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Flat wormholes from straight cosmic strings

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Abstract. Special multi-cosmic string metrics are analytically extended to describe configurations of Wheeler-Misner wormholes and ordinary cosmic strings. I investigate in detail the case of flat, asymptotically Minkowskian, Wheeler-Misner wormhole spacetimes generated by two cosmic strings, each with tension $-1/4G$.

1 Introduction

The wormholes I shall discuss are traversable, Lorentzian wormholes [1]. These are best defined by a simple example. Remove from Euclidean space R^3 a volume Ω. Take a second, identical copy of $R^3 - \Omega$, and either identify these two excised spaces along the boundaries $\partial\Omega$, or connect them by a cylindrical tube $\Gamma \times \partial\Omega$. The resulting spatial geometry is a model of an Einstein-Rosen [2] traversable wormhole, with two asymptotically flat regions (actually, these are everywhere flat except on $\partial\Omega$). One may similarly construct a model of a Wheeler-Misner [3] wormhole, with only one asymptotically flat region, by removing from R^3 two non-overlapping "identical" volumes Ω and Ω', and connecting the boundaries $\partial\Omega$ and $\partial\Omega'$ by a cylindrical tube. It has been speculated that travelling through such hypothetical wormholes could shorten intergalactic journeys.

Visser [4] discussed the special case where Ω is a polyhedron, and showed that in this case the curvature of the boundaries $\partial\Omega$ is concentrated on the edges. The resulting Visser wormholes are thus generated by arrays of cosmic strings (the polyhedra). In this talk I shall consider the case of parallel cosmic strings, following an analytical approach complementary to the geometrical approach just outlined. I shall first show how special multi-cosmic string metrics may be analytically extended to Einstein-Rosen or Wheeler-Misner multi-wormhole, multi-cosmic string metrics. Then I shall focus on the special cases of Wheeler-Misner wormholes generated by two cosmic strings, or by a single cosmic string.

2 Wormholes from cosmic strings

The well-known multi-cosmic string metric is

$$ds^2 = dt^2 - d\sigma^2 - dz^2, \tag{2.1}$$

where the 2-metric

$$d\sigma^2 = \prod_i |\zeta - a_i|^{-8Gm_i}\, d\zeta\, d\zeta^* \tag{2.2}$$

($\zeta \equiv x + iy$) is everywhere flat:

$$d\sigma^2 = dw\, dw^* = du^2 + dv^2\,, \tag{2.3}$$

except for conical singularities (branch points of $\zeta(w)$) with deficit angles $8\pi Gm_i$, located at the points $\zeta = a_i$.

Consider the special case of the bicone with $m_1 = m_2 = 1/8G$,

$$d\sigma^2 = \frac{d\zeta\, d\zeta^*}{|\zeta^2 - b^2|}\,. \tag{2.4}$$

This may be analytically extended to a geodesically complete surface: a cylinder. To show this [5], pinch the cylinder along a parallel. We thus obtain two identical bicones with deficit angles π at the two vertices, joined along the pinch. These two bicones are diffeomorphic to the two sheets of the Riemann surface of the metric (2.4) cut along the segment connecting the two branch points. The diffeomorphism is implemented by the transformation

$$\zeta = b\cosh w\,, \tag{2.5}$$

leading back to the cylinder metric (2.3) ($v = \mathrm{Im}\, w$ is an angular variable from (2.5)).

The cylinder with its two circles at infinity is the basic building block for Einstein-Rosen wormholes. The metric for a flat space-time with n wormholes and $2p$ ordinary cosmic strings is (2.1) with

$$d\sigma^2 = \frac{\prod_{i=1}^{p} |\zeta - c_i|^{-8Gm_i}}{|\zeta_n^2 - b^{2n}|}\, d\zeta\, d\zeta^*, \qquad \zeta_n = \prod_{j=1}^{n} (\zeta - a_j)\,, \tag{2.6}$$

analytically extended to the Riemann surface made of two sheets joined along the n-component cut. The deficit angle at infinity corresponds to a total mass per unit length

$$M = \frac{n}{4G} + \sum_{i=1}^{p} m_i\,. \tag{2.7}$$

In the case $n = 2$, $p = 0$, Eq. (2.7) gives $M = 1/2G$, corresponding to compact spatial two-sections. These being also regular and orientable can only be tori $S^1 \times S^1$. To recover the symmetrical Riemann surface, pinch the torus along two opposite circles; this yields two tetracones with deficit angles π at each vertex, joined along the two pinches, which correspond to the two cuts of the Riemann surface. The transformation from the tetracone metric to the torus metric is (in the case $a_1 + a_2 = 0$)

$$\zeta = c\,\mathrm{sn}(cw, k) \tag{2.8}$$

(c and k constant), where sn is a bi-periodical Jacobi function.

