(\lambda)$ is just the right hand side of (29), we have verified that $M(\lambda)$ does indeed lie in the interior of C_N, for N sufficiently large.

Standard Weyl theory implies that $m(z)$ also is in the interior of C_N. Hence the distance, measured in the upper half plane, between $m(z)$ and $M(\lambda)$, cannot exceed $R_N(\lambda+i\epsilon)$, for large N.

Noting the inequality $\sinh\theta > \theta$ for $\theta > 0$, it follows from eq. (28) that $R_N(\lambda+i\epsilon) < {}^1/C$ for N sufficiently large.

We have, then,

$$|m(z) - M(\lambda)| < {}^1/C , \tag{30}$$

provided ϵ is sufficiently small and positive. (We can here drop all reference to N, since the left hand side of (30) is now independent of N).

Since C in (30) is an arbitrary positive constant, eq. (27) follows immediately, and the Theorem is proved.

Theorem 2 allows us to define a family of solutions $f(x,z)$ of eq. (2), with z in the upper half plane, having boundary value $f(x,\lambda)$ given by eq. (7), as z approaches a point λ on the real axis. Thus, we can write

$$f(x,z) = u(x,z) + m(z)v(x,z) . \tag{31}$$

Following Theorem 2, we can prove several remarkable asymptotic formulae satisfied by $f(x,z)$.

Corollary to Theorem 2

Define $\mathcal{M}$ as in Theorem 2. Then for $\lambda \in \mathcal{M}$, with $z = \lambda + i\epsilon$, we have

$$\left.\begin{array}{ll}\lim_{\epsilon\to 0+} \epsilon\int_0^x (f(t,z))^2 dt = 0 & \text{(i)}\\ \lim_{\epsilon\to 0+} \int_0^\infty ((f(x,z))^2 dx / \int_0^\infty |f(x,z)|^2 = 0 & \text{(ii)}\\ \lim_{\epsilon\to 0+} \epsilon \| f(\cdot,z) - (u(\cdot,\lambda) + M(\lambda)v(\cdot,\lambda))\exp\{-\epsilon B(\lambda)\int_0^{\cdot} (v(t,\lambda))^2 dt\}\|^2 = 0 & \text{(iii)}\end{array}\right\} \quad (32)$$

Moreover, the limit in (32)(i) is uniform in x for $0 < x < \infty$.

PROOF

Since $m(z) \to M(\lambda)$ as $z \to \lambda$, the asymptotics of $f(x,z)$ in $L^2(O,N)$ are as for the function $u(x,z) + M(\lambda)v(x,z)$. Proceeding as in the proof of eqs. (23), and using eq. (25), it is then straightforward to verify (i) of eqs. (32) for $0 < x < N(\epsilon)$, and that the limit is uniform over x in this interval. To verify (i) or $x > N(\epsilon)$, we have to obtain a bound for $\epsilon\int_N^\infty f(t,z))^2 dt$. Following the proof of Theorem 2, with $m(z)$ for $M(\lambda)$ on the left hand side of (29) , we have already shown that

$$\lim_{\substack{N\to\infty\\ \epsilon\to 0+}} \epsilon\int_0^N |f(t,z)|^2 dt = B(\lambda)\,(1 - \exp\{-2CB(\lambda)\}) \ .$$

Moreover, from the Weyl theory we have the standard formula

$$\epsilon\int_0^\infty |f(t,z)|^2 dt = \operatorname{Im} m(z) \ . \tag{33}$$

Taking the limit $z \to \lambda$, it follows that

$$\lim_{\substack{N\to\infty\\ \epsilon\to 0+}} \epsilon\int_0^\infty |f(t,z)|^2 dt = B(\lambda)\exp\{-2CB(\lambda)\} \tag{34}$$

By taking C large, we can therefore ensure that $\epsilon\int_N^\infty (f(t,z))^2 dt$ is arbitrarily small in the limit $\epsilon\to 0+$, with $N = N(\epsilon)$. Hence the limit in (32)(i), initially proved only for $0<x<N(\epsilon)$, now extends to all x in the range $0<x<\infty$, and is, moreover, uniform in that interval. From the uniformity of convergence, it follows immediately that $\epsilon\int_0^\infty (f(x,z))^2 dx$ converges to zero in the limit $\epsilon\to 0+$. From eq. (33), we see that the denominator of eq. (ii) approaches $B(\lambda)/\epsilon$ asymptotically, and the second equation of the Corollary is proved. Finally, note that (iii) of eqs. (32) is a simple consequence of eq. (25), provided that we had $\|\cdot\|_N$ instead of $\|\cdot\|$. As in the proof of (ii) of the Corollary, we therefore need to estimate the contribution to the norm coming from $x > N$. Such an estimate for $f(\cdot,z)$ is provided by eq. (34), and it remains only to obtain a similar estimate for

$$\epsilon\int_N^\infty |u(t,\lambda) + M(\lambda)v(t,\lambda)|^2 \exp\{-2\epsilon B(\lambda)\int_0^t (v(s,\lambda))^2 ds\}\, dt \ .$$

