

**Zeitschrift:** Helvetica Physica Acta  
**Band:** 67 (1994)  
**Heft:** 6

**Artikel:** Zero modes of Gauss' constraint in gaugeless reduction of Yang-Mills theory  
**Autor:** Khvedelidze, A. / Pervushin, V.  
**DOI:** <https://doi.org/10.5169/seals-116661>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 12.07.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

# Zero Modes of Gauss' Constraint in Gaugeless Reduction of Yang - Mills Theory

By A. Khvedelidze,<sup>1</sup> V. Pervushin

Bogoliubov Theoretical Laboratory, Joint Institute for Nuclear Research,  
Dubna, Russia

(24.III.1994, revised 15.VIII.1994)

*Abstract.* The physical variables for pure Yang - Mills theory in four - dimensional Minkowskian space time are constructed *without using a gauge - fixing condition* by the explicit resolving of the non - Abelian Gauss constraint. Gauge invariant variables are obtained by the use of the Bogoliubov transformation that diagonalizes the kinetic term in reduced action (action on constraint shell). As a result, the reduced action apart from a gauge invariant field variables depends on an additional global (only time - dependent) variable, describing zero mode dynamics of the Gauss constraint. >From the point of view of the gauge fixing method the additional variable gives the characteristic of the Gribov ambiguity: it is a collective variable corresponding to the group of transformation remaining after imposing the gauge condition.

The perturbation theory in terms of quasiparticles with the new stable vacuum, which is defined through the zero mode configuration, is proposed. The Green functions for quasiparticles are determined after the averaging over the collective variable. It is conjectured that the degeneracy of quasiparticles states with respect to zero mode of constraint can be cause of the color confinement.

---

<sup>1</sup>Permanent address : Tbilisi Mathematical Institute, Tbilisi, Georgia  
Electronic address: khved@theor.jinrc.dubna.su

# 1 Introduction

The identification of physical degrees of freedom of the non - Abelian gauge theory is a crucial point for understanding physical phenomena hidden in its structure. The procedure of identification of physical variables and their separation >from nonphysical ones has been called the reduction procedure. There are two ways for the realization of the reduction : *gaugeless* and *gauge fixing*. In the former, independent physical variables are constructed by the explicit resolution of constraints. In this case nonphysical variables disappear and the remaining gauge invariant variables describe a usual unconstrained system. To avoid the difficulties with the resolution of complete constraints, one commonly uses the general method of Dirac [1], gauge - fixing approach, based on the introduction into the theory of some new “ gauge constraints ” and on replacement of the Poisson bracket by Dirac’s one. However, there are also some problems in the Dirac approach. In particular, the gauge - fixing scheme is based on “gauge equivalence ” theorem. The rigorous proof of gauge independence is known in the assumption of existence of asymptotically free states of elementary particles [2], [3]. The extension of this result to more general cases, including a nonperturbative one, is quite problematic. Another problem of gauge - fixing procedure for the non - Abelian gauge theory is Gribov’s ambiguity [4] : there are many equivalent gauge field configurations obeying the same gauge condition. After the study of the space of orbits ( the space of gauge fields modulo the group of gauge transformations ) [5] it has been clear that the gauge - fixing procedure is not quite painless and it needs more rigorous treatment. For these reasons from time to time attempts have been undertaken to deal with the gaugeless method [6] - [11].

In this article we would like to demonstrate that the gaugeless method of quantization allows us to describe the gauge invariant content of Gribov’s ambiguity taking into account the collective variable inherent in gauge theory with nontrivial topological properties of gauge group.

According to the gaugeless approach [6], [7], [11], the physical variables in the Yang - Mills theory is constructed in the following way : the non - Abelian Gauss’ constraint is solved with respect to the nondynamical time component of gauge field, and then a new gauge invariant variables are constructed with the help of the Bogoliubov transformation [12] in terms of the constraint solution. The solution of Gauss’ equation for nondynamical variable is definite within the arbitrary time - dependent functions. These functions for simple cases (e.g., electrodynamics in infinite space time) are excluded from consideration owing to boundary conditions for gauge fields. In the general case these functions arise in the kinetic term of the constrained action and thus these zero modes are physical variables. In particular, Bogoliubov’s transformation is defined within the arbitrary time - dependent phase. The appearance of this phase in the framework of gaugeless method corresponds to the Gribov ambiguity in the gauge - fixing approach, where it can be treated as a collective variable for the group of transformations remaining after imposing the gauge condition.

The paper will be organized as follows. In Section 2 we shall briefly describe the gaugeless

method by example of QED in the four - dimensional Minkowskian space - time. Section 3 is devoted to the construction of physical variables in  $(1+3)$  dimensional  $SU(2)$  Yang - Mills theory with zero mode sector of Gauss' law. The connection is shown of the phase zero mode variable with the winding number functional. We prove a no - go theorem about the local realization of the representation of a homotopy group without the collective mode, and show that the presence of a zero mode leads to another realization different from the " instanton " one [13, 14]. In the last section the perturbation theory in terms of quasiparticles around the stable vacuum, corresponding to zero mode configuration, is proposed. All observables are determined after the averaging over the collective variable. It is noted, that the degeneration of physical states with respect to this collective variable can be cause of the color confinement.

