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Spacetime Models from the Electromagnetic Field*

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Abstract. A geometric treatment of the electromagnetic field leads to the introduction of spacetime models representing the dynamics of the system. The manifold structure of these models and differential geometric structure such as a conformal metric are induced from Maxwell's equations. Such models take into account the qualitative difference between time and space while retaining notions such as Lorentz covariance. The conformal metric allows a natural decomposition of the electromagnetic field into transverse (photon) and longitudinal (force field) components.

1 Introduction

Lorentz covariance is one of the structural cornerstones of physical theory. However a point that is often overlooked is that such covariance is only dynamical: Lorentz transformations act on the motion of the system and not on its state. Experimentally it is time and space that are primitive notions, not spacetime. This fact is mirrored in formalisms such as axiomatic quantum field theory where one must choose an explicit representation of the Poincaré group to formulate the theory. This implies in particular a choice of time direction, that is separating time and space. This does not contradict relativity in spacetime since time and space as separate entities lead directly to dynamical covariance as we will see later.

In this paper we consider the simple case of the electromagnetic field, showing how

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the search for a geometric representation of its evolution leads to the introduction of spacetime models. It is important to note that these models are secondary constructions; space and time being the primitive objects. Each "observer" uses a concrete Galilean frame considered to be at rest to construct his own spacetime from space and time. In the future we will call such constructions spacetime models. In this model he can then use Lorentz transformations to map one motion into another, however to interpret the equations that he finds he must retain the same spacetime model throughout. Finally, the geometric properties of the spacetime models describe properties of the dynamics of the system. For example, the Lorentz matrix is induced by the so-called phenomenological relations for the electromagnetic field.

The rest of this paper is organised as follows. In section 2 we discuss the notion of physical space and its corresponding manifold structure. The geometric representation of the state of the electromagnetic field is then analysed in section 3. As space has no orientation one must distinguish even forms representing intensive quantities and odd forms representing extensive quantities. In section 4 we consider the physical time and the nature of dynamical covariance. In section 5 we show how spacetime models are induced from a geometric consideration of the dynamics of the electromagnetic field. One finds that such models come in pairs, allowing for the introduction of antiparticles in a natural way. The resulting geometric structure is then applied in section 6 to the decomposition of the electromagnetic field. This decomposition gives a canonical splitting into transverse and longitudinal potentials, hence allowing the possibility of a gauge-free formulation of the quantum theory of the electromagnetic field.

2 Space

We study space S as the physical 3-dimensional vacuum space of everyday experience. As we are interested in modelling a physical system of interest and not the entire universe we will often consider a part of space, for example the interior of a superconducting cavity. We are then interested in the properties of the system, that is what one can do with certainty with the system [Aerts 1982, Piron 1990a]. Such properties are nothing more than elements of reality in the sense of Einstein [Einstein *et al.* 1935]. The properties of space are geometry and fields: as we will see in the following, a study of electromagnetic fields gives much information on space, and in particular its geometric aspects.

It is important to note that a property is not simply linked to the measuring apparatus: the electromagnetic field exists independently of whether or not we perform any experiment. The distinction can be made clear by the following examples. A piece of chalk has the property that it is breakable. However once we perform the experiment the chalk is broken; in performing an experiment we disturb the properties of the entity. Similarly the existence of a gravitational field corresponds to the assertion that *if* we were to introduce matter it would fall. The field exists independently and in its own right so that the usual notion of a test particle is redundant.

Similarly the properties of space manifest the existence of space as a physical object. The vacuum space is not the void. There is then no need to fill it with some ether to have something and give reality to geometrical and mechanical properties. Every time an ether is introduced into physics one has problems which are only resolved once it is banished. However the notion of ether remains firmly entrenched in the way of thinking of most physicists. Einstein went so far as to say that all we can do is avoid the use of the word [Einstein and Infeld 1947, p184]! For example, Maxwell introduced an ether to support his electromagnetic waves which lead to well-known problems such as ether drag. Dirac then quantised the photon as a harmonic oscillator, making the implicit assumption that there was a substance which oscillates. This lead to infinities which can only be removed in a rather *ad hoc* fashion. Finally Feynman adopted the Descartian view that interactions were caused by the contact or exchange of particles. His sea of four-component photons strongly resembles the sea of "particules ultramondaines" used by Le genevois Le Sage to explain gravitation as early as 1818. Let us recall that this theory was finally abandonned only when Poincaré calculated the unacceptable heating the earth would suffer from the impact of these particles needed to explain gravity. It is interesting to compare this problem to the renormalisation required in quantum field theories.