This we can do in the same manner as for previous asymptotic estimates, setting

$$|u(t,\lambda) + M(\lambda)v(t,\lambda)|^2 = \frac{d}{dt}\int_0^t |u(s',\lambda) + M(\lambda)v(s',\lambda)|^2 ds'$$

and integrating by parts; here once again we make use of the comparisons (13) between integrals, and an asymptotic bound as on the right hand side of (34) is obtained. Since again the value of the constant C may be chosen arbitrarily large, the final part of eqs. (32) is verified, and the proof of the Corollary is complete.

Eq. (32)(i) is an important ingredient in establishing necessary and sufficient conditions for Condition (A), in terms of the boundary behaviour of the m-function $m(z)$. In the following result, which as before assumes limit point case at infinity, $f(\cdot,z)$ is again defined by eq. (31).

THEOREM 3

For any given $\lambda \in \mathbb{R}$, with $z = \lambda + i\epsilon$, Condition (A) holds if and only if both

(i) $\lim_{\epsilon \to 0+} \epsilon \int_0^x (f(t,z))^2 dt = 0$,

uniformly in x for $0 < x < \infty$, and

(ii) $\lim_{\epsilon \to 0+} m(z)$ exists and has strictly positive imaginary part.

PROOF

If Condition (A) holds at λ, then (ii) of Theorem 3 follows from eq. (27) together with the observation, in Lemma 1, that $\mathrm{Im}M(\lambda) > 0$. Moreover (i) of Theorem 3 is one of the consequences of the Corollary to Theorem 2.

Conversely, suppose (i) and (ii) of Theorem 3 are satisfied. We have to deduce Condition (A). The argument is sufficiently close to that of Theorem 1 that it will be enough to give a sketch of the proof. Whereas in Theorem 1 we used asymptotics for integrals of solutions $u(x,\lambda)$, $v(x,\lambda)$ of the real λ Schrödinger equation to derive corresponding asymptotics for the Schrödinger equation at complex z, here we put the argument into reverse to derive real λ asymptotics of a solution $f(x,\lambda)$ from the postulated behaviour of $f(x,z)$. To this end, we start from the integral equation

$$f(x,\lambda) = f(x,z) - i\epsilon f(x,z) \int_0^x v(t,z)f(t,\lambda)dt + i\epsilon v(x,z) \int_0^x f(x,z)f(t,\lambda)dt \quad . \tag{35}$$

The solution $f(x,\lambda)$ of eq. (35) satisfies the differential equation (2)′ and from the initial values of f and $\frac{df}{dx}$ we find that

$$f(x,\lambda) = u(x,\lambda) + m(z)v(x,\lambda) \; . \tag{36}$$

Strictly, we should write $f(x,\lambda;z)$ to indicate the dependence of this solution, through

$m(z)$, on the value of the complex parameter $z = \lambda + i\epsilon$; for simplicity of notation, and to emphasise that we are dealing with a solution of the real λ differential equation, we prefer to suppress this dependence on z .

We shall determine the asymptotics of $f(\cdot,\lambda)$ in the Hilbert space $L^2(0,N)$, where for $\epsilon > 0$ and a fixed positive constant γ . the relationship $N = N(\epsilon)$ between ϵ and N will be defined by the condition (cf. eq. (21))

$$\epsilon(\|v(\cdot,z)\|_N)^2 = \gamma .$$

Since, by hypothesis, $m(z)$ is bounded as z approaches λ , eq. (33) implies also the norm estimate

$$\epsilon(\|f(\cdot,z)\|_N)^2 \leq \text{const.}$$

Writing eq. (35) in the operator form

$f(\cdot,\lambda) = f(\cdot,z) + K_\epsilon f(\cdot,\lambda)$, the above bounds for $v(\cdot,z)$ and $f(\cdot,z)$ may be used, as in Lemma 2, to derive norm estimates for powers of K_ϵ as operators in $L^2(0,N)$. In particular, this guarantees that $\|(1-K_\epsilon)^{-1}\|_N$ is uniformly bounded in the limit $\epsilon \to 0+$, and the solution of eq. (35) for $f(\cdot,\lambda)$ may be obtained by iteration. Now apply the integral operator K_ϵ to the function