## 2 Reduction of QED without gauge fixing

To introduce the method of gaugeless reduction, let us begin with an example of electrodynamics in the instant form

$$W[A, \psi, \bar{\psi}] = \int d^4x \left( \frac{1}{2} [(\partial_0 A^i - \partial^i A_0)^2 - B_i^2] + \bar{\psi}(i\hat{\partial} - m)\psi + eJ_\mu A^\mu \right), \quad (2.1)$$

$$B_i = \epsilon_{ijk} \partial^j A^k.$$

In the action (2.1) in accordance with the choice of time axis  $\eta \cdot x = x_0$ , the time component of vector field is distinguished. The main point of gaugeless reduction for electrodynamics is to resolve explicitly the Lagrangian constraint

$$\Delta A_0 = \partial^i \partial_0 A_i + J_0. \quad (2.2)$$

Within the zero modes of operator  $\Delta$  (explanation will be given in the next section ) we can write down the solution for (2.2) in the form of the following decomposition of  $A_0$ :

$$A_0 = A_0^{tr} + A_0^J, \quad A_0^{tr} = \frac{1}{\Delta} (\partial_0 \partial_j A^j), \quad A_0^J = \frac{1}{\Delta} J_0, \quad (2.3)$$

where  $A^{tr}$  is varied under the gauge transformations

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda,$$

while  $A_0^J$  remains invariant. The kinetic term in eq.(2.1) according to this decomposition can be diagonalized with the help of the Bogoliubov transformation to the new variables [6]:

$$\begin{aligned} A_k^I[A] &= v[A] (A_k - i \frac{1}{e} \partial_k) (v[A])^{-1}, \\ \psi^I[A] &= v[A] \psi, \end{aligned} \quad (2.4)$$

where

$$v[A] = \exp \{ i \int^t dt' A_0^{tr} \} = \exp \{ i e \frac{1}{\Delta} \partial^j A_j \}. \quad (2.5)$$

These variables are the gauge invariant functionals from initial gauge fields

$$A^I[A + \partial\Lambda] = A^I[A]$$

and satisfy, by the construction, the identity

$$\partial_i A_i^I[A] = 0. \quad (2.6)$$

Note that this identity is the consequence of the explicit resolution (2.3). Thus, the functional  $A^I[A]$  contains only two observable transverse fields

$$A_i^I = \sum_{a=1,2} e_i^a A_a^I$$

without selecting the Coulomb gauge as the initial supposition. The initial action (2.1) on the constraint shell (2.2) in terms of the new variables gets the form:

$$W^{Red}[A^I, \psi^I] = \int d^4x \left[ \frac{1}{2} \sum_{a=1,2} (\partial_\mu A_a^I \partial^\mu A_a^I) + \frac{1}{2} j_0^I \frac{1}{\Delta} j_0^I - j_i^I A_i^I + \bar{\psi}^I (i\hat{\partial} - m) \psi^I \right]. \quad (2.7)$$

So, we are ready to pass to the Hamiltonian form for our theory by using the conventional Legendre transformation for physical coordinates  $A_a^I$ . The quantization is achieved by imposing the canonical equal-time commutation relation between conjugate variables:

$$[A_a^I(x), E_b^I(y)] = \delta_{ab} \delta^3(x - y).$$

One can write down the generating functional for Green's function of the obtained unconstrained system in the form

$$Z_\eta^{Red}[s^I, \bar{s}^I, J^I] = \int \prod_a DA_a^I D\psi^I D\bar{\psi}^I e^{iW^{Red}[A^I, \psi^I, \bar{\psi}^I] + iS^I}, \quad (2.8)$$

with the external source term

$$S^I = \int d^4x \left( \bar{s}^I \psi^I + \bar{\psi}^I s^I + J_a^I A_a^I \right). \quad (2.9)$$

As to about on gauge and relativistic covariance in gaugeless scheme, there is subtle realization of Poincare symmetry – mixing of this rigid symmetry with gauge one

$$\begin{aligned} U_L^{-1} \psi(x) U_L &= S(L) \exp(ie\Lambda(x, L)) \psi(Lx), \\ U_L^{-1} A_\mu(x) U_L &= (L)_\mu^\nu \exp(ie\Lambda(x, L)) \left( A_\nu(Lx) + \frac{i}{e} \partial_\mu \right) \exp(-ie\Lambda(x, L)). \end{aligned} \quad (2.10)$$

In particular, it has been proved that the infinitesimal Lorentz transformation of coordinates with parameters  $\varepsilon^k$  corresponds to the transformation law for physical variables [10]:

$$A_i^I[A + \delta_L A] - A_i^I[A] = \delta_L A^I + \partial_i \Lambda[A^I],$$

with the conventional Lorentz variation  $\delta_L A = LA$  supplemented by the gauge one:

$$\Lambda[A^I, J] = \varepsilon^k \frac{1}{\Delta} [(\partial_0 A_k^I) + \partial_k J_0] .$$

This form was interpreted by Heisenberg and Pauli [15] (with reference to the unpublished note by von Neumann) as the transition from the Coulomb gauge with respect to the time axis in the rest frame  $\eta_\mu^0 = (1, 0, 0, 0)$  to the Coulomb gauge with respect to the time axis in the moving frame

$$\eta_\mu = \eta_\mu^0 + \delta_L \eta_\mu^0 = (L\eta^0)_\mu .$$

The Lorentz covariance of the reduced theory was proved in the quantum theory by B. Zumino [16] and means

$$Z_{L\eta}^{Red}[s^I, \bar{s}^I, J^I] = Z_\eta^{Red}[Ls^I, L\bar{s}^I, LJ^I]. \quad (2.11)$$