We also emphasise that the concepts of space and time are distinct. There are many manifestations of this distinction. For example time always flows in the same direction and only one instant of time is actual. In contrast all of space exists and is actual independently of the time. As we will see in section 4, this fact does not interfere with relativity, as covariance is a dynamical symmetry of the set of possible evolutions and not just a symmetry of the actual state of the system.

To exhibit the properties of space we must build coordinate systems: the Cartesian product $\mathbb{R}^3$ which appears in our equations describes our chosen coordinates and not space itself. As coordinates are understood as classical observables we can provide coordinate functions $f : S \to \mathbb{R}$ [Piron 1976, p12] which then possess a differential structure as defined by Sikorski [1972]. We repeat that coordinates must be *added* to the vacuum space. In the simplest cases, such as that considered here, such a differential structure defines a differential manifold. In general one has a differential space, a class of structure which includes manifolds with boundaries or singularities as special cases.

Defining a particular set of coordinates defines an associated orientation, however as we do not select an *a priori* set of coordinates, the resulting manifold S is not oriented. Note that the existence of interactions which do not preserve parity does not select an orientation since parity is a dynamical symmetry. This is in sharp contrast to time, where the coordinate representing the time must be taken to increase with the flow of time. An important consequence of this fact is that when we construct a spacetime model by pasting time and space together we end up with a pair of models. The two resulting 4-dimensional models are homotopic in the sense that we can pass from one to the other by a continuous family of infinitesimal transformations. We can then relate the laws of physics valid in one model to the corresponding laws valid in the other by continuity. The generalised Galilei principle then affirms that the laws of physics can be written with any such model. This defines the CPT map and provides for the introduction of antiparticles [Barut *et al.* 1993].

The fact that we have no given orientation reinforces the distinction between differential forms which are odd or even as defined by G. de Rham. For example, the coefficients of an even 1-form transform as a covector in the usual way, however the coefficients of an odd 1-form have an extra factor of the sign of the Jacobian of the transformation [de Rham 1984, p19]. To integrate an even form we must provide an oriented submanifold. The given form with the orientation then provides an odd volume form on the submanifold, which can be integrated.

One can clearly see the physical difference between the two types of form by considering a simple example. If one integrates a mass density over a volume one finds the mass inside the volume. However this volume is not oriented and so a mass density is an odd 3-form. It is only in this way that one can define, for example, the mass of a non-orientable surface such as a Möbius strip. On the other hand the integral of a force over a line gives the work done in moving a particle along the line. The sign of this quantity depends on the direction in which one follows the curve. Thus we integrate over an oriented curve: a force is defined as an even 1-form. Such considerations will be used in the next section to discuss the representation of the electromagnetic field in terms of differential forms on space.

3 The Electromagnetic Field

In this section we consider the electromagnetic field at each instant of time. A geometric consideration of the dynamics of this object will then lead to spacetime models in section 5. The electromagnetic field is that property of space related to electromagnetic phenomena. For example, the presence of an electromagnetic field is indicated by the acceleration of a charged sphere or the induction of a current in a conducting ring. We emphasise that the field is defined before the introduction of any such object, it being the affirmation "*If* one were to introduce a charged object it would be accelerated".

The electromagnetic field is described by the electric field $\mathbf{E}$, the electric displacement $\mathbf{D}$, the magnetic field $\mathbf{H}$ and the magnetic flux $\mathbf{B}$. The source is described by the charge density ρ and the current $\mathbf{J}$. For more details see for example R. S. Ingarden and A. Jamiołkowski [1985, pp36-47]. We now show that the physical interpretation of these objects allows us to describe them as time-dependent forms on space.

We start with the field sources. The integral of ρ over a volume is the charge contained within it. The sign is related to the nature of the charges in the volume and not on any orientation of it. Thus ρ is an odd 3-form. Further, the integral of $\mathbf{J}$ over the non-oriented surface which bounds a volume is the flux of electric charge out of the volume. Hence $\mathbf{J}$ is an odd 2-form. We will write $J := (\mathbf{J}, \rho)$.