However the metric (2.6) with $n = 2$, $p = 0$ admits a more economical analytical extension to a topologically non trivial Riemann surface with only one sheet. The torus may be pinched only once into a single tetracone joined to itself by an identification of the two edges. This identification corresponds to an identification of the two cuts, leading to the identification $\zeta \to -\zeta$ of the two sheets of the Riemann surface for the complex variable $\zeta(w)$. Such a one-sheeted extension is possible whenever the distribution of both the n cuts and the p conical singularities of the metric (2.6) is invariant under the isometry $\zeta \to -\zeta$, so that the two sheets of the symmetrical extension may be identified together. In the case $n = 2$ the resulting surface —a topological torus with a single point at infinity and p conical singularities— is a flat Wheeler-Misner wormhole. This may be asymptotically Minkowskian if $M = 0$, *i.e.* $\sum_{i=1}^{p} m_i = -1/2G$.

3 Two-string and one-string wormholes

The asymptotically Minkowskian Wheeler-Misner wormhole generated by two cosmic strings ($n = p = 2$) with negative mass/tension $m_1 = m_2 = -1/4G$ is the one-sheeted extension of the metric

$$d\sigma^2 = \frac{|\zeta^2 - c^2|^2}{|(\zeta^2 - a^2)^2 - b^4|}\, d\zeta\, d\zeta^*. \tag{3.1}$$

Depending on the values of the parameters a, b, c, we obtain two basic geometries, DD or Q:

1) *DD wormhole* ("dipole-dipole"). The geodesic pattern leads to the following geometrical construction of the $t =$ const., $z =$ const. sections Σ. Remove from the Euclidean (u, v) plane a rectangular strip, and glue together two opposite edges of length $2d$ of the resulting boundary. Then glue the remaining two edges of length $2l$ to the two ends of a truncated cylinder of circumference $2l$ and length $2L$.

We define a "path through the wormhole" as a non-contractible closed path in $\overline{\Sigma}$. There are here two such kinds of paths corresponding to the two circles of $S^1 \times S^1$. Paths crossing once the two identified segments of length $2d$ are "shorter" (than they would be in Euclidean space) by $2l$. Paths crossing once the two circular junctions of length $2l$ may be "shorter" if $L < d$, but are always "longer" if $L > d$.

2) Q *wormhole* ("quadrupole"). This two-dimensional geometry may be obtained by incising the Euclidean plane along a segment of length $2d$, also incising a torus of circumferences $2l$ and $2L$ along a matching segment of a small circle ($2l > 2d$), and gluing together the torus and the plane along the two edges of the cuts. In this case, paths "through the wormhole" (paths crossing once the two junctions) are always "longer" by at least $2L$, and may be arbitrarily long, due to the possibility of multiple windings around the large circle.

One-cosmic string wormholes may be obtained from the preceding geometries by taking limits such that the two cosmic strings (conical singularities) coincide. There are again two basic geometries. The DD_0 *wormhole* is obtained by gluing together, first two opposite edges, then the other two edges, of a rectangular hole in the Euclidean plane. To construct the *8 wormhole*, incise the plane along a segment, bring together the two vertices so that the two edges make a figure 8, and glue these two edges to the two ends of a truncated cylinder. These two geometries have a common limit, which corresponds simply to a plane with two points identified.

In the geometrical optics approximation, light is not scattered by these flat, asymptotically Minkowskian wormholes. However, light rays following inequivalent paths may be shifted relative to each other. This effect, which is similar to that of a parallel plate, multiple reflections being replaced by multiple turns around the cylinder (or the torus), gives rise to a one–dimensional (for the DD or *8* wormholes), or two–dimensional (for the Q wormhole) array of images of a point source. A fuller wave–optics treatment is under progress; one expects a non–trivial effect arising both from diffraction by the topological defects (cosmic strings) and resonance due to periodicity conditions in the cylinder (or torus).

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