## 2.1 Gauge equivalence theorem

The usual form of the gauge – fixing Faddeev-Popov integral for generating functional of Green functions in the gauge  $F(A) = 0$  is

$$\begin{aligned} Z^F[s^F, \bar{s}^F, J^F] &= \int \prod_\mu DA_\mu^F D\psi^F D\bar{\psi}^F \Delta_{FP}^F \delta(F(A^F)) e^{iW[A^F, \psi^F, \bar{\psi}^F] + iS^F}, \\ S^F &= \int d^4x (\bar{s}^F \psi^F + \bar{\psi}^F s^F + J_\mu^F A_\mu^F) \end{aligned} \quad (2.12)$$

where  $W$  is the initial action and  $\Delta_{FP}$  is the Faddeev – Popov (FP) determinant.

As was proved [2], [3] by the changing variables of integration of the type of (2.4), this representation coincides with the gauge invariant reduced result (2.8) only for the choice of the following form of the source term

$$s^F = (v[A^F])^{-1} s^I, \quad \bar{s}^F = \bar{s}^I (v[A^F]), \quad J_0^F = 0, \quad \partial_i J_i^F = 0, \quad (2.13)$$

with the gauge transformation  $v[A^F]$  defined in (2.5).

In the Feynman diagrams for Green function this gauge factor  $v[A^F]$  leads to so-called spurious diagrams (SD). Just these spurious diagrams restore the Feynman rules (FR) for the reduced functional (2.8). Thus, one can present the identity

$$(FR)^F + (SD) \equiv (FR)^I \quad (\text{for Green's functions}) \quad (2.14)$$

as a consequence of independence of the functional integral (2.12) of the choice of variables. It can be verified that in calculation of the elements of  $S$ - matrix transition between the elementary asymptotical states these spurious diagrams disappear on the mass shell surface. As a result, we get

$$(FR)^F = (FR)^I \quad (\text{for elementary particles } S\text{- matrix}). \quad (2.15)$$

This statement is known as the gauge equivalence or independence theorem [2], [3]. When the asymptotic states contain a composite particle or some collective excitations, the equation (2.15) is quite problematic and we are sure only in the identity (2.14)

$$(FR)^F + (SD) \equiv (FR)^I \quad (\text{For } S - \text{matrix with composite particles}). \quad (2.16)$$

The violation of the gauge equivalence theorem (2.15) in this case does not mean the gauge noninvariance and relativistic noncovariance. This violation reflects the nonequivalence of the different definitions of the sources (2.9) and (2.13) because of nontrivial boundary conditions and residual interactions forming asymptotical composite, or collective states. In this context, the notion "gauge", in fact, is the gauge of sources in the FP - functional integral, but not only the choice of definite Feynman rules. As the gaugeless scheme takes into account explicitly the whole physical information from constraints, it is more correct to use this gauge - invariant and relativistic - covariant scheme for description of composite particles and collective excitations, rather than the Dirac approach with an arbitrary relativistic invariant "gauge of sources". Below, we would like to demonstrate the preference of the gaugeless scheme with an example of collective excitations in the Yang - Mills theory.

### 3 GAUGELESS REDUCTION of YANG – MILLS THEORY

#### 3.1 Zero mode of Gauss' law

Now we pass to the reduction of the Yang - Mills theory with the local  $SU(2)$  group in four - dimensional Minkowskian space - time

$$W[A_\mu] = -\frac{1}{4} \int d^4x F_{\mu\nu}^a F_a^{\mu\nu} = \frac{1}{2} \int d^4x (F_{0i}^{a2} - B_i^{a2}), \quad (3.1)$$

with the usual definitions of non-Abelian electric tension  $F_{0i}^a$

$$F_{0i} = \partial_0 A_i^a - \nabla(A)_i^{ab} A_0^b, \quad \nabla_i^{ab} = (\delta^{ab} \partial_i + e \epsilon^{acb} A_i^c),$$

and magnetic one  $B_i^a$

$$B_i^a = \epsilon_{ijk} \left( \partial_j A_k^a + \frac{e}{2} \epsilon^{abc} A_j^b A_k^c \right).$$

The reduction consists in the explicit resolution of non-Abelian Gauss' law

$$\frac{\delta W}{\delta A_0^a} = 0 \implies [\nabla^2(A)]^{ac} A_0^c = \nabla^{ac}_i(A) \partial_0 A_i^c \quad (3.2)$$

and next in dealing with the initial action (3.1) on the surface of these solutions

$$W^{Red} = W[A_\mu] \Big|_{\frac{\delta W}{\delta A_0^a} = 0}. \quad (3.3)$$



Let us choose some particular solution of the constraint (3.2) with the property

$$\lim_{|\vec{x}| \rightarrow \infty} a_0^c(\vec{x}, t) = 0 \quad (3.4)$$

and write down the general solution as a sum of this particular solution  $a_0$  and the general solution  $\Phi$  (zero mode field) of the homogeneous equation:

$$A_0^c = -\Phi^c + a_0^c, \quad [\nabla^2(A)]^{ac} \Phi^c = 0. \quad (3.5)$$

In the next step the QED gauge invariant variables (2.4) can be generalized as in ref. [6]:

$$\begin{aligned} \hat{A}_k^I &= v^I[A](\hat{A}_k + \partial_k)(v^I[A])^{-1}, \\ \hat{\Phi}^I &= v^I[A]\hat{\Phi}(v^I[A])^{-1}, \quad \hat{A} = e^{\frac{A^a \tau^a}{2i}}, \end{aligned} \quad (3.6)$$

where the non - Abelian Bogoliubov transformation  $v^I[A]$  is given with the help of the “good” solution  $a_0(\vec{x}, t)$

$$v^I[A] = T \exp\left\{\int^t dt' \hat{a}_0\right\}. \quad (3.7)$$

In terms of these variables the reduced action takes the form

$$W^{Red}[A^I \Phi^I] = \frac{1}{2} \int d^4x \left[ (\partial_0 A_i^I + \nabla_i(A^I) \Phi^I)^2 - B_i^2 \right], \quad (3.8)$$

where the fields  $\partial_0 A_i^I$ ,  $\Phi^I$ ,  $B_i$  satisfy the geometrical constraints

$$\begin{aligned} \nabla_i^{ab}(A^I) \partial_0 A_i^{bI} &= 0, \\ \nabla_i^{ab}(A^I) \nabla_i^{bc}(A^I) \Phi^{Ic} &= 0, \\ \nabla_i^{ab}(A^I) B_i^b(A^I) &= 0. \end{aligned} \quad (3.9)$$

It is clear that due to (3.9) the reduced action depends on the zero mode field  $\Phi$  only through the surface terms on spatial infinity  $I_E$  and  $I_\Phi$ :

$$I_E = \int d^3x \partial_0 A_i^I \nabla_i \Phi \equiv \int d^3x \partial_i (\partial_0 A_i \Phi) = \oint ds_i (\partial_0 A_i^I \Phi) \Big|_{|\vec{x}| \rightarrow \infty}, \quad (3.10)$$

$$I_\Phi = \frac{1}{2} \int d^3x (\nabla_i \Phi)^a (\nabla_i \Phi)^a \equiv \frac{1}{4} \int d^3x \Delta (\Phi^a)^2 = \frac{1}{4} \oint ds_i \partial_i (\Phi)^2 \Big|_{|\vec{x}| \rightarrow \infty}, \quad (3.11)$$

$$W^{Red}[A, \Phi] = \frac{1}{2} \int d^4x \left[ (\partial_0 A_i^I)^2 - (B_i)^2 \right] + \int dt (I_E + I_\Phi). \quad (3.12)$$

Emphasize that in accordance with the property (3.4) the field  $A^I$  at the spacial infinity can be only stationary

$$A_i^I(\vec{x}, t) \Big|_{|\vec{x}| \rightarrow \infty} = b_i(\vec{x}), \quad (3.13)$$



that means the diagonalization of the kinetic term in the action (3.8): ( $I_E = 0$ ).

The background field  $b(\vec{x})$  can be treated as the zero mode of the time derivative operator in the Gauss law  $\partial_0 b_i^I = 0$ . As a consequence, we have the factorizable form for the zero mode field :

$$\Phi(t, \vec{x})^a \Big|_{|\vec{x}| \rightarrow \infty} = \varphi_0(t) \Phi_0^a(\vec{x}). \quad (3.14)$$

To clear up the meaning of  $\varphi_0(t)$  recall that by definition the variables  $A^I$  describe local excitation, while zero mode field  $\Phi$  is associated with some global (collective) dynamics of gauge fields. Such global properties of the theory are connected with the well - known topological invariant [13], [14]

$$\nu = \frac{e^2}{64\pi^2} \int d^4x F^{\nu\mu} \tilde{F}_{\mu\nu}. \quad (3.15)$$

On the constraint shell the quantity  $\nu$  can be represented as

$$\nu^{Red} = \nu \Big|_{\text{constraint}} = \int d^4x \left( \partial_0 A_i^I + \nabla_i \Phi \right) \bar{B}_i, \quad \bar{B}_i^a = \frac{e^2}{8\pi^2} B_i^a. \quad (3.16)$$

We can find the connection between the time - dependent part of the zero mode  $\varphi_0(t)$  and this topological invariant after using the following decomposition for  $\nu$  on the local and global parts:

$$\nu^{Red} = \int dt \partial_0 N_T[A^I, \Phi], \quad N_T[A^I, \Phi] = N_L[A^I] + N_0, \quad (3.17)$$

$$N_L[A] = \frac{e^2}{16\pi^2} \int d^3x \epsilon_{ijk} \left( A_i^a \partial_j A_k^a + \frac{1}{3} \epsilon^{abc} A_i^a A_j^b A_k^c \right), \quad (3.18)$$

$$\partial_0 N_0 = \oint d^3x \bar{B}_i \nabla_i \Phi = \oint ds_i \left( \Phi \bar{B}_i \right) \Big|_{|\vec{x}| \rightarrow \infty}. \quad (3.19)$$