We can make similar arguments for the other quantities defining the electromagnetic field. The pair $B := (\mathbf{E}, \mathbf{B})$ of intensive objects gives the Lorentz force so that the electric field and magnetic flux are even forms. On the other hand the pair $H := (\mathbf{H}, \mathbf{D})$ of extensive objects couple to the source so that the magnetic field and electric displacement are odd forms. Since $\mathbf{B}$ is an even form the field B_{ij} defines an axial (orientation dependent) vector. On the other hand, since $\mathbf{D}$ is an odd form the field D^{ij} defines a polar vector. Hence $\int dv \operatorname{div}\vec{D}$ does not depend on any orientation and thereby defines the electric charge. The divergence of $\vec{D}$ can also be defined as a de Rham current when the field is not differentiable (see Weyl [1940, section 3]), as is the case for punctual sources. It is easy to see that, as for the source J, the two forms in each pair differ in degree by unity. It is this fact that will allow us to define spacetime models by representing the dynamics of J, B and H as differential forms on a suitable 4-manifold. The fields B and H are related by the phenomenological relations. These express the fact that the field H acts on the vacuum to generate the field B. It is this field that then acts on the particles.

The difference in character between the pairs of forms B and H is usually hidden by the use of vector notation and the mistaken identification of **H** with **B** and **D** with **E** in the vacuum. This assumes not only a choice of reference frame but also of canonical orientation. In this case one can pass from even forms to odd forms by multiplying by the orientation, a smooth odd function which takes values ± 1 [de Rham, 1984, p19]. In light of the preceding discussion we must guard the distinction here.

4 Time

As in the case of the space S, to exhibit the properties of the time T we must build a parameter $\mathbb{R}$ and its 1-dimensional manifold structure. First let us reemphasise that time and space are qualitatively different. Time flows in a single direction and so any time coordinate we add should respect the orientation induced by the flow of time. This fact is often confused by the fact that since 1983 the SI units for time and length measurements have been linked; c is now defined to be exactly 299 792 458 m/s. However different physical phenomena are used to measure time and space. A time interval is measured by counting the passage of a fixed number of oscillations of a travelling light wave, for instance the second was defined in 1967 as the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the cesium-133 atom. However a distance is measured by counting the number of troughs of a stationary wave in a cavity.

Having defined the second as a unit one can then define a corresponding time coordinate scale such as the International Atomic Time (TAI) which was defined in 1980 to be a coordinate time scale defined at a geocentric datum line and having as its unit one SI second as obtained on the geoid in rotation. It is also useful to define other coordinate frames such as the heliocentric frame. These must then be related back to the TAI. To do this one corrects for the effective gravitational fields acting on the ideal clock. One can then calculate the time coordinate in the reference frame of interest [Guinot 1986]. Such corrections have a typical magnitude of several hundred nanoseconds and allow one to synchronise clocks in a neighbourhood of the Earth to within a current accuracy of at least 2ns/day.

One then observes that there is no dispersion of particles in time, a fact in conflict with any fully covariant theory treating particles as points in spacetime [Piron 1990b]. It is also for this reason that we consider space and time as separate objects. It is interesting to note that the prescription of the International Bureau of Weights and Measures for defining reference frames is exactly the same as the procedure we will introduce in the next section to build spacetime models: one pastes a spatial coordinate system to the time scale defined by some standard measurement. The theory developed here is thus more than a mere abstraction, it closely parallels the actual practice of physicists in the laboratory.

Given T, we can now discuss the structure of the set of possible field motions. This will lead to the introduction of spacetime models. As discussed in the last section, the state of the electromagnetic field is given by pairs of forms on space, both of the same type and differing in degree by unity. Hence the dynamics of a given dynamical variable will be described by a pair of infinitely differentiable maps from T into the set of either even or odd forms on space, that is an element of one of the sets

$$\Omega_T^{k,\pm}(S) := \mathrm{C}^\infty(T;\Lambda^{k-1,\pm}(S)) \oplus \mathrm{C}^\infty(T;\Lambda^{k,\pm}(S)),$$

where $\Lambda^{k,+}(S)$ and $\Lambda^{k,-}(S)$ are the sets of even and odd k-forms on space respectively. For example the motion of B is an element of the set $\Omega_T^{2,+}$. The total set of possible motions is then just the collection of all such sets:

$$\Omega_T(S) := \oplus_{k,\pm}\Omega_T^{k,\pm}(S),$$

where k runs from 0 to 4.

Now time has a natural orientation provided by the flow of time: given a 1-form on T one integrates from the past to the future. This is important as one can then define a coordinate t which increases with the direction of the flow of time. This provides a nowhere-vanishing even 1-form dt and corresponding vector field ∂_t which will be used in the next section to define spacetime models. Essentially one splits the exterior algebra of a 4-manifold M using ∂_t. The two pieces then represent the two pieces of $\Omega_T(S)$. This will imply that the spacetime model M is diffeomorphic to $T \times S$.