$$f(x,z) \exp \{-i\epsilon \int_0^x v(t,z)f(t,z)dt\}$$

Using (i) of the Theorem and integrating by parts, we arrive at the asymptotic formula

$$(1-K_\epsilon) (f(x,z) \exp \{-i\epsilon \int_0^x v(t,z)f(t,z)dt\})$$
$$= f(x,z) + o_N(v(\cdot,z)),$$

which on comparison with the equation for $f(\cdot,\lambda)$ and using the bound on $(1 - K_\epsilon)^{-1}$, leads to

$$f(x,\lambda) = f(x,z) \exp \{-i\epsilon \int_0^x v(t,z)f(t,z)dt\} + o_N(v(\cdot,z)) \tag{37}$$

Now use (ii) of the Theorem to define $m_+(\lambda) = \lim_{\epsilon\to 0+} m(z)$, so that $\text{Im } m_+(\lambda) > 0$. Since $\|v(\cdot,\lambda\|_N = \|(1-K_\epsilon)^{-1} v(\cdot,z)\|_N \leq \text{const } \|v(\cdot,z)\|_N$, the convergence of $m(z)$ to $m_+(\lambda)$ in eq. (36) allows us to replace $f(x,\lambda)$ on the left hand side of (37) by $f_+(x,\lambda)$ where

$$f_+(x,\lambda) = u(x,\lambda) + m_+(\lambda)v(x,\lambda) .$$

Notice that here $f_+(x,\lambda)$ is indeed independent of z .

It remains to use (37), with f_+ for f, to estimate the integral $\int_0^N (f_+(x,\lambda))^2 dx$. Here the norm estimates for $v(\cdot,z)$ and $f(\cdot,z)$ imply that the argument of the exponential function remains bounded and that we have, on integrating by parts and using once more (i) of the hypotheses of the Theorem,

$$\lim_{\epsilon\to 0+} \epsilon\int_0^N (f_+(x,\lambda))^2 dx = \lim_{\epsilon\to 0+} \epsilon\int_0^N (f(t,z))^2 \exp\{-2i\epsilon \int_0^t v(s,z)f(s,z)ds\}dt$$
$$= 0 .$$

On the other hand, $\|v(\cdot,z)\|_N = \|(1 - K_\epsilon)v(\cdot,\lambda)\|_N \le \text{const.}\ \|v(\cdot,\lambda)\|_N$, from which it follows that

$$\epsilon\int_0^N (v(x,\lambda))^2 dx \ge \text{const.}\ \epsilon\ \|v(\cdot,z)\|^2 > \text{const.} > 0 .$$

Combining these bounds, we now have, with $N = N(\epsilon)$,

$$\lim_{\epsilon\to 0+} \int_0^N (f_+(x,\lambda))^2 dx \Big/ \int_0^N (v(x,\lambda))^2 dx = 0 .$$

The relationship $N = N(\epsilon)$ maps any neighbourhood $0 < \epsilon \le \epsilon_0$ of zero continuously on to a corresponding semi-infinite interval $N_0 \le N < \infty$ in such a way that ϵ becomes arbitrarily small for large N. Thus general properties of the mapping imply that the limit $\epsilon \to 0+$ may be replaced by a limit $N \to \infty$. Hence eq. (9) holds with $M(\lambda) = m_+(\lambda)$. By Lemma 1, we have now verified Condition (A), and with this the proof of Theorem 3 is complete

5 Further comments and developments

The set of all $\lambda \in \mathbb{R}$, for which $m(z)$ has a boundary value having positive imaginary part, defines an essential support for the absolutely continuous spectrum. From Theorem 2 we see that all λ at which Condition (A) holds lie in this essential support. However, the characterisation of Condition (A) provided by Theorem 3 entails an additional property (i), and leaves open the question of whether the set of λ at which Condition (A) holds is itself an essential support for the absolutely continuous spectrum. Although in all cases which we have analysed in detail Condition (A) is found to hold at almost all $\lambda \in \mathbb{R}$, we are as yet unable to find a definitive answer to the question of whether this will be true in general. We conjecture that the set $\mathcal{M}$ defined in Theorem 2, is indeed an essential support for the absolutely continuous spectrum. To verify this conjecture, it will be necessary to show that (i) of Theorem 3 holds, as a uniform limit in x, for almost all λ. As a first step in this direction, one may show that property (i) does hold, for almost all λ, in the $x\to\infty$ limit ; namely, we have, almost everywhere,