Comparing (3.14) and (3.19) we get the desirable connection

$$\varphi_0 = \partial_0 N_0 I_B^{-1}, \quad (3.20)$$

where the constant  $I_B$  is determined via the stationary part of fields (3.13) at spatial infinity

$$I_B = \oint ds_i \left( \Phi_0 \bar{B}_i \right) \Big|_{|\vec{x}| \rightarrow \infty}. \quad (3.21)$$

Finally, our reduced action gets the form:

$$\begin{aligned} W^{Red}[A^I, N_0] &= W_L^{Red}[A^I] + W_G^{Red}, \\ W_L^{Red}[A^I] &= \frac{1}{2} \int d^4x \left[ \left( \partial_0 A_i^I \right)^2 - (B_i)^2 \right], \\ W_G^{Red} &= \int dt (\partial_0 N_0)^2 I, \quad I = I_\Phi / I_B^2. \end{aligned} \quad (3.22)$$

>From the reduced action (3.22) with zero mode we get an unexpected result:

*There is a static solution of pure Yang - Mills theory with finite energy in four - dimensional Minkowskian space - time [11].*

One can show that one of such static solutions coincides with the well - known Bogomoln'yi - Prasad - Sommerfield (BPS) monopole solution of the Yang - Mills - Higgs system [17]. To prove this statement, it is enough to observe that the zero mode in the action (3.8) plays the role of the Higgs stationary field. Indeed, the Prasad - Sommerfield solution of the Bogomoln'yi equation

$$\nabla_i^{ac}(A)\Phi_0^c = \frac{2\pi}{\mu}B_i^a(A), \quad (3.23)$$

$$\begin{aligned} A_i^a &= \frac{1}{e}\epsilon^{abi}m^e \left[ \frac{\mu}{\sinh(\mu r)} - \frac{1}{r} \right]; \quad m^l = \frac{x^l}{r}; \quad r = |\vec{x}|, \\ \Phi_0^a &= \frac{2\pi}{e}m^a \left[ \coth(\mu r) - \frac{1}{\mu r} \right], \end{aligned} \quad (3.24)$$

with  $\mu$  being the parameter of the mass dimension automatically satisfies the zero mode equation (3.5) and our boundary conditions. For these field configurations, the constants (3.11), (3.21) are the following

$$I_B = 1, \quad I = I_\Phi = \frac{2(2\pi)^3}{\mu e^2}. \quad (3.25)$$

It is interesting to note that there are arguments [18], [19] in favour of stability of this perturbation theory under small deformations around this vacuum background.

### 3.2 Zero mode and homotopy group

It is usually assumed that all nontrivial topological properties of the gauge theory are connected with the existence of classical solutions of the Euclidean Euler - Lagrange equations with finite action — instantons. Recall that instanton calculations are based on consideration of the topological nontrivial gauge symmetry group [13], [14]:

$$\hat{A}_\mu \rightarrow \hat{A}_\mu^g = g(\hat{A}_\mu + \partial_\mu) g^{-1},$$

with restriction on the class of stationary transformations  $\{g(\vec{x})\}$  with the asymptotical property

$$\lim_{|\vec{x}| \rightarrow \infty} g(\vec{x}) = 1.$$

One can use the assumption of compactification of three - dimensional space into a three-sphere  $S^3$ . In this case all maps  $\{g(x)\} : S^3 \rightarrow SU(2)$  can be split into the disjoint homotopy classes characterized by the integer index  $n$ :

$$n = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{tr} [\hat{V}_i \hat{V}_j \hat{V}_k]; \quad \hat{V}_i = g \partial_i g^{-1}.$$

Thus, we can speak about the homotopy group  $\pi_3(SU(2)) = Z$ . The configurations belonging to the different classes cannot be deformed continuously into each other. The gauge transformations which are deformable to identity are called small, while homotopically non-trivial ones  $n \neq 0$  are large. For large gauge transformations the local topological variable  $N_L[A]$  (3.18) varies as

$$N_L[A^g] = N_L[A] + n. \quad (3.26)$$

The group of these transformations is usually considered in the context of instanton approximation [13] for the Green functional in the Euclidean space

$$G^{Euc}(A_1, A_2; T) = \int_{A_1}^{A_2} [DA] e^{-W_{Euc}},$$

according to which the contribution from the self - dual fields ( $E = \pm B$ ) dominates in the vacuum sector. One can write the spectral decomposition of this functional

$$G^{Euc}(A_1, A_2; T) = \sum_{\epsilon} e^{-\epsilon T} \Psi_{\epsilon}(A_1) \Psi_{\epsilon}^*(A_2),$$

with wave functions satisfying the following set of equations

$$H_L \Psi_{\epsilon} = \epsilon \Psi_{\epsilon}, \quad (3.27)$$

$$\nabla_i E_i \Psi_{\epsilon} = 0, \quad (3.28)$$

$$T_L \Psi_{\epsilon} = e^{i\theta} \Psi_{\epsilon}. \quad (3.29)$$

The first equation is the stationary Schrödinger equation with the Hamiltonian

$$H_L[A, E] = \int d^3x \frac{1}{2} (E_i^2 + B_i^2).$$

Eq.(3.28) reflects the invariance of the theory under the small gauge transformations, while eq.(3.29) describes the covariance properties of the wave function under a large gauge transformation with the topological shift operator  $T_L$  represented in the following form :

$$T_L = \exp \left\{ \frac{d}{dN_L[A]} \right\},$$

where  $N_L$  is the functional (3.18). This form is justified in refs. [13] by representing the solution of (3.27) – (3.29) in the form of the Bloch wave function