5 Spacetime Models

As discussed in the last section, the dynamics of the electromagnetic field can be described by the bigraded algebra

$$\Omega_T(S) := \oplus_{k,\pm}\Omega_T^{k,\pm}(S).$$

Here the bigrading refers to the indices k and $\pm$. We will show that Maxwell's equations induce a canonical manifold structure for $\Omega_T(S)$, a dynamical object. To do this we note that Maxwell's equations give a natural derivation d on $\Omega_T(S)$. One can then represent $\Omega_T(S)$ by the set of differential forms on some 4-manifold M, unique up to diffeomorphism, in such a way that d becomes the exterior derivative.

Explicitly, with $B = (\mathbf{E},\mathbf{B})$, $H = (\mathbf{H},\mathbf{D})$ and $J = (\mathbf{J},\rho)$ we find that Maxwell's equations take the form $dB = 0$, $dH = J$ if we define the linear operator

$$d = \begin{bmatrix} d_S & w d_T \\ 0 & d_S \end{bmatrix}.$$

Here d_T is the derivative with respect to the time and d_S is the application of the exterior derivative on space pointwise in time. Explicitly, if we choose to write $\mathrm{C}^\infty(T;\Lambda^k(S))$ as $\Lambda^0(T)\otimes\Lambda^k(S)$ then $d_T = i_{\partial_t} d \otimes 1$

The operator w is defined by $w\mathbf{P} = (-1)^k\mathbf{P}$ for $\mathbf{P} \in \Lambda^{k,+}(S)$ and $w\mathbf{P} = (-1)^{n-k}\mathbf{P}$ for $\mathbf{P} \in \Lambda^{k,-}(S)$, where $n = 3$ is the dimension of the space manifold. The difference between the two is due to the fact that w is derived from the tensorial character of the object and not directly from its degree as a form. For example, as we have seen an odd 3-form on space is a scalar density. The trivial lifting of this operator to $\Omega_T(S)$ takes the matrix form

$$w = \begin{bmatrix} -w & 0 \\ 0 & w \end{bmatrix}.$$

Since d_T has degree 0 and d_S degree 1, we can treat $\hat{d} := (d_T, d_S)$ as an operator valued element of $\Omega_T^{1,+}(S)$. Multiplication of a form by this operator is equivalent to the action of d if we define the graded product

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \wedge \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} \alpha \wedge w\beta' + \beta \wedge \alpha' \\ \beta \wedge \beta' \end{bmatrix}.$$

The graded product $\wedge$ and derivation d provide $\Omega_T(S)$ with the structure of a cochain complex of degree 4, the map d being nilpotent as d_T and d_S commute. It is then natural to ask whether this complex can be realised as the exterior algebra of some 4-manifold M. This manifold would then provide a geometrical representation of the dynamics of the system and can thus be called a spacetime model. That this can in fact be done, and in only two ways, is given by the following theorem:

Theorem 5.1 *There exists a cochain complex isomorphism $\sigma : \Lambda(M) \to \Omega_T(S)$ if and only if there is a diffeomorphism $\mu : T \times S \to M$. In this case there are two dual diffeomorphisms $\mu_\pm$.*

Proof: We sketch a proof, discussing the physical ideas involved afterwards. Let $\mu : T \times S \to M$ be a diffeomorphism. Using the natural projections from $T \times S$ to T and S respectively we can construct continuous surjections $\pi_T : M \to T$ and $\pi_S : M \to S$ from the inverse diffeomorphism μ^{-1}. This allows us to define an even 1-form $\tau_\pm$ on M by setting $\tau = \pm\pi_T^* dt$. Note that by definition $d\tau_\pm = \pm\pi_T^* ddt = 0$. There then exists a unique vector field $X_\pm$ such that $\tau_\pm X_\pm = 1$ and $X_\pm \pi_S^* \mathbf{P} = 0$ for all $\mathbf{P} \in \Lambda(S)$. Define the operators $P_\pm = e(\tau_\pm) i_{X_\pm}$ and $Q_\pm = i_{X_\pm} e(\tau_\pm)$, where $e(\tau_\pm)$ is the exterior product with $\tau_\pm$ and $i_{X_\pm}$ is the interior product with $X_\pm$. One can then show that $P_\pm$ and $Q_\pm$ are dual projection operators on the exterior algebra $\Lambda(M)$. We write $\Lambda(M) = V_\pm(M) \oplus H_\pm(M)$, where $V_\pm(M) = P_\pm \Lambda(M)$ and $H_\pm(M) = Q_\pm \Lambda(M)$. A quick calculation shows that for all k, the linear spaces $H_\pm^k(M)$ and $V_\pm^{k+1}(M)$ are isomorphic and the linear spaces $H_\pm^k(M)$ and $C^\infty(T; \Lambda^k(S))$ are isomorphic. The cochain complexes $\Lambda(M)$ and $\Omega_T(S)$ are then isomorphic under the map