$$\lim_{\epsilon\to 0+} \epsilon\int_0^\infty f(t,z)^2 dt = 0\,. \tag{38}$$

To verify (38), start from the identity

$$\epsilon\int_0^\infty f(t,z_1)f(t,z_2)dt = \frac{m(z_1) - m(z_2)}{z_1 - z_2}\ ,$$

in which the integral has been evaluated by standard arguments involving Wronskians, and proceed to the limit $z_1 \to z_2$.

Then eq. (38) is equivalent to

$$\lim_{\epsilon\to 0+} \epsilon\frac{dm(z)}{dz} = 0\,. \tag{38$'$}$$

Using the Herglotz representation (4) or (4)$'$ for the m–function in terms of the spectral function $d\rho$, we can restate (38)$'$ as

$$\lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} \frac{\epsilon\, d\rho(t)}{(t-z)^2} = 0\,. \tag{38$''$}$$

In the limit as $z = \lambda + i\epsilon \to \lambda$, the integral in (38)$''$ is governed by the behaviour of the spectral function $\rho(t)$ near $t = \lambda$. In particular, if $\rho(t)$ is differentiable at λ , we can integrate by parts and make the change of variable $t = \lambda + \epsilon s$. Substituting the asymptotic form for the spectral function near λ and using the fact that $\int_{-\infty}^{\infty} \frac{s^2-1}{(s^2+1)^2}\, ds = 0$, we can verify that (38)$''$ holds at all points of differentiability. Indeed, identities for spectral integrals may be used to show that $\epsilon\frac{dm(z)}{dz}$ has zero boundary value wherever $m(z)$ has a (finite) boundary value.

It follows that property (i) of Theorem 3 is almost everywhere equivalent to (i)$'$:

$$\lim_{\epsilon\to 0+} \epsilon\int_x^\infty (f(t,z))^2 dt = 0\,.$$

This integral too may be evaluated using Wronskian methods. Thus

$$\epsilon\int_x^\infty (f(t,z))^2 dt = \epsilon\{f(x,z)\frac{\partial}{\partial z}f'(x,z) - f'(x,z)\frac{\partial}{\partial z}f(x,z)\}\ ,$$

where the prime denotes differentiation with respect to x .

Let us define an m–function $m_x(z)$, with corresponding spectral function $\rho_x(t)$, for the differential operator in the interval $[x,\infty)$, where now the initial conditions (3) are applied at the endpoint x rather than at zero. We take for simplicity the case $\alpha = 0$. Condition (i)$'$ may then be written

$$\lim_{\epsilon\to 0+} \epsilon(f(x,z))^2 \int \frac{d\rho_x(t)}{(t-z)^2} = 0\,.$$

If the measure $d\rho$ is absolutely continuous, then $d\rho_x$ will have a density function $\frac{1}{\pi}\operatorname{Im} m_x^{+}(t)$, where m_x^{+} is the complex boundary value of m_x. In that case Condition (i)′,
with change of integration variable $t = \lambda + \epsilon s$, becomes

$$\lim_{\epsilon \to 0+} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} m(\lambda+\epsilon s)\,|f(x,\lambda+i\epsilon)|^2 ds}{(s-i)^2 |f_+(x,\lambda+\epsilon s)|^2} = 0 .$$

This Condition will hold, in particular, if the m-function is continuous at λ, if $|f(x,z)|^2$ is uniformly bounded near $z = \lambda$, for z in the upper half-plane, and if $|f_+(x,t)|^2$ is uniformly continuous in t at $t = \lambda$. (Note also, from the proof of Theorem 3, that to establish Condition (A) it is enough to restrict attention to x in the range $0 \leq x \leq N$, where $\epsilon(\|v(\cdot,z)\|_N)^2 = \gamma$.)

These arguments, which relate asymptotic behaviour of solutions in the complex lane more directly to local properties of the spectral measure ρ and related measures ρ_x, are the first steps in the establishment of further links between asymptotics and spectral behaviour. In particular, it appears that further progress can be made in the analysis of potentials for which there is a uniform pointwise bound on solutions. This is an area of special interest in scattering theory, and will be the subject of continuing research.

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