$$\Psi_{\epsilon}(N_L, A) = e^{iP N_L} \Psi_{\epsilon}(A)$$

with the exact solution with energy  $\epsilon = 0$

$$\Psi_0 = \exp \left\{ \pm \frac{8\pi^2}{e^2} N_L[A] \right\},$$

which represents the quantum version of the instanton solution ( $\hat{E}\Psi_0 = \pm \hat{B}\Psi_0$ ). However, this solution is nonphysical (nonnormalizable). It is easy to check also that the operators  $H_L, T_L$  do not commute

$$[H_L, T_L] \neq 0, \quad [[H_L, T_L], T_L] \neq 0,$$

therefore they cannot have a common system of physical eigenstates. So, for such local realization of the topological shift operator  $T_L$  the following statement is valid:

No - go theorem: *There are no physical solutions for equations (3.27) – (3.29).*

These obstacles can be overcome in our gaugeless consideration where the winding number is represented according to (3.17) as a sum of the local functional  $N_L$  and some collective variable  $N_0$ . Introduction of this new variable allows a consistent description of the representation of the homotopy group. Indeed, starting with reduced action one can get the following Hamiltonian:

$$H^{Red} = \frac{1}{2I} \hat{P}_0^2 + H_L[A^I, E^I], \quad (3.30)$$

as the sum of the local Hamiltonian  $H_L$  and the global one with canonical momentum conjugated to the global variable  $N_0$

$$P_0 \equiv \frac{\delta W^{Red}(N_0)}{\delta \partial_0 N_0} = \partial_0 N_0. \quad (3.31)$$

The requirement of the invariance under the large gauge transformations (3.26) leads to the region of definition of the variable  $N_0$  —  $[0, 1)$ . In this case eqs. (3.27)-(3.29) transform as follows

$$\begin{aligned} H^{Red} \Psi_\epsilon &= \epsilon \Psi_\epsilon, \\ \nabla_i E_i \Psi_\epsilon &= 0, \end{aligned} \quad (3.32)$$

$$T_G \Psi_\epsilon = e^{i\theta} \Psi_\epsilon, \quad (3.33)$$

where

$$T_G = \exp(i\hat{P}) = \exp\left(\frac{d}{dN_0}\right). \quad (3.34)$$

These equations admit the factorization of the wave function on the plane wave describing the topological collective motion, with the momentum spectrum:

$$P_0 = 2\pi k + \theta$$

and on the oscillator - like part depending on transverse variables :

$$\Psi_\epsilon(N_0, A^I) = \langle P_0 | N_0 \rangle \Psi_L[A^I] \quad \langle P_0 | N_0 \rangle = e^{iP_0 N_0}. \quad (3.35)$$

Thus, the consistent solution of the problem of the quantization of the Yang – Mills theory with nontrivial homotopy group is achieved by introducing an additional global variable arising in the process of reduction as zero mode of Gauss' constraint.

### 3.3 Zero mode and topological confinement

Generally speaking, for constrained system in the "gauge"

$$\nabla_i^{ab}(A) \partial_0 A_i^b = 0 \quad (3.36)$$

the generating functional for Green's function can be constructed [7] by using the conventional Faddeev – Popov functional integral [3]:

$$Z[J] = \int \prod_{\mu} D^4 A_{\mu} [\det(\nabla^2(A_i))] \delta(\nabla_i(A) \partial_0 A_i) e^{iW[A] + iS[J]}.$$

After the integration over  $A_0$  it can be rewritten as

$$Z[J] = \int D^3 A_i [\det(\nabla^2(A_i))]^{1/2} \delta(\nabla_i(A) \partial_0 A_i) e^{iW^{Red}[A] + iS[J]}. \quad (3.37)$$

The calculation of this functional integral faces the problems of the existence of zeroes of FP determinant well known as the Gribov ambiguity of fixing variables [4]. In the gaugeless approach this ambiguity corresponds to the existence of the zero mode sector (3.5) and, accordingly, of two types of variables (3.6) and

$$\begin{aligned} A_k^{\Phi} &= v^{\Phi} (A_k^I + \partial_k)(v^{\Phi})^{-1}, \\ v^{\Phi} &= T \exp\left\{\int^t dt' \hat{\Phi}^I\right\}, \end{aligned} \quad (3.38)$$

satisfying one and the same gauge constraint (3.36). The above - introduced variables  $A^I$  (3.6) are invariant under the small gauge transformations, while the variables  $A^{\Phi}$  are invariant against the large one. By the construction, the gauge factor  $v^{\Phi}$  neutralizes the large gauge transformation, as the factor  $v^I$  neutralized the small one in (3.6) and the total topological variable has the form

$$N_T = N_L[A^I] + N_0 = N_L[A^{\Phi}] + \text{Invariant term}. \quad (3.39)$$

In particular, for the BPS fields (3.24), the second Bogoliubov transformation is

$$v^{\Phi} = e^{iN_0 \tau^a m^a \beta(r)}, \quad (3.40)$$

with function  $\beta(r) = 2\pi[\coth(\mu r) - \frac{1}{\mu r}]$ , and the invariant term in eq.(3.39) has the following form:

$$\text{Invariant term} = -\frac{\sin(2\pi N_0)}{2\pi}. \quad (3.41)$$

It is important to note that in the conventional Hamiltonian approach Gauss' constraint is considered as the generator of small gauge transformations. In the gaugeless approach one can be convinced that Gauss' constraint is responsible for both gauge transformations: small and large. The small gauge transformations are generated by Gauss' constraint without zero mode, while the large by the zero mode (3.34).