$$\sigma_\pm : \Lambda(M) \to \Omega_T(S) : \alpha \mapsto \begin{bmatrix} w\mu^* i_{X_\pm} P_\pm \alpha \\ \mu^* Q_\pm \alpha \end{bmatrix}.$$

This is the only class of suitable manifolds as two manifolds with isomorphic cohomology complexes are diffeomorphic, completing the proof. ■

The splitting argument used in the proof of the above theorem has a physical origin. The generalised Galilei principle of equivalence of reference frames asserts that the laws of physics should be expressible in any coordinate system. In particular we can choose to paste a right- or a left-handed orthogonal coordinate system for space to the time coordinate defined with the orientation of the flow of time. These two possibilities are nothing more than the two models given by the theorem and provide some justification for the two spacetime approach of A. Connes in the context of non-commutative geometry [Connes 1992].

The passage from one spacetime model to the other is achieved by the CPT map [Barut *et al.* 1993]. Hence one model describes the dynamics of particles and the other the dynamics of the corresponding antiparticles. Further the fact that dynamical covariance arises naturally in the context of such models means that the separate existence of space and time as physical entities is not in disagreement with relativity.

We stress that $\Omega_T(S)$ is a dynamical object. In the rest of this section we will use further dynamical considerations to induce additional structure on the manifold derived above. In particular, we will show how the phenomenological relations lead to the existence of a pseudo-metric. Note that the phenomenological relations describe the equilibrium response of space to electromagnetism; the current induces a field H which induces a field B as a property of the vacuum space. This point of view, where geometry arises from the dynamics of the system, is also discussed in terms of a different set of axioms by J. Audretsch and C. Lämmerzahl [1993] and A. L. L. Videira *et al.* [1985].

We proceed by showing that the phenomenological relations induce a natural pseudo-Hodge map $*$. The easiest way to see this is to express the pair of even forms $B = (\mathbf{E}, \mathbf{B})$ and the odd pair of forms $H = (\mathbf{H}, \mathbf{D})$ in the coordinates of a spacetime model. We write

$$B = \tfrac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu,$$
$$H = \tfrac{1}{4} H^{\mu\nu} \varepsilon_{\mu\nu\rho\lambda} dx^\rho \wedge dx^\lambda.$$

As B is even, the coefficients $B_{\mu\nu}$ are 2-covariant. However H is odd so that $H^{\mu\nu}$ transforms with a factor $|||\alpha|||$ and is 2-contravariant. Here $\|\alpha\|$ is the determinant of the transformation α and the extra factor appears so that, when combined with the factor of $\|\alpha\|^{-1}$ that comes from $\varepsilon_{\mu\nu\rho\lambda}$, the odd form H changes sign with the determinant of α just as the volume form. Note that for Yang-Mills fields these two forms are taken to have the same type. This leads to the existence of magnetic instead of electric monopoles, a very unphysical conclusion.

As discussed in section 3, at equilibrium these two forms are connected by the phenomenological relations, which then provide the conformal geometry of the spacetime model [see for example Sternberg 1978]. In the vacuum the phenomenological relations take the general coordinate form

$$B_{\mu\nu} = \tfrac{1}{2} \mu_{\mu\nu\rho\lambda} H^{\rho\lambda},$$

where the pseudotensor $\mu_{\mu\nu\rho\lambda}$ transforms with a factor $|||\alpha|||^{-1}$ and is 4-covariant. We will show that the phenomenological tensor $\mu_{\mu\nu\rho\lambda}$ can be written in terms of a symmetric matrix $\hat{g}_{\mu\nu}$ also acting in the tangent space which turns out to be invariant under the Weyl group. This matrix will allow us to write Maxwell's equations in terms of the generalised d'Alembertian of a canonical vector potential A, expliciting the quantum aspects of the theory.

Experiment shows that in a freely falling orthogonal reference frame the phenomenological relations of interest take the coordinate form

$$B_{ij} = \mu_0 H^{ij},$$
$$B_{0i} = \varepsilon_{ij}^{-1} H^{j0},$$

where μ_0 is a positive constant and ε_{ij}^{-1} is a positive symmetric matrix which is then such that at each point we can find a local coordinate system in which it is equal to a diagonal matrix with single eigenvalue ε_0. The gravitational equivalence principle then asserts that this local reference frame also diagonalises the mechanical inertia tensor. Far away from material bodies we have by definition that $\mu_0 = 4\pi.10^{-7}$ and $\varepsilon_0\mu_0 = c^{-2}$ in SI units.

In such a reference frame we have the following theorem:

Theorem 5.2 *There exists a matrix $\hat{g}_{\mu\nu}$, unique up to a sign, such that $\mu_{\mu\nu\rho\lambda} = \hat{g}_{\mu\rho}\hat{g}_{\nu\lambda} - \hat{g}_{\mu\lambda}\hat{g}_{\nu\rho}$, in fact*

$$\hat{g}_{\mu\nu} = \pm\mu_0^{1/2} \begin{bmatrix} \varepsilon_0^{-1}\mu_0^{-1} & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}.$$

Proof: The equations $B_{ij} = \frac{1}{2}(\hat{g}_{i\rho}\hat{g}_{j\lambda} - \hat{g}_{i\lambda}\hat{g}_{j\rho})H^{\rho\lambda} = \mu_0 H^{ij}$ and $B_{0i} = \frac{1}{2}(\hat{g}_{0\rho}\hat{g}_{i\lambda} - \hat{g}_{0\lambda}\hat{g}_{i\rho})H^{\rho\lambda} = \varepsilon_0^{-1}H^{i0}$ give the following two sets of relations

(i) $\hat{g}_{ii}\hat{g}_{jj} - \hat{g}_{ij}^2 = \mu_0$ for $i \neq j$

(ii) $\hat{g}_{00}\hat{g}_{ii} - \hat{g}_{0i}^2 = -\varepsilon_0^{-1}$

and for the off-diagonal part

(iii) $\hat{g}_{0j}\hat{g}_{ik} - \hat{g}_{0k}\hat{g}_{ij} = 0$

(iv) $\hat{g}_{00}\hat{g}_{ij} - \hat{g}_{0i}\hat{g}_{0j} = 0$ for $i \neq j$.

From (iii) we have that $\hat{g}_{0j}^2\hat{g}_{ik}^2 = \hat{g}_{0k}^2\hat{g}_{ij}^2$. If we eliminate the off-diagonal $\hat{g}_{\mu\nu}$ using (i) and (ii) we find that $(\hat{g}_{00}\hat{g}_{ii} + \varepsilon_0^{-1})(\hat{g}_{ii}\hat{g}_{kk} - \mu_0) = (\hat{g}_{00}\hat{g}_{kk} + \varepsilon_0^{-1})(\hat{g}_{ii}\hat{g}_{jj} - \mu_0)$. Expanding and regrouping leads to $(\hat{g}_{ii} - \hat{g}_{kk})(\varepsilon_0^{-1}\hat{g}_{jj} + \mu_0\hat{g}_{00}) = 0$. Hence either $\hat{g}_{ii} = \hat{g}_{kk}$ or $\hat{g}_{jj} = -\varepsilon_0\mu_0\hat{g}_{00}$. Thus at least three of the values $\alpha = \hat{g}_{11}$, $\beta = \hat{g}_{22}$, $\gamma = \hat{g}_{33}$ and $\delta = -\varepsilon_0\mu_0\hat{g}_{00}$ must be equal.

Suppose that $\alpha = \beta = \gamma$ (the other cases being treated by permutation). Then from (i) the three $\hat{g}_{ij}^2$ with $i \neq j$ must be equal and from (iii) so must the three $\hat{g}_{0i}^2$. Now, setting

$i = j$ in (iii) we have $\hat{g}_{0i}^2 \hat{g}_{ik}^2 = \hat{g}_{0k}^2 \hat{g}_{ii}^2$. However from (i) we see that $g_{ik}^2 \neq g_{ii}^2$ since μ_0 does not vanish and so $\hat{g}_{0i} = 0$. Since ε_0^{-1} does not vanish either, from (ii) we find that $\hat{g}_{00} \neq 0$. Using (iv) we then have that $\hat{g}_{ij} = 0$ for $i \neq j$. Finally (i) and (ii) give $\hat{g}_{ii} = \pm\mu_0^{1/2}$ and $\hat{g}_{00} = \mp\varepsilon_0^{-1}\mu_0^{-1/2}$. It is trivial to verify that this does indeed give a solution. Hence there is a diagonal matrix $\hat{g}_{\mu\nu}$, unique up to a sign, such that $\mu_{\mu\nu\rho\lambda} = (\hat{g}_{\mu\rho}\hat{g}_{\nu\lambda} - \hat{g}_{\mu\lambda}\hat{g}_{\nu\rho})$. ∎