In the gaugeless approach, the requirement of gauge invariance of the observable quantities under the whole group of transformations (small and large) means that we should work in terms of variables  $A^\Phi$ . The gaugeless reduced configuration space besides the variables  $A^\Phi$  contains also the above - introduced topological variable  $N_0$  and, therefore, in the expression of the corresponding generating functional there is an additional functional integral over it. To write down the generating functional we must separate the stationary asymptotic part  $b$ , which is accompanied by  $N_0$ , from the dynamical one

$$\hat{A}_k^\Phi(\vec{x}, t) = v^\Phi \left( \hat{b}_k^I(\vec{x}) + \partial_k + \hat{a}_k^I(\vec{x}, t) \right) v^\Phi = v^\Phi \left( \hat{b}_k^I(\vec{x}) + \partial_k \right) v^\Phi + a^\Phi(\vec{x}, t).$$

where

$$(a^\Phi)^c = \Omega^{cd}(\vec{x}|N_0(t))(a^I)^d,$$

the matrix  $\Omega^{cd}(\vec{x}|N_0(t))$  realizes the transformation (3.38) in the adjoint representation of the color group

$$v^\Phi \hat{\tau}^a (v^\Phi)^{-1} = \Omega^{ab} \tau^b.$$

In the following we shall call  $b$  the condensate and  $a$  the quasiparticle excitation.

Thus, instead of (3.37) in the gaugeless reduce scheme we have the representation for the generating functional of Green's functions:

$$\begin{aligned} Z_{PQ}^{Red}[J^\Phi] &= \int_{P_0(0)=P}^{P_0(T)=Q} DP_0 DN_0 e^{W_G^{Red}[P_0, N_0]} \int [D^3 a_i] e^{iW_L^{Red}[b^I + a] + iS^\Phi}, \\ \int [D^3 a_i] &= \int D^3 a_i [\det(\nabla^2(b^I + a))]^{1/2} \delta(\nabla_i(b^I + a) \partial_0 a_i) \end{aligned} \quad (3.42)$$

with

$$S^\Phi[J^\Phi, \Phi] = \int d^4x J_b^\Phi a_b^\Phi = \int d^4x J_b^\Phi a_c \Omega^{cb}(x|N_0(t)).$$

The generating functional (3.42) is free from zeroes of FP determinant and corresponds to gauge invariant Green's functions of quasiparticles

$$G_\Phi(1, \dots, n) = \langle vac P | T [a^\Phi(1) \cdots a^\Phi(n)] | vac Q \rangle. \quad (3.43)$$

In Eq.(3.43) the vacuum vector  $|vac Q\rangle$  means the state without quasiparticles and with definite topological momentum  $Q$ .

The perturbation theory with respect to the quasiparticles in the background  $b$  is constructed by the decomposition of action

$$W^{Red} = W_G + W_0[b] + \frac{1}{2} \int d^4x [(\partial_0 a)^2 - a(\tilde{\Delta})a] + W_{int}[a, b], \quad (3.44)$$

where  $\tilde{\Delta}$  is the differential operator :

$$(\tilde{\Delta})_{ij}^{cd} = \delta_{ij}(\nabla^2(b))^{cd} + 2eF_{ij}^a(b)\epsilon^{cad}.$$

After the introducing the complete set of eigenfuctions

$$(\tilde{\Delta})_{ij}^{cd} f_j^d(\vec{x}|w) = w^2 f_i^c(\vec{x}|w) \epsilon^{cfd},$$

we can write the following expansion for the field  $a^I$  :

$$a^I(\vec{x}, t) = \sum_w \left( c^+(w) f_j^d(\vec{x}|w) e^{+iwt} + c^-(w) f_j^d(\vec{x}|w) e^{-iwt} \right).$$

In the canonical operator quantization the coefficient  $c^+(w)(c^-(w))$  is the creation (annihilation) operator of quasiparticles with the asymptotical Hamiltonian

$$H_L^{Asympt} = \sum_w w c_w c_w^+.$$

The function  $f_j^d(\vec{x}|w)$  is the amplitude of probability to find the quasiparticle with the energy  $w$  at the point  $\vec{x}$ . For observable quasiparticle  $a^\Phi$  this amplitude has the form :

$$\langle vac P | a_j^{\Phi c}(\vec{x}, 0) | vac Q \rangle = \int_0^1 dN e^{iN(P-Q)} \Omega^{cd}(\vec{x} | N_0(0) = N) f_j^d(\vec{x}|w).$$

The factor  $\Omega$  reflects the degeneration of the quasiparticle energy under the topological variable and leads to the sum of Kronecker symbols  $\delta_{P,Q \pm \pi\beta(r)}$ . This result can be treated as confinement of color states [7], [10]. For colorless states topological degeneration factors disappear and we get the conventional expressions for corresponding matrix elements. This scheme of topological confinement will be discussed in more detail in the forthcoming publication . It is worth to note that there are the values of the coupling constant  $e^2 = 2(|2\pi k + \theta|)^{-1}$  for which the background part  $W_0[b]$  of the action (3.44) is compensated by the collective motion one  $W_G$ .