The invariance properties of $\hat{g}_{\mu\nu}$ can then be deduced from the following lemma:

Lemma 5.3 *The matrix $\hat{g}_{\mu\nu}$ at a point is invariant under a linear transformation in the tangent space at this point if and only if $\mu_{\mu\nu\rho\lambda}$ is left invariant.*

Proof: According to a well-known theorem a linear transformation of the matrix $\hat{g}_{\mu\nu}$ cannot change its signature. Hence one cannot map $\hat{g}_{\mu\nu}$ into $-\hat{g}_{\mu\nu}$ and so $\mu_{\mu\nu\rho\lambda}$ is invariant if and only if $\hat{g}_{\mu\nu}$ is. ∎

One can easily see that $\hat{g}_{\mu\nu}$ transforms with a factor $|||\alpha|||^{-1/2}$ and is 2-covariant. It is then invariant at a given point under the Weyl group, that is the Lorentz transformations and the dilations acting on the tangent space at this point. To get a corresponding conformal group on the manifold itself one must integrate the corresponding Lie algebra. This depends on the global topology of the manifold as one must consider tangent vectors, which are defined over the whole manifold and not in the tangent space over one point. For instance, it is well known that the usual special conformal transformations have a singularity and so do not strictly speaking define a diffeomorphism of $\mathbb{R}^3$. The elements g of the conformal group act as coordinate transformations $g : x^\mu \mapsto x'^\mu$ and can be interpreted physically in a passive way. They map each motion of the field into another via the pullback g^*.

In resumé we can then write $B_{\mu\nu} = \frac{1}{2}(\hat{g}_{\mu\rho}\hat{g}_{\nu\lambda} - \hat{g}_{\mu\lambda}\hat{g}_{\nu\rho})H^{\rho\lambda} = \hat{g}_{\mu\rho}\hat{g}_{\nu\lambda}H^{\rho\lambda}$. This relation is very suggestive of the definition of the Hodge map for a Riemannian metric [de Rham 1984, p121] and we will write $B = *H$ in the following. This relation can be inverted to give $H = \tilde{*}B$, where $H^{\mu\nu} = \hat{g}^{\mu\rho}\hat{g}^{\nu\lambda}B_{\rho\lambda}$ and $\hat{g}^{\mu\nu}$ is the inverse matrix of $\hat{g}_{\mu\nu}$. For $B = dA$ we can then write $\Box A = J$ where $\Box = d\tilde{*}d$ is the generalised d'Alembertian acting on 1-forms. In local coordinates $\Box^{\mu\nu}A_\nu = J^\mu$ with $\Box^{\mu\nu} = \partial_\rho\hat{g}^{\mu\lambda}\hat{g}^{\rho\theta}\delta^{\pi\nu}_{\lambda\theta}\partial_\pi$, where $\delta^{\pi\nu}_{\lambda\theta}$ is the antisymmetrised Kronecker symbol. Note that the generalised d'Alembertian contains a factor of μ_0 arising from the factor of $\mu_0^{1/2}$ in $\hat{g}_{\mu\nu}$.

6 Decomposition of the Field

In this section we show that the fields introduced above allow a natural decomposition into transverse and longitudinal parts. This is of much importance when we search for a quantum theory of the photon, which must reduce to Maxwell's equations for coherent states. The transverse part of the field A is described by a de Rham current which can then be consistently treated as a quantum object. The longitudinal part of A is just a function and is thus a superselection variable [Piron 1969, D'Emma 1980].

To build a spacetime model we must choose a coordinate system in space to paste to the time coordinate defined with the orientation of time. Having chosen a particular coordinate system we can use ε_{ij}^{-1} to define a Riemannian metric on the manifold S. This provides us with a Hodge operator $*_S$ and a coderivative δ_S on space S in the usual way. If we consider a compact domain in space, such as the interior of a superconducting cavity, then the Hodge-Kodaira-de Rham theorem allows us to decompose the electromagnetic field in a canonical way.

The Hodge-Kodaira-de Rham theorem is based on the existence of a Green's function G for the generalised Laplacian $\Delta_S := d_S\delta_S + \delta_S d_S$ and a projector Π_Δ onto the set of harmonic forms. Note that a form is called harmonic if its generalised Laplacian vanishes. Such forms are functions. Each form which is closed and coclosed is harmonic, the reverse implication only holding in special cases such as when S is complete at infinity [de Rham 1984, p158]. Explicitly we have the following result [de Rham 1984, p134]:

Lemma 6.1 *Let S be a compact Riemann manifold. Then there exist unique linear operators Π_Δ and G acting on the exterior algebra $\Lambda(S)$ such that $\Pi_\Delta \circ d = d \circ \Pi_\Delta = 0$, $\Pi_\Delta \circ \delta = \delta \circ \Pi_\Delta = 0$, $\Pi_\Delta \circ * = * \circ \Pi_\Delta$, $\Pi_\Delta^2 = \Pi_\Delta$, $G \circ d = d \circ G$, $G \circ \delta = \delta \circ G$, $G \circ * = * \circ G$, $G \circ \Pi_\Delta = \Pi_\Delta \circ G = 0$ and $G \circ \Delta = \Delta \circ G = 1 - \Pi_\Delta$.*

In the proof the compactness of S is required so that the vector space of harmonic forms will be a finite-dimensional space of functions, allowing the algebraic construction of the corresponding projector. For non-compact spaces we must limit ourselves to the decomposition of square integrable forms, treating the problem by considering the limit of a sequence of forms with compact support. This approach appears naturally in the construction of the quantum mechanical interpretation of the theory, as the Hilbert space is generated by completing the set of forms of compact support in exactly this way. It is interesting to note that this idea is also equivalent to the usual approach of physicists when they enclose a system in a box of ever increasing size.

On $\Omega_T(S)$ we can then define the linear operators

$$P_\perp = \begin{bmatrix} \delta_S d_S G & 0 \\ 0 & \delta_S d_S G \end{bmatrix}, \qquad P_\| = \begin{bmatrix} d_S\delta_S G & 0 \\ 0 & d_S\delta_S \end{bmatrix}, \qquad P_\Delta = \begin{bmatrix} \Pi_\Delta & 0 \\ 0 & \Pi_\Delta \end{bmatrix}.$$

Using the lemma one readily shows that $P_\perp$, $P_\|$ and P_Δ form a set of mutually orthogonal projectors. Further one can define the operator

$$\theta = \begin{bmatrix} \delta_S G & 0 \\ 0 & \delta_S G \end{bmatrix}.$$

A quick calculation shows that $d\theta + \theta d = 1 - P_\Delta$ and $P_\perp\theta = \theta$, where we use the fact that d_T commutes with d_S and δ_S and thus with G and Π_Δ. If we are given an electromagnetic field B we can define the corresponding potential $A = \theta B$ and a direct computation shows that $dA = B - P_\Delta B$ and $P_\perp A = A$. Note that in general $P_\Delta B = 0$. This is the unique such field as $P_\perp = \theta d P_\perp$ since $\theta P_\perp = \theta P_\Delta = 0$. Hence we have a canonical description of B in terms of the physical field $A = (-V, \mathbf{A}_\perp)$.

The time-dependent 1-form $\mathbf{A}_\perp$ on S (which is a de Rham current in general) satisfies $\delta_S \mathbf{A}_\perp = 0$ and so is the transverse vector potential. The other component $-V$ is a function by construction. $\mathbf{A}_\perp$ is defined via its scalar product with a form. It is therefore nonlocal and defines in a natural way a coherent state whose mean values satisfy Maxwell's equations. On the other hand $-V$ possesses definite values which must be interpreted as superselection variables. Hence when we build an interpretation of the field A it is only $\mathbf{A}_\perp$ that is quantum. As there are only two linearly independent coexact 1-forms on space we find that there are only two helicities as required. Further the transverse vector potential does not depend on any particular gauge. In this way one demonstrates the identity of solid state photons quantised in the Coulomb gauge and the transverse photon quantised in the Lorentz gauge. The transverse current $\mathbf{J}_\perp = *_S \mathbf{J} + \partial_t \varepsilon_0 dV$ is responsible for the reversible production of the coherent photon state whereas the longitudinal current is responsible for the production of bremsstrahlung photons, a process which is irreversible and random.

7 Conclusion

We have shown how a geometric treatment of the dynamics of the electromagnetic field leads to the introduction of pairs of spacetime models. The resulting geometric structure can then be used to provide an intrinsic description of the transverse electromagnetic field, allowing the possibility of a gauge-free formulation of quantum electrodynamics.

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