## 4 Summary

We have discussed the method of quantization of the Yang - Mills theory in which the non-dynamical variables are eliminated by the explicit resolving of the classical Gauss constraint

$$\nabla^2(A) A_0 - \nabla_i(A) \partial_0 A_i = 0,$$

and then quantum theory is built up for the action on the constraint shell. We take into account the zero modes of all operators in the constraint:

– the condensate  $b(x)$  as a zero mode of the operator  $\partial_0$

$$\partial_0 b(x) = 0,$$

– the phase angle  $\Phi(x)$  as a zero mode of the operator  $\nabla^2(A)$

$$\nabla^2(A) \Phi = 0.$$



This zero mode is associated with the collective coordinate which restores the gauge invariance of the theory, broken in any particular solution of this constraint. At this point the situation is quite similar to the case of semiclassical soliton quantization, based on the introduction of collective coordinates [20] for taking into account the breaking of rigid global symmetries (e.g., translation, rotation, etc.). The collective variable allows us to separate zero modes of the Faddeev - Popov determinant and to solve the problem of gaugeless version of Gribov's copies.

There is a significant difference between the Yang - Mills theory with the collective variables and the conventional one. In particular, it has been shown that in the former there is a static stable solution that corresponds to the well - known Prasad - Sommerfield solution in the conventional Yang -Mills theory interacting with Higgs' field. In the gaugeless reduced Yang - Mills the zero mode  $\Phi(x)$  plays the role of the Higgs field .

The collective variable associated with the phase angle  $\Phi$  describes the dynamical realization of the homotopy group  $\pi_3(SU(2)) = Z$  in the Minkowskian space. In contrast to the instanton version with integer winding number functional, in the dynamical realization the collective variable as winding number is continuous. Just the averaging over the continuous winding number leads to confinement of gauge invariant color fields configuration due to the phenomenon of the complete destructive interference.

### Acknowledgments

The authors thank Profs. P.I. Fomin, B. Ovrut, S. Sawada, E. Seiler, M. K. Volkov and Drs. S. Gogilidze, A. Kvinikhidze for useful discussions. The work was supported in part by the Russian Foundation of Fundamental Investigations, Grant No 94-02-14411.

### References

- [1] P.A.M. Dirac, Phys. Rev. 114 (1959) 924;
- [2] J.C. Taylor, Nucl. Phys. B33 (1971) 436;  
A.A.Slavnov, Teor. Mat. Fiz. 10 (1972) 99;
- [3] L.D.Faddeev, A.A.Slavnov, *Introduction in the Quantum Theory of Gauge Fields* ,  
Moscow, Nauka, 1984;
- [4] V.N.Gribov, Nucl. Phys. B 139 (1978) 1;
- [5] I.M.Singer, Commun. Math. Phys. 60 (1978) 7;  
Physica Scripta. 24 (1981) 817;
- [6] V.N.Pervushin, Teor. Mat. Fiz. 45 (1980) 327;

- [7] V.N.Pervushin, Riv. Nuovo Cimento 8 (1985) N 10, 1;
- [8] N.P.Ilieva, V.N.Pervushin, Sov. J. Par. Nucl. 22 (1991) 573;
- [9] L.D. Faddeev, R. Jackiw, Phys. Rev. Lett. 60 (1988) 1692;  
R.Jackiw, (*Constrained*) *Quantization without tears* , preprint CTP - 2215 MIT, Cambridge, 1993;
- [10] Nguyen Suan Han, V.N. Pervushin, Fortsch. Phys. 37 (1989) 611;
- [11] A.M. Khvedelidze, V.N. Pervushin, *Zero modes of first class constraints in gauge theories*, preprint JINR E2-93-439, Dubna, 1993 ;
- [12] N.N.Bogoliubov, Sov. J. Phys. JETP 1 (1958) 51;
- [13] A.A.Belavin et al., Phys.Lett 59 (1975) 85;  
R.Jackiw, C.Rebbi, Phys. Lett. 63B (1976) 172;  
R.Callan, R.Dashen, D.Gross, Phys. Rev. D17 (1977) 2717;
- [14] L.D.Faddeev, Proc. Fourth Intern. Symposium on Nonlocal Field Theory, 1976, D1-9788, Dubna, (1977), 267;
- [15] W. Heisenberg, W. Pauli, Z.Phys. 56 (1929)1; ibid 59 (1930) 166;
- [16] B.Zumino, J. Math. Phys. 1 (1960) 1;
- [17] M.K.Prasad, C.M.Sommerfield, Phys.Rev.Lett. 35 (1975) 760,  
E.B. Bogomol'nyi Yad. Fiz 24 (1976) 449;
- [18] H. Leutwyler, Nucl.Phys B179, (1981), 129;
- [19] R.Jackiw, C.Rebbi, Phys. Rev. D13 (1975) 3398;  
A.S.Goldhaber, Phys. Rev. Lett. 36 (1976) 1122;  
P.Hasenfratz, G.'t Hooft, Phys. Rev. Lett. 36 (1976) 1119;
- [20] J.L.Gervais, A.Jevicki, B.Sakita, Phys. Rep. 23C (1976) 281;